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2009

MIMS EPrint: **2009.71**

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ISSN 1749-9097

CONVOLUTIONS OF CANTOR MEASURES WITHOUT RESONANCE

FEDOR NAZAROV, YUVAL PERES AND PABLO SHMERKIN

ABSTRACT. Denote by μ_a the distribution of the random sum $(1-a)\sum_{j=0}^{\infty}\omega_j a^j$, where $\mathbf{P}(\omega_j = 0) = \mathbf{P}(\omega_j = 1) = 1/2$ and all the choices are independent. For $0 < a < 1/2$, the measure μ_a is supported on C_a , the central Cantor set obtained by starting with the closed unit interval, removing an open central interval of length $(1-2a)$, and iterating this process inductively on each of the remaining intervals. We investigate the convolutions $\mu_a * (\mu_b \circ S_{\lambda}^{-1})$, where $S_{\lambda}(x) = \lambda x$ is a rescaling map. We prove that if the ratio $\log b / \log a$ is irrational and $\lambda \neq 0$, then

$$D(\mu_a * (\mu_b \circ S_{\lambda}^{-1})) = \min(\dim_H(C_a) + \dim_H(C_b), 1),$$

where D denotes any of correlation, Hausdorff or packing dimension of a measure.

We also show that, perhaps surprisingly, for uncountably many values of λ the convolution $\mu_{1/4} * (\mu_{1/3} \circ S_{\lambda}^{-1})$ is a singular measure, although $\dim_H(C_{1/4}) + \dim_H(C_{1/3}) > 1$ and $\log(1/3)/\log(1/4)$ is irrational.

1. INTRODUCTION AND STATEMENT OF RESULTS

Given $0 < a < 1/2$, let C_a be the Cantor set obtained by starting with the closed unit interval, removing a central open interval of length $1-2a$, and continuing this process inductively on each of the remaining intervals. Formally,

$$C_a = \left\{ (1-a) \sum_{j=0}^{\infty} \omega_j a^j : \omega_j \in \{0, 1\} \text{ for all } j \right\}.$$

The set C_a supports a natural probability measure μ_a which assigns mass 2^{-n} to each interval of length a^n in the construction. The measure μ_a can be defined in several alternative ways. For example, it is the

Date: October 14, 2009.

2000 Mathematics Subject Classification. Primary 28A80.

Key words and phrases. Sums of Cantor sets, Hausdorff measures, correlation dimension, convolution of measures, resonance.

P.S. acknowledges support from the Academy of Finland, Microsoft Research, EPSRC grant EP/E050441/1 and the University of Manchester.

normalized restriction of $\dim_H(C_a)$ -dimensional Hausdorff measure to C_a . It is also the distribution of the random infinite sum

$$(1-a) \sum_{j=0}^{\infty} \omega_j a^j, \quad (1.1)$$

where $\mathbf{P}(\omega_j = 0) = \mathbf{P}(\omega_j = 1) = 1/2$ and all choices are independent. These equivalences are well known and easy to verify.

In this paper we study convolutions of the form $\mu_a * (\mu_b \circ S_\lambda^{-1})$, where $S_\lambda(x) = \lambda x$ scales by a factor of λ . We will show that, under a natural irrationality condition, $\mu_a * (\mu_b \circ S_\lambda^{-1})$ has “fractal dimension” equal to the sum of the Hausdorff dimensions of C_a and C_b , provided this is at most one. What we mean for fractal dimension is made precise below; we will in fact show that this is true for several commonly used concepts of dimension of a measure.

The study of these convolutions goes back to Senge and Straus [14] who, answering a question that Salem posed in [12], characterized all the pairs a, b such that $\phi_{a,b}(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$, where $\phi_{a,b}$ denotes the Fourier transform of $\mu_a * \mu_b$. Senge and Straus showed that this happens only if $1/a$ and $1/b$ are Pisot numbers and $\log b / \log a$ is rational (Recall that a Pisot number is an algebraic integer larger than one, such that all its algebraic conjugates are smaller than one in modulus).

Let us write

$$\nu_{a,b}^\lambda = \mu_a * (\mu_b \circ S_\lambda^{-1}).$$

There are two sharply different cases in the study of $\nu_{a,b}^\lambda$: The subcritical case $d_a + d_b < 1$ and the supercritical case $d_a + d_b > 1$ (the critical case $d_a + d_b = 1$ is often analyzed separately).

