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SDELab—stochastic differential equations with MATLAB

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Abstract

We introduce SDELab, a package for solving stochastic differential equations (SDEs) within MATLAB. SDELab features explicit and implicit integrators for a general class of Itô and Stratonovich SDEs, including Milstein's method, sophisticated algorithms for iterated stochastic integrals, and flexible plotting facilities.

1 Introduction

MATLAB is an established tool for scientists and engineers that provides ready access to many mathematical models. For example, ordinary differential equations (ODEs) are easily examined with tools for finding, visualising, and validating approximate solutions [20]. The main aim of our work has been to make stochastic differential equations (SDEs) as easily accessible. We introduce SDELab, a package for solving SDEs within MATLAB. SDELab features explicit and implicit integrators for a general class of Itô and Stratonovich SDEs, including Milstein's method and sophisticated algorithms for iterated stochastic integrals. Plotting is flexible in SDELab and includes path and phase plane plots that are drawn as SDELab computes. SDELab is written in C. SDELab and installation instructions are available from either

`http://www.ma.man.ac.uk/~sdelab`
`http://wws.mathematik.hu-berlin.de/~gilsing/sdelab`

This article is organised as follows: §2 introduces SDEs and the examples we work with. §3 describes the numerical integrators implemented in SDELab, including methods for Itô and Stratonovich equations, and the special case of small noise. §4 discusses approximation of iterated integrals. §5 shows how SDELab is used and includes the code necessary to approximate geometric Brownian motion. §6 uses the explicit solution for geometric Brownian motion to test the SDELab integrators. We also show that SDELab is much faster when dynamic libraries are used to specify the SDE rather than m-files. §7 uses SDELab to investigate the bifurcation behaviour of the van der Pol Duffing system.

2 Stochastic Differential Equations (SDEs)

An SDE is an important model in science and engineering when noise affects behaviour. For example, SDEs can model the trajectory of a distinguished particle subject to impacts by gas particles or the

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rapidly fluctuating prices of the stock market. A detailed review of SDEs and their mathematical theory can be found in many texts, including [16, 17], and we only cover the bare essentials. An SDE is a differential equation forced by white noise $\xi(t) \in \mathbf{R}^m$. Consider the equation for initial data $y_0 \in \mathbf{R}^d$

$$\frac{dY(t)}{dt} = f(t, Y(t)) + g(t, Y(t))\,\xi(t), \qquad Y(t_0) = y_0, \tag{2.1}$$

where $f\colon \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ is known as the drift and $g\colon \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^{d\times m}$ is the diffusion function. The white noise $\xi(t)$ only exists in an integral sense and is usually interpreted through Brownian motion. An $\mathbf{R}^m$ Brownian motion $W(t)$ is a process with increments $W(t) - W(s)$ for $s < t$ that are independent on disjoint intervals $[s, t]$ and have distribution $N(0, |t-s|I_m)$ (the Gaussian distribution with mean 0 and covariance $|t - s|I_m$, where I_m is the $m \times m$ identity matrix). Let $\mathbf{E}$ denote the average over samples of the Brownian motion: if $W(0) = 0$ then $\mathbf{E}W(t)W(t)^T = t\,I_m$ and we see the rescaled Brownian motion $\alpha^{-1/2}W(\alpha t)$ is also a Brownian motion. In other words, $W(t)$ scales locally like $\sqrt{t}$, which is made precise by the law of iterated logarithms, and it is not surprising that $W(t)$ has no derivative. To have $\xi(t) = dW(t)/dt$ in (2.1), consider the integral form

$$Y(t) = y_0 + \int_{t_0}^{t} f(s, Y(s))\,ds + \int_{t_0}^{t} g(s, Y(s))\,dW(s), \tag{2.2}$$

where the second integral is *the Itô Integral* defined as the limit as $N \to \infty$ of

$$\sum_{i=0}^{N} g(s_i, Y(s_i))(W(s_{i+1}) - W(s_i)),$$

for a partition $t_0 = s_0 < s_1 < \cdots < s_N = t$ of the interval $[t_0, t]$. Notice the integrand is evaluated at s_i, the left hand point of the interval $[s_i, s_{i+1}]$. Replacing $g(s_i, Y(s_i))$ by $g(\hat{s}_i, Y(\hat{s}_i))$ where $\hat{s}_i = \frac{1}{2}(s_i + s_{i+1})$ yields the Stratonovich integral, which is normally denoted $\displaystyle\int_{t_0}^{t} g(s, Y(s)) \circ dW(s)$. These are the two main ways of interpreting stochastic integrals, others are available by modifying $\hat{s}_i$, and they lead to a well defined theory of stochastic integrals and differential equations.

We will consider the Itô SDE

$$dY = f(t, Y)\,dt + g(t, Y)\,dW, \quad Y(t_0) = y_0, \tag{2.3}$$

and also the Stratonovich SDE

$$dY = f(t, Y)\,dt + g(t, Y) \circ dW, \quad Y(t_0) = y_0, \tag{2.4}$$

where Y and W are evaluated at time t. This is common short hand for the Itô integral form (2.2) (and the corresponding Stratonovich integral form) We will assume that f, g are sufficiently regular that the SDEs have a unique solution $Y(t)$ on $[t_0, T]$ for each y_0. In general this is hard to establish, though some general theory is available. For example [16], if f and g are continuous in (t, Y) and globally Lipschitz in Y, there is a unique solution.

We will test `SDELab` with the following examples.

Geometric Brownian motion. The basic model of stock prices in mathematical finance is geometric Brownian motion, the solution of the following one dimensional SDE ($d = m = 1$)

$$dY = r\,Y\,dt + \sigma\,Y\,dW, \quad Y(0) = y_0 \tag{2.5}$$

where r is the interest rate and σ is the volatility. Because of the roughness of $W(t)$, the deterministic chain rule does not hold and $dY^2 \neq 2Y dY$. Even though Brownian motion is nowhere differentiable

almost surely, it is Hölder continuous up to exponent $1/2$ and this provides boundedness of variations of second order. These second order terms are significant and change the rules of calculus, most famously in the Itô formula: for smooth $\phi\colon \mathbf{R}\times\mathbf{R}^d\to\mathbf{R}$

$$\begin{aligned} d\phi(t,Y) &= \Big(\phi_t(t,Y)+\phi_Y(t,Y)f(t,Y) \\ &+\tfrac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{m}\phi_{Y_iY_j}(s,Y)g_{ik}(t,Y)g_{jk}(t,Y)\Big)\,dt+\phi_Y(t,Y)g(t,Y)dW. \end{aligned} \tag{2.6}$$

Applying this formula with $\phi(t,Y)=\ln Y$, it is easy to verify that geometric Brownian motion

$$Y(t)=y_0\exp((r-\tfrac{1}{2}\sigma^2)t+\sigma W(t)). \tag{2.7}$$

The Stratonovich integral is the choice of $\hat{s}_i$ for which the chain rule has no second order term and the solution of

$$dY=r\,Y\,dt+\sigma\,Y\circ dW,\quad Y(0)=y_0$$

is easily found to be

$$Y(t)=y_0\exp(r\,t+\sigma W(t)). \tag{2.8}$$

An alternative approach to this is the following transformation of the drift (see [17], page 125): The solution $Y(t)$ of (2.3) solves the Stratonovich SDE

$$dY=\Big[f(t,Y)-\frac{1}{2}\Big(\sum_{j=1}^{m}\frac{\partial}{\partial y}g_j(t,Y)g_j(t,Y)\Big)\Big]\,dt+g(t,Y)\circ dW, \tag{2.9}$$

with initial data $Y(t_0)=y_0$ and where $g_j(t,Y)$ is the jth column of $g(t,Y)$. Thus, the solution $Y(t)$ of (2.5) obeys the Stratonovich equation

$$dY=(r-\tfrac{1}{2}\sigma^2)Y dt+\sigma Y\circ dW.$$

The Stratonovich solution (2.8) is now available using the Itô solution (2.7).