In the subcritical case, the measure $\nu_{a,b}^\lambda$ is always singular, as it is supported on $C_a + \lambda C_b$, which has Hausdorff dimension at most $d_a + d_b < 1$ (see (1.6) below). Thus, in this case the interest lies in the degree of singularity of $\nu_{a,b}^\lambda$, as measured by some concept of fractal dimension. In particular, one is interested in whether there is a “dimension drop”, i.e. whether the dimension of $\nu_{a,b}^\lambda$ is strictly smaller than the dimension of $\mu_a \times \mu_b$. We will prove that if $\log b / \log a$ is irrational, then there is no dimension drop for any $\lambda \neq 0$, for several different concepts of dimension of a measure; see Theorem 1.1 and the discussion afterwards. One motivation comes from the results in [9], where it is proved that for all pairs $0 < a, b < 1/2$ such that $\log b / \log a$ is irrational and all $\lambda \neq 0$,

$$\dim_H(C_a + \lambda C_b) = \min(\dim_H(C_a) + \dim_H(C_b), 1),$$

where \dim_H stands for Hausdorff dimension. The proofs in [9] involve the construction of an ad-hoc measure supported on $C_a + \lambda C_b$, which is not related in a natural way to $\nu_{a,b}^\lambda$. In this paper we base the arguments on more conceptual ergodic-theoretical ideas.

In the case $d_a + d_b > 1$, one would expect $\nu_{a,b}^\lambda$ to be absolutely continuous as long as $\log b / \log a \notin \mathbb{Q}$. However, we will show that this is not always the case. More precisely, we will prove that whenever $1/a$ and $1/b$ are Pisot numbers and $\log b / \log a$ is irrational, there is a dense G_δ set of parameters λ , such that the Fourier transform of $\nu_{a,b}^\lambda$ does not go to zero at infinity; see Theorem 4.1 in Section 4.

In order to state our main result about the fractal dimensions of $\nu_{a,b}^\lambda$, we start by recalling the definition of correlation dimension of a measure. Given a Borel measure ν on \mathbb{R}^n and $r > 0$, let

$$\mathcal{C}_\nu(r) = \int \nu(B(x, r)) d\nu(x),$$

where $B(x, r)$ denotes the closed ball with center x and radius r . The **lower correlation dimension** of ν is defined as

$$\underline{D}(\nu) = \liminf_{r \downarrow 0} \frac{\log \mathcal{C}_\nu(r)}{\log r}.$$

The **upper correlation dimension** $\overline{D}(\nu)$ is defined analogously by taking the lim sup. If $\underline{D}(\nu) = \overline{D}(\nu)$ we say that the **correlation dimension** $D(\nu)$ exists, and is given by the common value. Other definitions of correlation dimension are often used. For example, the lower correlation dimension of ν is the supremum of all α such that

$$I_\alpha(\nu) = \int \int |x - y|^{-\alpha} d\nu(x) d\nu(y) < \infty. \quad (1.2)$$

This is well-known, see e.g. [13, Proposition 2.3]. We can now state our main result:

Theorem 1.1. *Let $0 < a, b < 1$. If $\log b / \log a$ is irrational, then*

$$D(\nu_{a,b}^\lambda) = \min(d_a + d_b, 1), \quad (1.3)$$

for all $\lambda \neq 0$.

Let us make some comments on this statement. Observe that the measure $\nu_{a,b}^\lambda$ is, up to affine equivalence, the push-down of the product measure $\mu_a \times \mu_b$ by orthogonal projection onto the line

$$\{t(\cos(\theta), \sin(\theta)) : t \in \mathbb{R}\},$$

where $\theta = \arctan(\lambda)$. The potential-theoretic proof of Marstrand's projection theorem (see e.g. [8, Chapter 9]) implies that (1.3) holds for

almost every λ . Our contribution is to prove this for *every* $\lambda \neq 0$, and in particular for $\lambda = 1$, which yields the correlation dimension of the convolution measure $\mu_a * \mu_b$.