The choice of stochastic integral is part of the modelling process and has significant impact on the solution. As in (2.7)–(2.8), the Itô interpretation of the integral includes an $\exp(-\frac{1}{2}\sigma^2 t)$ factor, which stabilises the solution $Y(t)$ in comparison to the Stratonovich solution. Broadly speaking, the Stratonovich integral arises from a rough but *absolutely continuous* noise process, whereas the Itô interpretation results from external noise in a system that is independent of the current state. Experiments in physics with noisy electric circuits are best modelled in the Stratonovich sense [11]. The Itô integral is a martingale (so that the stochastic integrals have zero mean) and this is decisive in finance as it corresponds to the common theoretical no-arbitrage assumption of ideal markets. `SDELab` supports both types of model.

Van der Pol Duffing. Consider the van der Pol Duffing system [1] ($d=2, m=1$) where $Y=(Y_1,Y_2)^T$,

$$dY=\begin{pmatrix} Y_2 \\ \alpha Y_1+\beta Y_2-AY_1^3-BY_1^2Y_2\end{pmatrix}\,dt+\begin{pmatrix}0\\ \sigma Y_1\end{pmatrix}dW, \tag{2.10}$$

where α,β,A,B are parameters and σ is noise intensity. This second order system is typical of many problems in physics where the noise impinges directly only on Y_2, which represents the momentum of an oscillator. In constant temperature molecular dynamics, the Langevin equations [15] models have this character. The Itô notation is used, but in this case Stratonovich and Itô interpretations are the same.

This system does not have an explicit solution and numerical approximations are required. There are two types of approximation that we may be interested in. The first is *pathwise or strong approximation*: for a given sample of the Brownian path $W(t)$, compute the corresponding $Y(t)$. This is of interest for example in understanding how a change in one of the parameters affects behaviour. The second is *weak approximation*: for a given test function $\phi\colon \mathbf{R}^d \to \mathbf{R}$, compute the average of $\phi(Y(t))$. For example, we may like to know the average kinetic energy at time t, case $\phi(Y) = \frac{1}{2}Y_2^2$. Release 1 of `SDELab` focuses on strong approximations and we illustrate its use in understanding the dependence of trajectories on parameters in §7.

Geometric Brownian motion in $\mathbf{R}^d$. Consider the following generalisation of geometric Brownian motion to d dimensions [7, Page 151]:

$$dY = AY\,dt + \sum_{i=1}^{m} B_i Y dW_i(t), \tag{2.11}$$

where $A, B_i \in \mathbf{R}^{d\times d}$ and W_i are independent scalar Brownian motions for $i = 1, \ldots, m$. If the matrices A, B_i all commute (so that $AB_i = B_iA$ and $B_iB_j = B_jB_i$ for $i, j = 1, \ldots, m$),

$$Y(t) = \exp\Big(\Big(A - \tfrac{1}{2}\sum_{i=1}^{m} B_i^2\Big)t + \sum_{i=1}^{m} B_i W_i(t)\Big)y_0. \tag{2.12}$$

We take advantage of the exact solution in §6 to demonstrate the convergence of the methods in `SDELab`.

3 Integrators

3.1 Itô SDEs – Euler methods

The basic integrators for the ODE $dY/dt = f(t, Y)$ are

$$Z_{n+1} = Z_n + \Big[(1-\alpha)f(t_n, Z_n) + \alpha f(t_{n+1}, Z_{n+1})\Big]\Delta t, \quad Z_0 = y_0,$$

where α is a parameter in $[0, 1]$, Δt is the time step, and $t_n = t_0 + n\Delta t$. It is well known that this method converges to the exact solution on $[t_0, T]$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbf{R}^d$ and $\mathcal{O}(\Delta t^p)$ denote a quantity bounded by $K\Delta t^p$, where K is independent of Δt but dependent on the differential equation, the time interval, and initial data. If f is sufficiently smooth, $\|Y(t_n) - Z_n\| = \mathcal{O}(\Delta t^p)$ when $t_0 \le t_n \le T$ for $p = 1$ (respectively, $p = 2$) if $\alpha \ne 1/2$ (resp., $\alpha = 1/2$).

These methods are extended to Itô SDEs as follows:

$$Z_{n+1} = Z_n + \Big[(1-\alpha)f(t_n, Z_n) + \alpha f(t_{n+1}, Z_{n+1})\Big]\Delta t + g(t_n, Z_n)\Delta W_n, \tag{3.1}$$

where initial data $Z_0 = y_0$ and $\Delta W_n = W(t_{n+1}) - W(t_n)$. We call this the Strong Itô Euler method and in the explicit case ($\alpha = 0$) it is often called the Euler-Maruyama method following [10].

The Strong Itô Euler methods provide accurate pathwise solutions for small time steps if the drift and diffusion are well behaved. Use $\|\cdot\|$ also to denote the Frobenius norm on $\mathbf{R}^{d\times m}$ and denote by $\mathbf{E}$ the expectation over samples of the Brownian motion. Suppose [6, Theorem 10.2.2] for a constant $K > 0$ that f and g obey

$$\begin{aligned} \|f(t, Y_1) - f(t, Y_2)\| + \|g(t, Y_1) - g(t, Y_2)\| &\le K\|Y_1 - Y_2\|, \\ \|f(t, Y)\| + \|g(t, Y)\| &\le K(1 + \|Y\|), \\ \|f(s, Y) - f(t, Y)\| + \|g(s, Y) - g(t, Y)\| &\le K(1 + \|Y\|)|s - t|^{1/2} \end{aligned} \tag{3.2}$$

for $t_0 \le t \le T$ and $Y, Y_1, Y_2 \in \mathbf{R}^d$. Then, the solution Z_n of (3.1) converges to the solution $Y(t)$ of (2.3) and has strong order 1/2; i.e., $(\mathbf{E}\|Y(t_n) - Z_n\|^2)^{1/2} = \mathcal{O}(\Delta t^{1/2})$ for $t_0 \le t_n \le T$. The conditions (3.2) are restrictive and do not apply for instance to the van der Pol Duffing system. Theory is available [5] for systems with locally Lipschitz f if the moments can be controlled, but it is hard to characterise completely when the methods will converge. The user should be aware that integrators in `SDELab` may fail if asked to approximate an SDE with poor regularity.

There are two main issues in implementing this class of method: the generation of random numbers and solution of nonlinear equations. To generate the increments ΔW_n, we must take m independent samples from the distribution $N(0, \Delta t)$. `SDELab` employs the Ziggurat method [9]. This method covers the Gaussian distribution curve with a set of regions, comprising rectangles and a wedge shaped area for the tail. By careful choice of the covering, a Gaussian sample is generated by choosing a region from the uniform distribution and rejection sampling on the chosen region. A very efficient implementation is provided [9] that uses 255 rectangles and is able to generate a Gaussian sample using only two look up table fetches and one magnitude test 99% of the time. For efficiency, the method is implemented in C within `SDELab`, rather than calling MATLAB's own random number generators.

For $\alpha \neq 0$, the integrator requires solution of a system of nonlinear equations for all but the most trivial drift functions. We employ Minpack [14], a library of FORTRAN routines freely available through `http://www.netlib.org/`, to solve the nonlinear equations. Minpack implements a variation of the Powell hybrid method [18] that can be used with exact or numerical derivatives.