Correlation dimension is just one of the several concepts of dimension of a measure which are often used. For example, the Hausdorff and packing dimensions of a finite Borel measure ν on \mathbb{R}^n are defined as

$$\begin{aligned}\dim_H(\nu) &= \inf\{\dim_H(E) : \nu(\mathbb{R}^n \setminus E) = 0\}, \\ \dim_P(\nu) &= \inf\{\dim_P(E) : \nu(\mathbb{R}^n \setminus E) = 0\}.\end{aligned}$$

(Here \dim_P denotes packing dimension; see e.g. [8, Chapter 5] for its definition and basic properties). It is then an easy consequence of Frostman's lemma (see [8, Chapter 8]) and the definitions that

$$\underline{D}(\nu) \leq \dim_H(\nu) \leq \dim_H(\text{Supp}(\nu)) \leq n, \quad (1.4)$$

where Supp denotes the support of the measure. It is also immediate that

$$\dim_H(\nu) \leq \dim_P(\nu) \leq \dim_P(\text{Supp}(\nu)) \leq n. \quad (1.5)$$

Since orthogonal projections do not increase either Hausdorff or packing dimension, and $C_a \times C_b$ has Hausdorff and packing dimension $d_a + d_b$, we get

$$\dim_H(C_a + \lambda C_b) \leq \dim_P(C_a + \lambda C_b) \leq \min(d_a + d_b, 1) \quad (1.6)$$

for *all* $0 < a, b < 1/2$. Hence we deduce from Theorem 1.1, (1.4) and (1.5) that, whenever $\log b / \log a$ is irrational,

$$D(\nu_{a,b}^\lambda) = \dim_H(\nu_{a,b}^\lambda) = \dim_P(\nu_{a,b}^\lambda) = \min(d_a + d_b, 1). \quad (1.7)$$

Given a measure ν on Euclidean space, the **local dimensions** of ν are defined as

$$\dim_{\text{loc}}(\nu)(x) = \lim_{r \downarrow 0} \frac{\log(\nu(B(x, r)))}{\log(r)},$$

whenever the limit exists; otherwise one speaks of lower and upper local dimensions. When the local dimension exists and is constant ν -almost everywhere, it is said that ν is **exact dimensional**. Always assuming that $\log b / \log a$ is irrational, it follows from (1.7) and [4, Theorems 1.2 and 1.4] that

$$\dim_{\text{loc}}(\mu_a * \mu_b)(x) = \min(d_a + d_b, 1) \quad \text{for } (\mu_a * \mu_b)\text{-a.e. } x.$$

We remark that in general all of the concepts of dimension of a measure that we discussed can differ; even if a measure is exact-dimensional, it may happen that its correlation dimension is strictly smaller than the almost-sure value of the local dimension.

We note that if $\log b / \log a$ is rational, and $d_a + d_b \leq 1$, then

$$\dim_H(\mu_a * \mu_b) \leq \dim_H(C_a + C_b) < d_a + d_b.$$

(See [9] for a proof). Borrowing the terminology of [9], we can summarize our results as saying that **algebraic resonance**, defined as the rationality of $\log b / \log a$, is equivalent to **geometric resonance** for the measures μ_a and μ_b , defined by the condition that there is a dimension drop for at least one orthogonal projection in a non-principal direction.

Compared to [9], one basic new ingredient in our proofs is the construction of a subadditive cocycle reflecting the structure of the orthogonal projections of $\mu_a \times \mu_b$ at different scales and for different angles. This is obtained by adapting the proof of the existence of L^q dimensions for self-conformal measures in [10]. We will also make use of a theorem of A. Furman on subadditive cocycles over a uniquely ergodic transformation.

The paper is structured as follows. In Section 2 we introduce notation and prove several lemmas in preparation for the proof of Theorem 1.1, which is contained in Section 3. In Section 4, we prove the singularity of $\nu_{1/3,1/4}^\lambda$ for an uncountable set of λ , and discuss some implications. We finish the paper with some remarks and open questions in Section 5.

2. NOTATION AND PRELIMINARY LEMMAS

In this section we collect several definitions and lemmas which will be used in the proof of Theorem 1.1.

2.1. Notation and basic facts. From now on we will assume that $0 < a < b < 1/2$ and $\log b / \log a$ is irrational.

Consider the product set $E = C_a \times C_b$, and let $\eta = \mu_a \times \mu_b$. This is, up to a constant multiple, $(d_a + d_b)$ -dimensional Hausdorff measure restricted to E . It is well known, and easy to verify, that η is Ahlfors-regular, i.e.

$$C^{-1}r^{d_a+d_b} \leq \eta(B(x, r)) \leq Cr^{d_a+d_b}, \quad (2.1)$$

for some constant $C > 1$, all $r > 0$ and all $x \in E$. This implies that

$$D(\eta) = \dim_H(\eta) = \dim_P(\eta) = d_a + d_b. \quad (2.2)$$

The set C_a can also be realized as the attractor of the iterated function system $\{f_{a,0}, f_{a,1}\}$, where

$$f_{a,i}(x) = ax + i(1 - a). \quad (2.3)$$

In other words,

$$C_a = f_{a,0}(C_a) \cup f_{a,1}(C_a).$$

Likewise, the measure μ_a satisfies the self-similarity relation

$$\mu_a(F) = \frac{1}{2}\mu_a(f_{a,0}^{-1}(F)) + \frac{1}{2}\mu_a(f_{a,1}^{-1}(F)), \quad (2.4)$$

where F is an arbitrary Borel set. This is well known; when F is a basic interval in the construction of C_a it follows from the scaling property of Hausdorff measure and self-similarity, and the case of general F is obtained by noting that basic intervals in the construction of C_a form a basis of closed subsets of C_a .

If $u = (u_1, \dots, u_k) \in \{0, 1\}^k$, let

$$f_{a,u} = f_{a,u_1} \circ \dots \circ f_{a,u_k};$$

analogously one defines $f_{b,u}$. Let

$$\mathbb{X} = \bigcup_{k,l=0}^{\infty} \{0, 1\}^k \times \{0, 1\}^l.$$

Given $\xi = (u, v) \in \mathbb{X}$ we will denote

$$f_{\xi}(x, y) = (f_{a,u}(x), f_{b,v}(y)).$$

Moreover, we will write $E(\xi) = f_{\xi}(E)$ and $Q(\xi) = f_{\xi}(Q)$, where $Q = I \times I$ is the unit square. If $\xi_i = (u_i, v_i)$, $i = 1, 2$, by $\xi_1 \xi_2$ we will mean the juxtaposition $(u_1 u_2, v_1 v_2)$. The empty word will be denoted by \emptyset . Finally, if $\xi = (u, v)$, the **length pair** $|\xi|$ of ξ is the pair $(|u|, |v|)$, where $|u|, |v|$ are the lengths of the corresponding words.

Let ℓ be any integer such that $b/a < 1/b^{\ell}$. Write $\alpha = \log(b/a)$, $\beta = \log(1/b^{\ell})$; notice that $0 < \alpha < \beta$, and that α/β is irrational because of our assumption that $\log b/\log a$ is irrational. Moreover, note that β can be made arbitrarily large by starting with an appropriately large ℓ . Endow $[0, \beta)$ with normalized Lebesgue measure \mathcal{L} . Let also $\mathbf{R} : [0, \beta) \rightarrow [0, \beta)$ be given by

$$\mathbf{R}(x) = x + \alpha \pmod{\beta}.$$

The irrationality of α/β implies that this is a uniquely ergodic transformation. This fact will be crucial in the proof: it is the only place where the irrationality of $\log b/\log a$ is used.

We inductively construct two families $\{\mathbb{X}_n\}_{n=0}^{\infty}$, $\{\mathbb{Y}_n\}_{n=0}^{\infty}$ of subsets of \mathbb{X} . We set $\mathbb{X}_0 = \{(\emptyset, \emptyset)\}$ (recall that \emptyset denotes the empty word). Once \mathbb{X}_n has been defined, we define \mathbb{X}_{n+1} in the following way:

- if $\mathbf{R}^n(0) + \alpha < \beta$, then

$$\mathbb{X}_{n+1} = \{(\xi)(i, k) : \xi \in \mathbb{X}_n, i, k \in \{0, 1\}\}.$$

- If $\mathbf{R}^n(0) + \alpha > \beta$, then

$$\mathbb{X}_{n+1} = \{(\xi)(i, v) : \xi \in \mathbb{X}_n, i \in \{0, 1\}, v \in \{0, 1\}^{\ell+1}\}.$$

Further, we let

$$\mathbb{Y}_n = \{(\xi)(\emptyset, v) : \xi \in \mathbb{X}_n, v \in \{0, 1\}^\ell\}.$$

One can readily check, from this inductive construction and the definition of \mathbf{R} , that the following properties are satisfied:

- (I) If $\xi \in \mathbb{X}_n$, then $Q(\xi)$ is a rectangle of size

$$a^n \times a^n \exp(\mathbf{R}^n(0)).$$

If $\xi' \in \mathbb{Y}_n$, then $Q(\xi')$ is a rectangle of size

$$a^n \times a^n \exp(\mathbf{R}^n(0) - \beta).$$

In particular, all elements of \mathbb{X}_n have the same length pair, as do all elements of \mathbb{Y}_n .