3.2 Milstein methods

The basic tool for developing integration methods of higher order is Taylor expansions. Taylor expansions for Itô equations may be developed as follows: Expand both drift and diffusion terms in (2.3) using the Itô Formula:

$$\begin{aligned} df(t,Y) =& f_t(t,Y)\,dt + f_Y(t,Y)f(t,Y)\,dt + f_Y(t,y)g(t,Y)\,dW \\ &+ \tfrac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{m} f_{Y_iY_j}(s,Y)g_{ik}(t,Y)g_{jk}(t,Y)\,dt \end{aligned}$$

and similarly for $g(t,Y)$. Substituting these expressions back into (2.3) yields

$$\begin{aligned} Y(T) - y_0 &= \int_{t_0}^{T}\Big[f(t_0,y_0) + \int_{t_0}^{t} f_t(s,Y)\,ds + \dots\Big]\,dt \\ &+ \int_{t_0}^{T}\Big[g(t_0,y_0) + \int_{t_0}^{t} g_t(s,Y)\,ds + \cdots + \int_{t_0}^{t} g_Y(r,Y(r))g(r,Y(r))dW(r)\Big]\,dW. \end{aligned}$$

Further iteration yields an expansion akin to the Taylor expansion that can be truncated to find new integrators in terms of iterated integrals

$$\int_0^{\Delta t}\int_0^{s_1}\cdots\int_0^{s_{p-1}} dW_{i_p}(s_p)dW_{i_{p-1}}(s_{p-1})\dots dW_{i_1}(s_1),$$

where $dW_0 = dt$ and $i_k \in \{0, 1, \dots, m\}$. These terms have order $\Delta t^{(p+q)/2}$, where q is the number of $i_j = 0$, and are the generalisation of the building blocks Δt^p of the deterministic Taylor expansion. It is difficult to compute these quantities. Usually, the work involved outweighs the benefits of high order convergence and `SDELab` provides integrators that depend on the first level of iterated integrals

only. The basic example is the Milstein method [12],

$$\begin{aligned} Z_{n+1} =& Z_n + \Big[(1-\alpha)f(t_n, Z_n) + \alpha f(t_{n+1}, Z_{n+1})\Big]\Delta t + g(t_n, Z_n)\Delta W_n \\ &+ \sum_{j=1}^{m} \frac{\partial}{\partial y} g_j(t_n, Z_n)\,\Big(g(t_n, Z_n)\xi_j\Big), \qquad Z_0 = y_0 \end{aligned} \tag{3.3}$$

where $g_j(t, Z)$ is the jth column of $g(t, Z)$, $\xi_j = (I_{1j,n}, \ldots, I_{mj,n})^T$, and

$$I_{ij,n} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{r} dW_i(s)\, dW_j(r).$$

We discuss how `SDELab` approximates ξ_j in §4. To implement this method without requiring the user to specify the derivative of g, we include derivative free versions,

$$\begin{aligned} Z_{n+1} =& Z_n + \Big[(1-\alpha)f(t_n, Z_n) + \alpha f(t_{n+1}, Z_{n+1})\Big]\Delta t + g(t_n, Z_n)\Delta W_n \\ &+ \sum_{j=1}^{m} Dg(n, j)\ \xi_j, \qquad Z_0 = y_0 \end{aligned} \tag{3.4}$$

where $Dg(n, j) = (g(t_n, Z^{aux}_{n,j}) - g(t_n, Z_n))/\Delta t^{1/2}$ and the support vectors $Z^{aux}_{n,j}$ can be set in `SDELab` as $Z^{aux}_{n,j} = Z_n + \Delta t^{1/2} g(t_n, Z_n)e_j$ or $Z^{aux}_{n,j} = Z_n + \Delta t f(t_n, Z_n) + \Delta t^{1/2} g(t_n, Z_n)e_j$ (where e_j denotes the jth standard basis function in $\mathbf{R}^m$). The Milstein methods converge with order 1, more rapidly than the order 1/2 convergence of the Euler methods. Further regularity on f and g is required, but details are not given here; see [6, page 345].

3.3 Small noise

Many SDEs of interest in science and engineering feature small noise and have the form

$$dY = f(t, Y)\, dt + \epsilon\, g(t, Y)\, dW, \quad Y(t_0) = y_0, \tag{3.5}$$

for a small parameter ϵ. Certain methods are especially useful in this context, as they give an improvement over the explicit Euler method when $\epsilon \ll \Delta t$ and this improvement does not depend on iterated integrals and therefore is efficient. This is true of the $\alpha = 1/2$ Euler method. `SDELab` also provides the second order Backward Differentiation Formula (BDF 2) method:

$$\begin{aligned} Z_{n+1} =& \frac{4}{3}Z_n - \frac{1}{3}Z_{n-1} + \frac{2}{3}f(t_{n+1}, Z_{n+1})\Delta t \\ &+ g(t_n, Z_n)\Delta W_n - \frac{1}{3}g(t_{n-1}, Z_{n-1})\Delta W_{n-1} \end{aligned} \tag{3.6}$$

for $n \geq 2$ and with starting values given by

$$Z_1 = Z_0 + \Big[\frac{1}{2}f(t_0, Z_0) + \frac{1}{2}f(t_1, Z_1)\Big]\Delta t + g(t_0, Z_0)\Delta W_0, \qquad Z_0 = y_0.$$

The solution Z_n from either BDF 2 (3.6) or Euler (3.1) with $\alpha = 1/2$ satisfies $(\mathbf{E}\|Y(t_n) - Z_n\|^2)^{\frac{1}{2}} = \mathcal{O}(\Delta t^2 + \epsilon\Delta t + \epsilon^2\Delta t^{\frac{1}{2}})$ for $t_0 \leq t_n \leq T$. See [2, 13] for further details. In the small noise case $\epsilon \ll \Delta t$, the $\mathcal{O}(\epsilon^2\Delta t^{1/2})$ term becomes negligible and the error is $\mathcal{O}(\epsilon\Delta t + \Delta t^2)$. The methods look like they have order 1 for certain Δt even though in the limit $\Delta t \to 0$ they are order 1/2.

3.4 Stratonovich SDEs

The Itô methods can be used to approximate Stratonovich SDEs by converting to the Itô formulation. To work directly with the Stratonovich SDE, `SDELab` provides the Euler Heun and Stratonovich Milstein methods. The Euler Heun method with parameter $\alpha \in [0,1]$ is

$$\begin{aligned} Z_{n+1} =& Z_n + \Big[(1-\alpha)f(t_n, Z_n) + \alpha f(t_{n+1}, Z_{n+1})\Big]\Delta t \\ &+ \tfrac{1}{2}\Big[g(t_n, Z_n^{aux}) + g(t_n, Z_n)\Big]\Delta W_n, \qquad Z_0 = y_0 \end{aligned}$$

with $Z_n^{aux} = Z_n + g(t_n, Z_n)\Delta W_n$. The Euler Heun method converges to the solution of (2.4) with order 1 when the noise is commutative and order 1/2 otherwise. The Stratonovich Milstein method with parameter α is

$$\begin{aligned} Z_{n+1} =& Z_n + \Big[(1-\alpha)f(t_n, Z_n) + \alpha f(t_{n+1}, Z_{n+1})\Big]\Delta t \\ &+ g(t_n, Z_n)\Delta W_n + \sum_{j=1}^{m} \frac{\partial}{\partial y} g_j(t_n, Z_n)\,\Big(g(t_n, Z_n)\xi_j\Big), \qquad Z_0 = y_0 \end{aligned}$$

where $\xi_j = (J_{1j,n}, \ldots, J_{mj,n})^T$ for the iterated Stratonovich integral $J_{ij,n} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{r} \circ dW_i(s) \circ dW_j(r)$. The Stratonovich Milstein method converges to the solution of (2.4) with order 1. Again `SDELab` includes versions that do not require user supplied derivatives.