- (II) The cylinders based at elements of \mathbb{X}_n form a partition of the symbolic space. In other words, if $\omega, \omega' \in \{0, 1\}^\mathbb{N}$ are infinite sequences, then there is exactly one $\xi = (u, v) \in \mathbb{X}_n$ such that ω starts with u and ω' starts with v . The same holds for \mathbb{Y}_n .

The rectangles $Q(\xi)$ are cartesian products of basic intervals of C_a and C_b ; by the first property, the logarithm of the ratio of the lengths stays bounded and, moreover, behaves like an irrational rotation. Property (II) guarantees that $\{Q(\xi) \in \mathbb{X}_n\}$ is an efficient covering of E , and likewise for \mathbb{Y}_n .

Heuristically, for \mathbb{X}_n , we start from the unit square, and then at each inductive step we always go one level further in the construction of C_a . With respect to the construction of C_b , we go one level further for as long as the eccentricity of the rectangles $Q(\xi)$ stays below $e^\beta = b^{-\ell}$ (note that, since $b > a$, going one step further in both constructions has the effect of increasing the eccentricity); otherwise, we go $\ell + 1$ levels further, which has the effect of reducing the eccentricity of $Q(\xi)$ back to a value between 1 and e^β . For the construction of \mathbb{Y}_n , we start from \mathbb{X}_n and go ℓ levels further in the construction of C_b , while keeping the same basic intervals in the construction of C_a ; this yields rectangles $Q(\xi')$ with width greater than height.

Let $h_s(x, y) = (x, e^s y)$, and $\Pi(x, y) = x + y$. We will write $\Pi_s = \Pi h_s$. The assignment $s \rightarrow h_s$ is an action of the additive group of real numbers by linear bijections of \mathbb{R}^2 .

The following observation will prove very useful: let $\xi \in \mathbb{X}_n$. Then f_ξ can be decomposed as

$$f_\xi(x) = a^n h_{\mathbf{R}^n(0)}(x) + d_\xi, \quad (2.5)$$

where d_ξ is a translation vector in \mathbb{R}^2 . From this we also obtain

$$f_\xi^{-1}(x) = a^{-n}h_{-\mathbf{R}^n(0)}(x) + \hat{d}_\xi, \quad (2.6)$$

where $\hat{d}_\xi = -a^{-n}h_{-\mathbf{R}^n(0)}(d_\xi)$. Of course, similar decompositions are valid for $f_{\xi'}, f_{\xi'}^{-1}$ for $\xi' \in \mathbb{Y}_n$.

We will denote the projected measure $\eta \circ \Pi_s^{-1}$ by η_s . Notice that $\eta_s = \mu_a * (\mu_b \circ S_{e^s}^{-1})$, where $S_\lambda(x) = \lambda x$. In particular, $\eta_0 = \mu_a * \mu_b$.

For $n \in \mathbb{N}$ we will denote

$$\mathcal{D}_n = \{[ja^n, (j+1)a^n) : j \in \mathbb{Z}\}.$$

Further, for $s \in \mathbb{R}$, we will denote by $\mathcal{D}_n(s)$ the subset of \mathcal{D}_n comprising all intervals which intersect $\Pi_s(E)$.

Given $s \in \mathbb{R}$, let us define

$$\tau_n(s) = \sum_{I \in \mathcal{D}_n} \eta_s(I)^2 = \sum_{I \in \mathcal{D}_n(s)} \eta_s(I)^2. \quad (2.7)$$

These functions will be a useful discrete analogue of $\mathcal{C}_{\eta_s}(a^n)$; indeed, both quantities are comparable up to a multiplicative constant.

The next definition is adapted from [10]. Given $n \in \mathbb{N}$ and $s \in \mathbb{R}$, we will say that a family of disjoint intervals \mathcal{C} is (n, s) -**good** if it is a minimal covering of $\Pi_s(E)$ (meaning that no proper subset is a covering of $\Pi_s(E)$), and each interval has length a^n .