4 Iterated Stochastic Integrals

We look at how `SDELab` generates second order iterated integrals. We work with Stratonovich iterated integrals on $[0, \Delta t]$ and use the notation

$$J_i = \int_0^{\Delta t} dW_i(s), \quad J_{ij} = \int_0^{\Delta t} \int_0^{r} \circ dW_i(s) \circ dW_j(r).$$

`SDELab` computes the second order Itô integrals from the Stratonovich version by $I_{ij} = J_{ij}$ for $i \neq j$ and $I_{ii} = J_{ii} - \frac{1}{2}\Delta t$ for $i = 1, \ldots, m$. There are a number of important special cases that are used by `SDELab` to improve efficiency. If the diffusion $g(t, Y)$ is *diagonal*, J_{ij} are not required for $i \neq j$. If

$$\frac{\partial}{\partial x_i} g_{kj}(t, Y) g(t, Y) = \frac{\partial}{\partial x_j} g_{ki}(t, Y) g(t, Y), \tag{4.1}$$

for $k = 1, \ldots, d$ and $i, j = 1, \ldots, m$, the diffusion is said to be *commutative* and the identity $J_{ij} + J_{ji} = J_i J_j$ is used to simplify the Milstein method. In particular, we compute only $J_i J_j$ and avoid the difficult off diagonal iterated integrals. If $g(t, Y)$ does has not have the above structures, we must approximate each J_{ij}. There are a number of efficient methods [19, 4] for sampling J_{ij} with $m = 2$. Unfortunately, J_{ij} cannot be generated pairwise for $m > 2$ because correlations are significant. Such specialist methods are not included in `SDELab` as we prefer algorithms that are widely applicable. `SDELab` generates samples using a truncated expansion of the Brownian bridge process with a Gaussian approximation to the tail.

The Brownian bridge process $W_j(t) - (t/\Delta t)W_j(\Delta t)$ for $0 \leq t \leq \Delta t$ has Fourier series

$$W_j(t) - \frac{t}{\Delta t} W_j(\Delta t) = \tfrac{1}{2} a_{j0} + \sum_{r=1}^{\infty} a_{jr} \cos \frac{2\pi r t}{\Delta t} + b_{jr} \sin \frac{2\pi r t}{\Delta t}, \tag{4.2}$$

for $j = 1, \ldots, m$, where (by putting $t = 0$)

$$a_{j0} = -2 \sum_{r=1}^{\infty} a_{jr} \tag{4.3}$$

and the coefficients b_{jr}, a_{jr} are independent random variables with distributions $N(0, \Delta t/2\pi^2 r^2)$ for $r = 1, 2, \ldots$, which is easily derived from the Fourier integrals

$$\begin{aligned} a_{jr} =& \frac{2}{\Delta t} \int_0^{\Delta t} \Big(W_j(s) - \frac{s}{\Delta t} W_j(\Delta t) \Big) \cos \frac{2\pi\, r\, s}{\Delta t}\, ds, \\ b_{jr} =& \frac{2}{\Delta t} \int_0^{\Delta t} \Big(W_j(s) - \frac{s}{\Delta t} W_j(\Delta t) \Big) \sin \frac{2\pi\, r\, s}{\Delta t}\, ds. \end{aligned}$$

This representation was developed [7] to express numerically computable formulae for iterated stochastic integrals and in particular J_{ij} by truncating the expansions to p terms. By integrating (4.2) over $[0, \Delta t]$ with respect to dt, $J_{i0} = \frac{1}{2}\Delta t(J_i + a_{i0})$, and using the symmetry relation $J_{0i} + J_{i0} = J_i J_0$, we see $J_{0i} = \frac{1}{2}\Delta t(J_i - a_{i0})$. Integrating (4.2) over $[0, \Delta t]$ with respect to $W_j(t)$ yields

$$J_{ij} = \tfrac{1}{2} J_i J_j - \tfrac{1}{2}(a_{j0} J_i - a_{i0} J_j) + \Delta t A_{ij}, \quad i, j = 1, \ldots, m, \tag{4.4}$$

where $A_{ij} = \dfrac{1}{\Delta t} \displaystyle\sum_{r=1}^{\infty} \zeta_{ir}^* \eta_{jr}^* - \eta_{ir}^* \zeta_{ji}^*$ and $\eta_{jr}^* = \sqrt{\pi r} a_{jr}$ and $\zeta_{jr}^* = \sqrt{\pi r} b_{jr}$. Because ζ_{jr}^*, η_{jr}^*, and $W_j(\Delta t)$ are independent, we easily find ζ_{jr}^* and η_{jr}^* by sampling from $N(0, \Delta t/2\pi\, r)$. We approximate A_{ij} by truncating the sum to p terms,

$$A_{ij}^p = \frac{1}{\Delta t} \sum_{r=1}^{p} \zeta_{ir}^* \eta_{jr}^* - \eta_{ir}^* \zeta_{ji}^*, \tag{4.5}$$

and define the approximate iterated integral $J_{ij}^p = \frac{1}{2} J_i J_j - \frac{1}{2}(a_{j0}^p J_i - a_{i0}^p J_j) + \Delta t A_{ij}^p$, where from (4.3)

$$a_{i0}^p = -\sum_{r=1}^{p} \frac{2}{\sqrt{\pi\, r}}\, \zeta_{ir}^*. \tag{4.6}$$

To understand the importance of the tail correction, consider the estimate

$$\Big(\mathbf{E}\Big[|\tilde{J}_{ij}^p - J_{ij}|^2 \Big] \Big)^{1/2} \le \frac{\Delta t}{\sqrt{2p}\, \pi},$$

which holds for approximations $\tilde{J}_{ij}^p = \frac{1}{2} J_i J_j - \frac{1}{2}(\tilde{a}_{j0}^p J_i - \tilde{a}_{i0}^p J_j) + \Delta t A_{ij}^p$ that include a higher order correction to $\tilde{a}_{i0}^p$,

$$\tilde{a}_{i0}^p = a_{i0}^p - 2\sqrt{\Delta t \rho_p}\, \mu_{jp}, \tag{4.7}$$

where $\mu_{jp} = (1/\sqrt{\Delta t \rho_p}) \sum_{r=p+1}^{\infty} a_{jr}$ and $\rho_p = (1/12) - (1/2\pi^2) \sum_{r=1}^{p} 1/r^2$. To use $\tilde{J}_{ij}^p$ in the Milstein scheme with time step Δt, we want errors of $\mathcal{O}(\Delta t^{3/2})$ and the number of terms in the expansions p should be $\mathcal{O}(1/\Delta t)$.

Wiktorsson [21] introduced a technique that reduces the number of terms p necessary to achieve an $\mathcal{O}(\Delta t^{3/2})$ error. Recall the Levy area

$$\mathcal{A}_{ij} = \tfrac{1}{2}(J_{ij} - J_{ji}) = a_{i0} J_j + a_{j0} J_i + \Delta t A_{ij}.$$

Wiktorsson uses the conditional (on $W_j(\Delta t)$) joint characteristic function of the Levy areas to derive a Gaussian approximation to the tail of A_{ij}. Sampling from this Gaussian provides a small correction to J_{ij}^p that improves the rate of convergence. `SDELab` implements the following algorithm:

(i) Fix a constant C. Choose the smallest number p such that

$$p \geq \frac{1}{C\pi}\sqrt{\frac{m(m-1)}{24\Delta t}}\sqrt{m + 4\sum_{j=1}^{m} W_j(\Delta t)^2/\Delta t}. \tag{4.8}$$

Note that the number of terms p grows like $1/\sqrt{\Delta t}$, not $1/\Delta t$ as in the first method, and that p depends on the path.

(ii) Using (4.5)–(4.6), compute approximations $J_{ij}^p = \frac{1}{2}J_iJ_j - \frac{1}{2}(a_{j0}^p J_i - a_{i0}^p J_j) + \Delta t A_{ij}^p$.