Lemma 2.1. *If \mathcal{C} is (n, s) -good, then*

$$4^{-1}\tau_n(s) \leq \sum_{C \in \mathcal{C}} \eta_s(C)^2 \leq 4\tau_n(s).$$

Proof. We only prove the right-hand inequality since the left-hand inequality is analogous. Since \mathcal{C} is (n, s) -good, each element C of \mathcal{C} intersects at most 2 elements of $\mathcal{D}_n(s)$, say $D(C, i), i = 1, 2$. Since $\mathcal{D}_n(s)$ is a covering of $\Pi_s(E)$,

$$C \cap \Pi_s(E) \subset (D(C, 1) \cup D(C, 2)) \cap \Pi_s(E).$$

Using this and Cauchy-Schwartz,

$$\eta_s(C)^2 \leq \left(\sum_{i=1}^2 \eta_s(D(C, i)) \right)^2 \leq 2 \sum_{i=1}^2 \eta_s(D(C, i))^2.$$

Therefore

$$\sum_{C \in \mathcal{C}} \eta_s(C)^2 \leq 2 \sum_{C \in \mathcal{C}} \sum_{i=1}^2 \eta_s(D(C, i))^2.$$

But since each element of \mathcal{D}_n intersects at most 2 elements of \mathcal{C} , any element of \mathcal{D}_n appears at most twice on the right-hand side, and the lemma follows. \square

Lemma 2.2. *For any $s \in \mathbb{R}$, $m, n \in \mathbb{N}$ and $\xi \in \mathbb{X}_n, \xi' \in \mathbb{Y}_n$, the families*

$$\begin{aligned}\mathcal{I} &= \{ \Pi_{\mathbf{R}^n(0)+s} f_\xi^{-1} \Pi_s^{-1} I : I \in \mathcal{D}_{m+n}, I \cap \Pi_s(E(\xi)) \neq \emptyset \}, \\ \mathcal{I}' &= \{ \Pi_{\mathbf{R}^n(0)+s-\beta} f_{\xi'}^{-1} \Pi_s^{-1} I : I \in \mathcal{D}_{m+n}, I \cap \Pi_s(E(\xi')) \neq \emptyset \},\end{aligned}$$

are $(m, \mathbf{R}^n(0) + s)$ -good and $(m, \mathbf{R}^n(0) + s - \beta)$ -good, respectively.

Proof. We will prove only that \mathcal{I} is $(m, \mathbf{R}^n(0) + s)$ -good, since the proof for \mathcal{I}' is analogous. Let $t = \mathbf{R}^n(0) + s$. Observe that

$$f_\xi(E) = E(\xi) \subset \bigcup_{I \in \mathcal{D}_{m+n}, I \cap \Pi_s(E(\xi)) \neq \emptyset} \Pi_s^{-1}(I),$$

and from this it follows that \mathcal{I} is a covering of $\Pi_t(E)$. Using (2.6), linearity of Π_t , and the action properties of $s \rightarrow h_s$, we get

$$\begin{aligned}\Pi_t f_\xi^{-1} \Pi_s^{-1} I &= \Pi_t \hat{d}_\xi + a^{-n} \Pi_t h_{-\mathbf{R}^n(0)} \Pi_s^{-1}(I) \\ &= \Pi_t \hat{d}_\xi + a^{-n} \Pi h_t h_{-\mathbf{R}^n(0)} h_s^{-1} \Pi^{-1}(I) \\ &= \Pi_t \hat{d}_\xi + a^{-n} I.\end{aligned}$$

This shows that the elements of \mathcal{I} are pairwise disjoint (since the elements of \mathcal{D}_{m+n} are), and have length a^m . Since, by definition, all elements of \mathcal{I} intersect $\Pi_t(E)$, the covering \mathcal{I} is optimal. This completes the proof of the lemma. \square

We finish this section by recalling the result of Furman alluded to in the introduction.

Theorem 2.3. *Let X be a compact metric space, and let T be a continuous homeomorphism of X with a unique invariant probability measure μ (which must be ergodic). Further, let $\{\phi_n\}$ be a continuous subadditive cocycle over (X, T) . In other words, each ϕ_n is a continuous real valued function on X , and*

$$\phi_{m+n}(x) \leq \phi_m(x) + \phi_n(T^m x),$$

for all $x \in X$. Let

$$A = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \phi_n(x) d\mu.$$

Note that A is well defined by subadditivity. Then for almost all $x \in X$,

$$\lim_{n \rightarrow \infty} \frac{\phi_n(x)}{n} = A,$$

and for all $x \in X$,

$$\limsup_{n \rightarrow \infty} \frac{\phi_n(x)}{n} \leq A.$$

Proof. The first assertion is Kingman's subadditive ergodic theorem [7]. The second one is proved in [5, Theorem 1]. \square

3. PROOF OF THEOREM 1.1

The proof consists of two main parts: in the first we construct a submultiplicative cocycle over $([0, \beta), \mathbf{R})$ related to the growth of $\tau_n(s)$. In the second part, we apply Theorem 2.3 to this cocycle and deduce Theorem 1.1 from the potential-theoretic proof of Marstrand's projection theorem.