(iii) We now define the tail approximation $\mathcal{A}^{p,tail}$. Let $x, y \in \mathbf{R}^m$, $M = \frac{1}{2}m(m-1)$, and e_i^m denote the ith standard basis element in $\mathbf{R}^m$. We introduce $P_m \colon \mathbf{R}^{m^2} \to \mathbf{R}^M$, the linear operator defined by $P_m(x \otimes y) = y \otimes x$, and $K_m \colon \mathbf{R}^{m^2} \to \mathbf{R}^M$, the linear operator defined by $K_m(e_i^m \otimes e_j^m) = e_{k(i,j)}^M$ and $K_m(e_j^m \otimes e_i^m) = 0 = K_m(e_j^m \otimes e_j^m)$, where $i < j$ and $k(i,j)$ is the position of (i,j) in the M term sequence

$$(1,2),(1,3),\ldots,(1,m),(2,3),\ldots,(2,m),\ldots,(m-1,m).$$

Denote by I_m the $m \times m$ identity matrix. The tail approximation is

$$\mathcal{A}^{p,tail} = (I_{m^2} - P_m)\, K_m^T\, \frac{\Delta t}{2\pi}\, a_p^{1/2}\, \sqrt{\Sigma_\infty} G_p, \tag{4.9}$$

where $a_p = \sum_{k=p+1}^{\infty} k^{-2}$, $G_p \in \mathbf{R}^M$ is chosen from the distibution $N(0, I_M)$,

$$\Sigma_\infty = 2I_M + \frac{2}{\Delta t} K_m(I_{m^2} - P_m)(I_m \otimes W(\Delta t)W(\Delta t)^T)(I_{m^2} - P_m)K_m^T,$$

and $W(\Delta t) = (W_1(\Delta t), \ldots, W_m(\Delta t))^T$. To compute (4.9), we use the following expression for the square root of Σ_∞ [21]:

$$\sqrt{\Sigma_\infty} = \frac{\Sigma_\infty + 2\alpha I_M}{\sqrt{2}(1+\alpha)}, \text{ where } \alpha = \sqrt{1 + \sum_{j=1}^{m} W_j(\Delta t)^2/\Delta t}.$$

(iv) Add the correction term to J_{ij}^p to define $J_{ij}^{p+tail} = J_{ij}^p + \mathcal{A}_{ij}^{p,tail}$.

Under the truncation condition (4.8), [21] proves that

$$\max_{ij} \mathbf{E}\Big[|J_{ij} - J_{ij}^{p+tail}|^2 \Big| W(\Delta t)\Big] \leq C^2 \Delta t^3,$$

where $\mathbf{E}\Big[\,\cdot\,\Big|W(\Delta t)\Big]$ denotes the expectation of $\cdot$ conditioned on $W(\Delta t)$. In terms of Gaussian samples, the tail expansion is justified in the limit $\Delta t \to 0$. The Euler methods (3.1) required $\mathcal{O}(1)$ Gaussians per time step, Milstein (3.3) with $\tilde{J}_{ij}$ requires $\mathcal{O}(1/\Delta t)$ Gaussians, and Milstein with $\tilde{J}_{ij}^{p+tail}$ requires $\mathcal{O}(\Delta t^{-1/2})$. On the other hand, rates of convergence are $\mathcal{O}(\Delta t^{1/2})$ for Euler and $\mathcal{O}(\Delta t)$ for Milstein. Hence, to achieve a particular level of accuracy ϵ both Euler and Milstein with $\tilde{J}_{ij}$ require ϵ^{-2} samples, whilst Milstein with $\tilde{J}_{ij}^{p+tail}$ requires only $\epsilon^{-3/2}$ samples.. Asymptotically in $\Delta t \to 0$, the use of the tail approximation means fewer Gaussian samples are required.

In practice, the method is expensive for large m, because the covariance matrix Σ_∞, which is an $M \times M$ matrix where $M = \frac{1}{2}m(m-1)$, is treated as a dense matrix with $\mathcal{O}(m^4)$ entries. This is impractical for very high m as it is hard to take Δt sufficiently small to see its benefits. To give some understanding, the table compares the two methods J_{ij}^p and J_{ij}^{p+tail} for different values of m. The time to compute 10^6 samples is given (in seconds) and the error in computing the variance (again using 10^6 samples) of J_{12}/dt, which is known to equal $1/2$, is given.

m	Δt	J_{ij}^{p} : time	error	J_{ij}^{p+tail} : time	error
5	0.1	1.9	0.1	3.5	3.7e-3
	0.01	4.8	2.3e-2	3.5	5.5e-4
10	0.1	4.2	0.9e-1	13.65	6.5e-3
	0.001	66.04	2e-3	12.39	6.3e-5
100	0.1	114	0.1	3420	0.04

5 The use of SDELab

We describe the most important features of SDELab with extensive documentation provided online. To start using SDELab within MATLAB, type sdelab_init. To find approximate paths for (2.3) or (2.4), one of the following is used

```
[t,y] = sdelab_strong_solutions(fcn, tspan, y0, m, opt, ...);
sdelab_strong_solutions(fcn, tspan, y0, m, opt, ...);
```

The return values give approximate solutions y(:,i) at times t(i). If [t, y] is omitted, a MATLAB figure appears and the approximate paths are plotted as they are computed. The arguments are

fcn SDELab provides the single structure fcn for specifying the drift f and diffusion g. The fcn fields may point to a variety of implementations, including m-files, mex files, and dynamic library routines. This flexibility allows users to prototype quickly using m-files and develop efficient code by linking to dynamic libraries of C or FORTRAN routines.

When using m-files, the fields drift and diff_noise, and optional fields drift_dy and diff_noise_dy contain the names of the m-files. An example is given later. When using dynamic libraries, the fields drift, diff_noise, etc. each have subfields Libname (name of dynamic library) and Init_fcn, Exec_fcn, and Cleanup_fcn (names of functions in the dynamic library that initialise, compute, and clean up).

tspan is a vector that indicates the time interval for integration $[t_0, T]$. If tspan has more than two points, it specifies the times at which $Y(t)$ is approximated and is returned in t.

y0 is the initial condition.

m equals dimension of the Brownian motion W.

opt is a structure whose fields set SDELab options.

... is an optional list of model parameters.

The following are specified by setting the corresponding field in opt.

MaxStepSize is an upper bound on Δt, chosen so an integer multiple of steps fits into $[t_0, T]$. A default value of $(T - t_0)/100$ is used.

IntegrationMethod specifies the type of equation (Itô/Stratonovich) as well as the integrator. The default is StrongItoEuler and the options are

```
StrongItoMilstein,      StrongItoBDF2
StrongStratoEulerHeun,  StrongStratoMilstein.
```

The parameter α is controlled by the following:

`StrongItoEuler.Alpha`, `StrongItoMilstein.Alpha`,
`StrongStratoEulerHeun.Alpha`, `StrongStratoMilstein.Alpha`.

RelTol is the relative error used by Minpack as a termination criterion.

MaxFeval controls the behaviour of the nonlinear Minpack solve. If positive, `MaxFeval` is the maximum number of function evaluations allowed by Minpack. If -1, the Minpack default value is chosen.

Stats controls the reporting of number of function calls and Minpack information. It should be set to `on` or `off`.

NoiseType indicates the structure of the diffusion term. If `NoiseType=1`, the diffusion is considered to be unstructured and second order iterated integrals are approximated using Wiktorsson's method. If `NoiseType=2`, the diffusion is treated as diagonal and if `NoiseType=3` as commutative (see (4.1)).

OutputPlot is set to `on` if plots are required; `off` otherwise.

OutputPlotType specifies the type of plot. The possibilities are
`sdelab_path_plot` (path plot; default);
`sdelab_phase_plot` (two dimensional phase plot);
`sdelab_time_phase` (two dimensional path plots against time);
`sdelab_phase3_plot` (three dimensional phase plot).

OutputSel controls which components of `y` are used in the plots.