Before starting the proof, we remark that due to the symmetry of $C_a \times C_b$, we only need to show that (1.3) holds for all $\lambda \geq 1$ (which corresponds to orthogonal projections for angles in $[\pi/4, \pi/2)$). Moreover, since β is arbitrarily large, it will be enough to establish (1.3) for all $\lambda \in [1, e^\beta)$.

A submultiplicative cocycle. The goal of this part is to show that $\tau_n(\cdot)$ defined in (2.7) satisfy

$$\tau_{m+n}(s) \leq A\tau_n(s)\tau_m(\mathbf{R}^n(s)), \quad (3.1)$$

for some $A > 1$ independent of $m, n \in \mathbb{N}$ and $s \in [0, \beta)$. In order to do this, we will follow the pattern of the proof of existence of L^q dimension in [10] in the case $q > 1$.

We will consider two cases, depending on whether $\mathbf{R}^n(0) + s < \beta$ or $\mathbf{R}^n(0) + s > \beta$. In the first case, we have $\mathbf{R}^n(s) = \mathbf{R}^n(0) + s$, while in the second, $\mathbf{R}^n(s) = \mathbf{R}^n(0) + s - \beta$. In the proof of the first case we will use the families $\{\mathbb{X}_n\}$, while the proof of the second is based on the families $\{\mathbb{Y}_n\}$. We will in fact only prove the first case; the second follows in the same way, so details are left to the reader.

Let $m, n \in \mathbb{N}$, and pick some $\xi \in \mathbb{X}_n$. Fix $s \in [0, \beta)$, and let

$$t = \mathbf{R}^n(s) = \mathbf{R}^n(0) + s.$$

It follows from Lemmas 2.1 and 2.2 that

$$\sum_{I \in \mathcal{D}_{m+n}, I \cap \Pi_s(E(\xi)) \neq \emptyset} \eta_t(\Pi_t f_\xi^{-1} \Pi_s^{-1}(I))^2 \leq 4\tau_m(t). \quad (3.2)$$

We claim that

$$\Pi_t^{-1} \Pi_t f_\xi^{-1} \Pi_s^{-1}(I) = f_\xi^{-1} \Pi_s^{-1}(I). \quad (3.3)$$

To see this, foliate \mathbb{R}^2 by fibers $\{\Pi_t^{-1}(x)\}$, and note that $\Pi_t^{-1}\Pi_t(F) = F$ if and only if F contains the fiber through all of its points. In the particular case $F = f_\xi^{-1}\Pi_s^{-1}(I)$, we have:

$$\begin{aligned} p \in F &\iff \Pi_s f_\xi(p) \in I \\ &\iff \Pi h_s(a^n h_{R^n(0)}(p) + d_\xi) \in I \\ &\iff \Pi(a^n h_t(p) + h_s d_\xi) \in I \\ &\iff a^n \Pi_t(p) + \Pi_s d_\xi \in I, \end{aligned}$$

where we used (2.5) in the second displayed line, and the linearity of Π in the fourth. Since the value of $\Pi_t(p)$ is constant on the leaf through p , (3.3) is proved.

Notice that if $I \cap \Pi_s(E(\xi)) = \emptyset$ then $f_\xi^{-1}\Pi_s^{-1}(I) \cap E = \emptyset$. Combining this with (3.2) and (3.3) yields

$$\sum_{I \in \mathcal{D}_{m+n}} \eta(f_\xi^{-1}\Pi_s^{-1}(I))^2 \leq 4\tau_m(t). \quad (3.4)$$

Let \mathcal{D}'_n be the family of unions of two consecutive elements of \mathcal{D}_n ; in other words,

$$\mathcal{D}'_n = \{[ja^n, (j+2)a^n) : j \in \mathbb{Z}\}.$$

Let also $\mathcal{D}''_n = \mathcal{D}_n \cup \mathcal{D}'_n$. To each $I \in \mathcal{D}_{m+n}$ we associate an element $\tilde{I} \in \mathcal{D}''_n$ in the following way: if I is contained in an element of \mathcal{D}_n , let \tilde{I} be this element; otherwise, I is contained in a unique element of \mathcal{D}'_n , and we let \tilde{I} be this element.