We consider how to approximate geometric Brownian motion (2.11) with `SDELab`. The drift is defined by the following m-file:

```
function [z] = drift(t, y, varargin)
 A = varargin{2}.A; % Extract parameters
 z = A*y;           % Compute drift
```

The specification of the diffusion is more involved. `SDELab` requires that we specify the diffusion in two ways: (1) as the product of the matrix $g(t, Y)$ with the Brownian increment, which is very beneficial for sparse diffusion matrices, and (2) as the matrix $g(t, Y)$. `SDELab` uses the two ways in its implementation of the Milstein methods to reduce the number of function calls.

```
function z = diff_noise(t, y, dw, flag, varargin)
 B = varargin{2}.B; % Extract parameter
 m = length(dw);
 d = length(y);
 B2 = zeros(d,m);   %  Compute the diffusion
 for (i=1:m)
    B2(:,i) = B(:,:,i) * y;
 end;
 if (flag==0)
    z = B2 * dw;
 else
    z = B2;
 end;
```

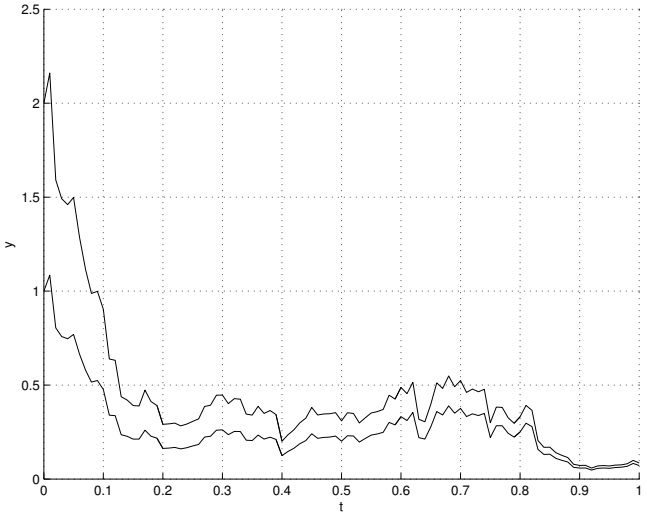

Figure 1: A realisation of geometric Brownian motion.

We may now use `SDELab` to approximates paths of (2.11). Set up the problem dimensions, time interval and initial data,

```
d = 2;          % dimension of y
m = d;          % dimension of W(t)
tspan = [0, 1]; % time interval
y0 = [1; 2];    % initial condition
```

and define the drift and diffusion functions with their parameters:

```
fcn.drift = 'drift';               % name of MATLAB m-files
fcn.diff_noise = 'diff_noise';
params.A = [-0.5, 0; 0, -1];       % parameters
params.B = zeros(d,d,m);
params.B(:, :, 1) = diag([1; 1]);
params.B(:, :, 2) = diag([1; 1]);
```

Run `SDELab` with the default method, Itô Euler with $\alpha = 0$.

```
opt.IntegrationMethod = 'StrongItoEuler';
opt.MSIGenRNG.SeedZig = 23;
sdelab_init;
sdelab_strong_solutions(fcn, tspan, y0, m, opt, params);
xlabel('t'); ylabel('y');
```

A window pops up automatically and you see the path plotted as it is computed. See Figure 1.

To assist the nonlinear solver, the user may provide derivatives of the drift function. `SDELab` can utilise a function `drift_dy` that returns the Jacobian matrix of f with entries $\partial f_i(t, Y)/\partial Y_j$ for $i, j = 1, \ldots, d$. For (2.11), the Jacobian is A.

```
function z = drift_dy(t, y, varargin)
 A = varargin{2}.A;
 z = A;
```

Let $g_{ij}(t, Y)$ denote the (i, j) entry of $g(t, Y)$ for $i = 1, \ldots, d$ and $j = 1, \ldots, m$. `SDELab` can utilise a function `diff_noise_dy` that returns the derivative of the jth column of $g(t, Y)$ with respect to Y in the direction $dy \in \mathbf{R}^d$; that is, $dg_j\, dy \in \mathbf{R}^d$, where dg_j is the $d \times d$ matrix with (i, k) entry $\partial g_{ij}/\partial Y_k$. For (2.11), $dg_j\, dy = B_j\, dy$.

```
function z = diff_noise_dy(t, y, dy, j, varargin)
 B = varargin{2}.B;
 d = length(y);
 z = B(:,:,j) * dy;
```

Finally define the `fcn` structure and compute the solution using Itô Milstein with $\alpha = 1$. Rather than plot, we store the results in `[t, y]`.

```
fcn.drift = 'drift';
fcn.drift_dy = 'drift_dy';
fcn.diff_noise = 'diff_noise';
fcn.drift_noise_dy = 'diff_noise_dy';
opt.IntegrationMethod = 'StrongItoMilstein';
opt.StrongItoMilstein.Alpha = 1.0;
[t, y] = sdelab_strong_solutions(fcn, tspan, y0, m, opt, params);
```

6 Geometric Brownian Motion

We consider the behaviour of five of `SDELab`'s Itô methods (Euler's method with $\alpha = 0, 0.5$, Milstein's method with $\alpha = 0, 0.5$, and BDF 2) for approximating geometric Brownian motion (2.11). The following matrices are used

$$A = -2I, \quad B_1 = \begin{pmatrix} 0.3106 & 0.1360 \\ 0.1360 & 0.3106 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.9027 & -0.0674 \\ -0.0674 & 0.9027 \end{pmatrix}.$$

These matrices are commutative and the exact solution (2.12) is available. Using the exact solution, we compute the strong error with L samples by

$$\Big(\frac{1}{L} \sum_{L \text{ trials}} \|Y(T) - Z_N\|^2\Big)^{1/2}, \qquad N\Delta t = T.$$

Figure 2 plots error against time step and run time (with drift and diffusion functions implemented as C dynamic library functions), with $L = 2000$. To test the approximations to the iterated integrals, we set `opt.NoiseType=1` during these calculations (rather than take advantage of the commutative structure). We see the order 1 convergence of the Milstein methods and the order 1/2 convergence of the Euler methods. Even allowing for the extra time to compute a single time step, it is more efficient to use the Milstein methods in this case. Figure 3 shows the same plot for the Stratonovich version of geometric Brownian motion. Here the Euler-Heun method has order 1 because the matrices are commutative. Figure 4 shows the same plot for the Itô equation with small noise; specifically, (2.11) with the diffusion matrices B_i replaced by ϵB_i with $\epsilon = 10^{-3}$. We clearly see the benefits of choosing the method carefully and the BDF 2 and Euler $\alpha = 0.5$ methods are most efficient. Figure 5 shows how the CPU time depends on problem dimension m. Matrices $A, B_1, \ldots, B_m$ are chosen and approximations computed using both the m-file and dynamic library implementation of the drift and diffusion functions. We see the cost of using Milstein methods scales badly with m due to the difficulties of computing the stochastic integrals. We also notice considerable speed improvements in using a dynamic library implementation.

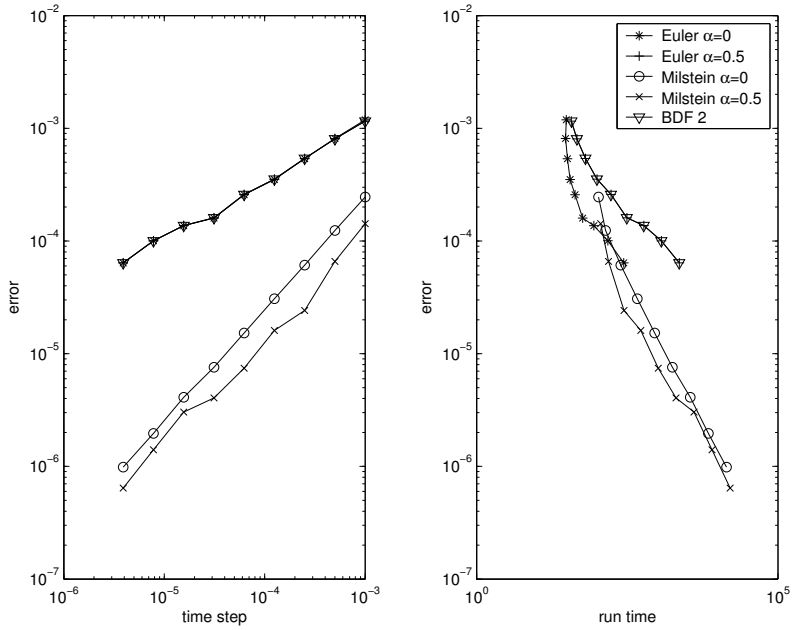

Figure 2: Plots of error (computed with 2000 samples) against time step and run time for Itô geometric Brownian motion. Notice Milstein methods have order 1 convergence, and the Euler and BDF 2 methods have order 1/2 convergence.

7 Van der Pol Duffing

Consider the van der Pol Duffing system (2.10) with parameters $A = B = 1$. We use the plotting facilities of `SDELab` to illustrate two bifurcations in this system. In order to use the same Brownian path for each plot, we set the seed of the random number generator at the start of each simulation. This is effective if we fix the time step for all our simulations. In the long run, we would like to add functionality to decrease the time step and refine the same Brownian motion.

The MATLAB m-files are given in Appendix A. Set the `fcn` structure as in the previous example, and set the dimensions, initial data, and time interval for the van der Pol Duffing system:

```
d = 2;  m = 1;            % problem dimensions
tspan = [0, 500];         % time interval
y0 = [0.0, 0.0001];       % inital condition
```

Define the problem parameters:

```
params.alpha = -1.0; params.beta = 0.1;
params.A = 1.0;      params.B = 1.0;
params.sigma = 0.1;
```

A phase plot with seed 23 and maximum time step 0.01 can be produced as follows:

```
opt.MaxStepSize = 1e-2;
opt.OutputPlotType = 'sdelab_phase_plot';
opt.MSIGenRNG.SeedZig = 23;
sdelab_strong_solutions(fcn, tspan, y0, m, opt, params);
```

It is now easy to explore the dynamical behaviour of the system and its response to varying α and β. Without noise (case $\sigma = 0$) and with $\beta < 0$, the system experiences a pitchfork bifurcation

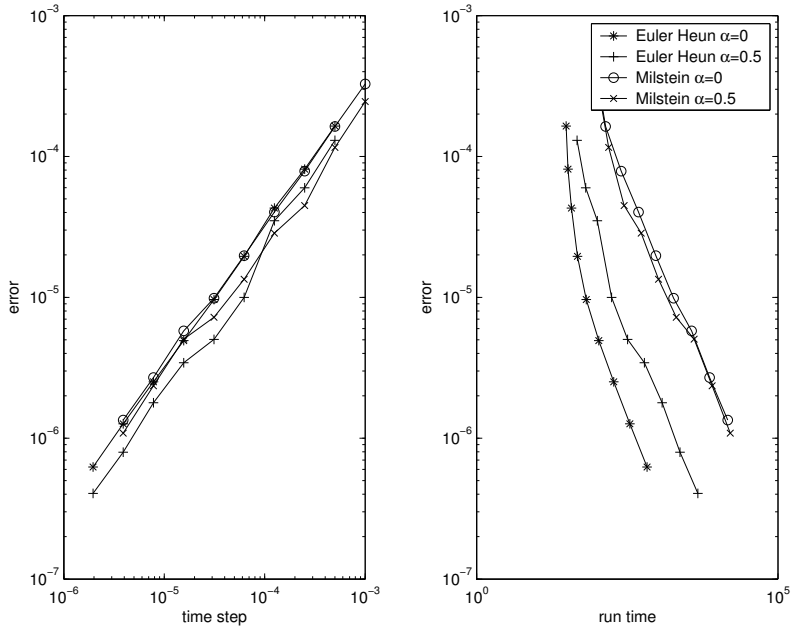

Figure 3: Plots of error (computed with 2000 samples) against time step and run time for Stratonovich geometric Brownian motion. As the noise is commutative, the Euler Heun method has the same order 1 convergence as the Milstein methods. The Milstein method is slower to compute, as this test was done *without* the commutative noise flag set.

(when a fixed point loses its stability and gives rise two to stable fixed points) as the parameter α crosses 0. We explore this situation for $\sigma = 0.4$ in Figure 6. We see the dynamics do not settle down to fixed points when there is noise, but oscillate near to meta stable states. Only the last plot shows the two meta stable created by the bifurcation (notice the change in scale on the plots), even though three of the figures have parameter $\alpha \geq 0$. This is a well known phenomenon [1]: noise delays a pitchfork bifurcation. In this case, the bifurcation is delayed until $\alpha \approx 0.1$.

If $\alpha < 0$, a Hopf bifurcation (or creation of a periodic orbit) can be found in the deterministic system as the parameter β crosses 0. With noise present, the bifurcation point is known to occur for $\beta < 0$. Figure 7 illustrates the behaviour of (2.10) with $\alpha = -1$ and $\sigma = 0.1$ for values of $\beta = -0.1, -0.01, 0, 0.1$. We see the trajectories are focused on a circle for $\beta \geq -0.01$, which shows that the bifurcation occurs for negative β.

8 Further directions

There are a number of ways we would like to develop `SDELab`.

(i) `SDELab` does not provide special algorithms for computing averages of $\phi(Y(t))$. This is an important problem and will be dealt with in future releases of `SDELab`.

(ii) One of the key features of the MATLAB ODE suite is its use of error estimation to select time step sizes. The theory of error estimation for SDE integrators is not well developed (see [8, 3] for recent work) and we are unaware of any technique robust enough for inclusion in a software package for (2.3) or (2.4). We hope the algorithms will mature and eventually be included in `SDELab`.

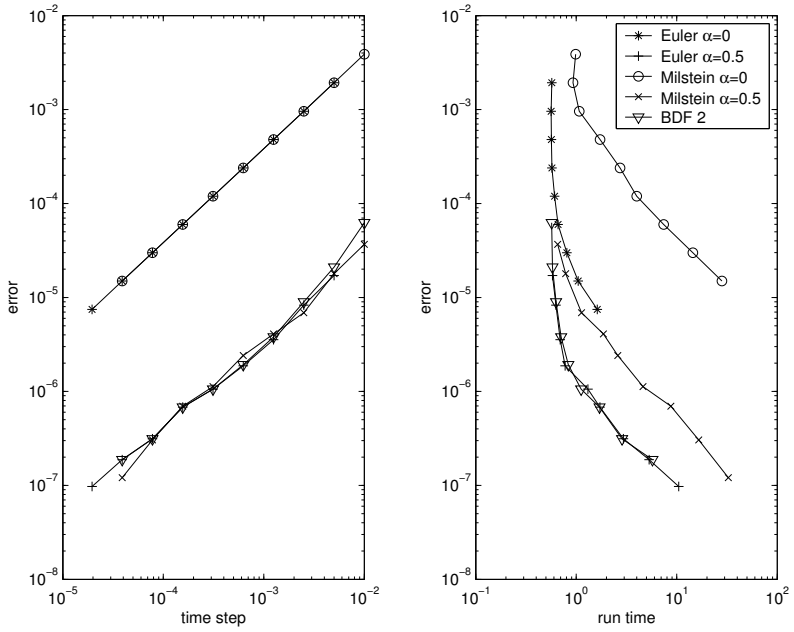

Figure 4: Plots of error (computed with 2000 samples) against time step and run time for the Itô equation (2.11) with small noise. Each method appears to have order 1.

(iii) It is frequently of interest to determine times at which certain events happen, such as $Y(t)$ crossing a barrier. At this time, no algorithms are included in `SDELab`.

(iv) Some classes of SDEs deserve special attention, such as Langevin equations, geometric Brownian motion, and the Ornstein-Uhlenbeck process, and we would like to address this within `SDELab`.

A Van der Pol Duffing

```
function z = drift(t,y,varargin)
alpha = varargin{2}.alpha; % Extract parameters
beta = varargin{2}.beta;
A = varargin{2}.A;
B = varargin{2}.B;
z=[ y(2); ...
    (beta-B*y(1)*y(1))*y(2)+ (alpha-A*y(1)*y(1))*y(1) ];

function z = drift_dy(t,y,varargin)
alpha = varargin{2}.alpha; % Extract parameters
A = varargin{2}.A;
B = varargin{2}.B;
z=[           0                                  1      ; ...
    alpha-(3*A*y(1)+2*B*y(2))*y(1)  beta-B*y(1)*y(1)];

function z = diff_noise(t,y,dw,flag,varargin)
sigma = varargin{2}.sigma;
if (flag)
```

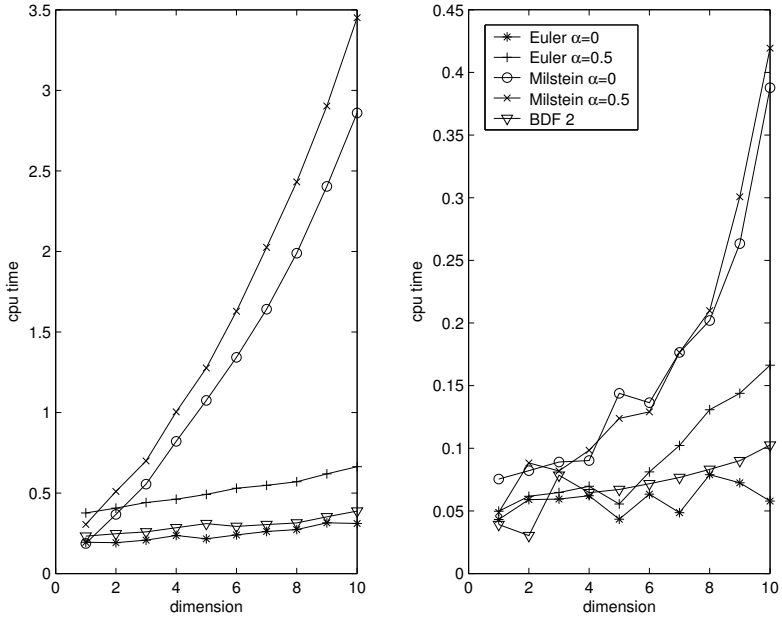

Figure 5: Plots of CPU time against problem dimension $m = d$ for the Itô equation (2.11) for both MATLAB m-files (left) and dynamic library functions (right).

```
  z =[0; sigma*y(1)];       % Return g(t,y)
else
  z =[0; sigma*y(1)*dw];    % Compute g(t,y) * dw
end;

function z = diff_noise_dy(t, y, dw, j, varargin)
sigma = varargin{2}.sigma;
z=[0; sigma * dw(1)];
```

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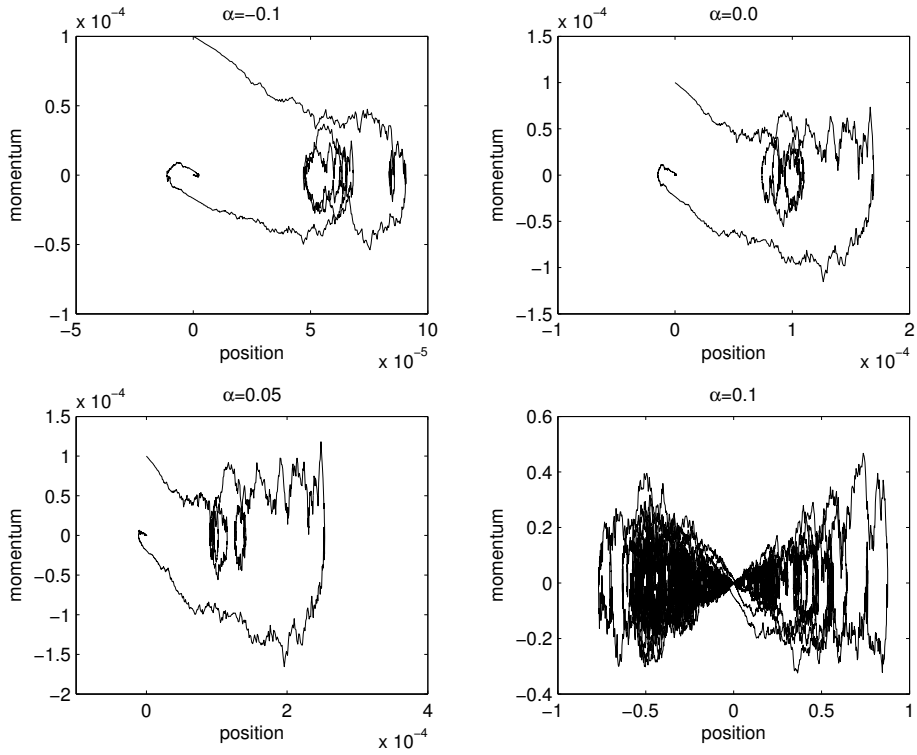

Figure 6: Paths of (2.10) for $\alpha = -0.2, 0, 0.05, 0.1$ with $\beta = -1.0$ over time interval $[500, 1000]$ with initial data $[0, 0.001]$ specified at $t_0 = 0$. Notice the change of scale in the bottom right plot.

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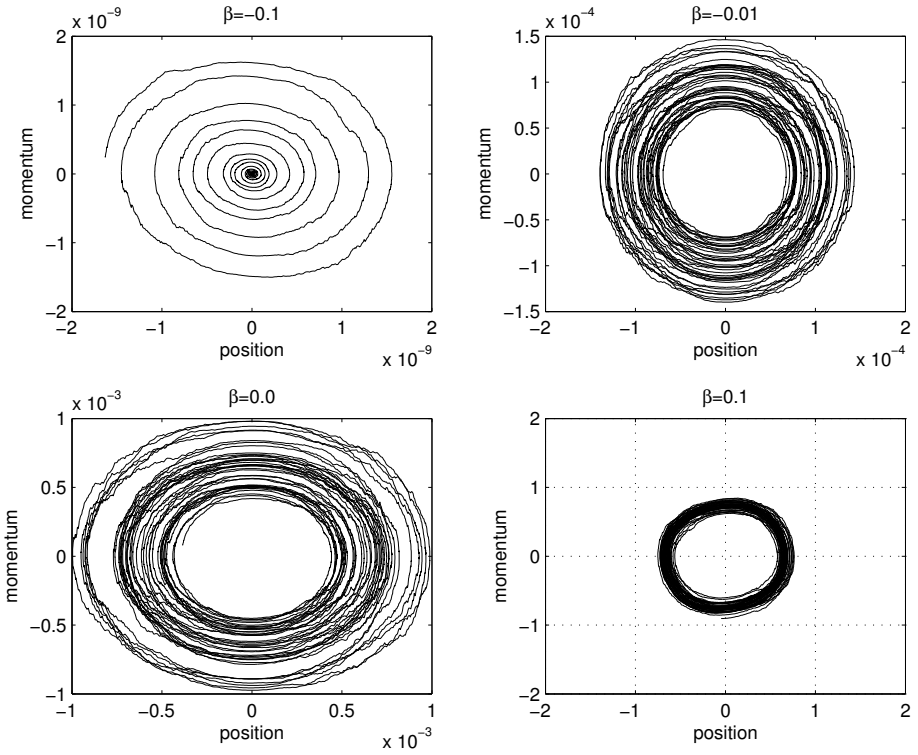

Figure 7: Paths of (2.10) for $\beta = -0.1, -0.01, 0, 0.1$ with $\alpha = -1.0$ over time interval $[250, 500]$ with initial data $[0, 0.001]$ specified at $t_0 = 0$. Notice the change of scales in the plots.

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