Iterating (2.4) and using properties (I)-(II) of $\{\mathbb{X}_n\}$, we see that

$$\eta(F) = |\mathbb{X}_n|^{-1} \sum_{\xi \in \mathbb{X}_n} \eta(f_\xi^{-1}F), \quad (3.5)$$

for any Borel set F . Using this we obtain

$$\eta_s(I) = \frac{1}{|\mathbb{X}_n|} \sum_{\xi \in \mathbb{X}_n} \eta(f_\xi^{-1}\Pi_s^{-1}I) = \frac{1}{|\mathbb{X}_n|} \sum_{\xi \in \mathbb{X}_n : \tilde{I} \cap \Pi_s(E(\xi)) \neq \emptyset} \eta(f_\xi^{-1}\Pi_s^{-1}I). \quad (3.6)$$

For $J \in \mathcal{D}''_n$, write

$$\mathbb{X}_n(J, s) = \{\xi \in \mathbb{X}_n : J \cap \Pi_s(E(\xi)) \neq \emptyset\}.$$

From (3.6), an application of Cauchy-Schwartz yields

$$\eta_s^2(I) \leq |\mathbb{X}_n|^{-2} |\mathbb{X}_n(\tilde{I}, s)| \sum_{\xi \in \mathbb{X}_n(\tilde{I}, s)} \eta(f_\xi^{-1}\Pi_s^{-1}I)^2.$$

For a fixed $J \in \mathcal{D}_n''$, we add over all $I \in D_{m+n}$ such that $\tilde{I} = J$, to get

$$\begin{aligned} \sum_{I \in \mathcal{D}_{m+n}, \tilde{I}=J} \eta_s^2(I) &\leq |\mathbb{X}_n|^{-2} |\mathbb{X}_n(J, s)| \sum_{\xi \in \mathbb{X}_n(J, s)} \sum_{\tilde{I}=J} \eta(f_\xi^{-1} \Pi_s^{-1} I)^2 \\ &\leq 4\tau_m(t) |\mathbb{X}_n|^{-2} |\mathbb{X}_n(J, s)|^2, \end{aligned}$$

where in the last inequality we applied (3.4). Adding over all $J \in \mathcal{D}_n''$, we obtain

$$\tau_{m+n}(s) \leq 4\tau_m(t) |\mathbb{X}_n|^{-2} \sum_{J \in \mathcal{D}_n''} |\mathbb{X}_n(J, s)|^2. \quad (3.7)$$

All the maps Π_s , $s \in [0, \beta)$, are Lipschitz with Lipschitz constant uniformly bounded by $2e^\beta$. Also, for $\xi \in \mathbb{X}_n$, the diameter of $Q(\xi)$ is bounded by $\sqrt{1 + e^{2\beta} a^n} < 2e^\beta a^n$. Given $J \in \mathcal{D}_n''$, denote by \hat{J} the interval with the same center as J and length $|J| + 16e^\beta a^n$. By our previous observations, if $\xi \in \mathbb{X}_n(J, s)$, then $\Pi_s^{-1}(\hat{J})$ contains a ball of radius $2e^\beta a^n$ centered at $E(\xi)$, and this implies that $E \subset f_\xi^{-1} \Pi_s^{-1}(\hat{J})$. It follows that

$$\begin{aligned} |\mathbb{X}_n|^{-1} |\mathbb{X}_n(J, s)| &\leq \eta(E)^{-1} |\mathbb{X}_n|^{-1} \sum_{\xi \in \mathbb{X}_n(J, s)} \eta(f_\xi^{-1} \Pi_s^{-1}(\hat{J})) \\ &\leq \eta(E)^{-1} \eta_s(\hat{J}) \leq \eta(E)^{-1} \sum_{J' \in \mathcal{D}_n, \text{dist}(J', J) < 8e^\beta a^n} \eta_s(J'), \end{aligned}$$

where we used (3.5). Using Cauchy-Schwartz once again, we deduce that

$$(|\mathbb{X}_n|^{-1} |\mathbb{X}_n(J, s)|)^2 \leq K_1 \sum_{J' \in \mathcal{D}_n, \text{dist}(J', J) < 8e^\beta a^n} \eta_s(J')^2,$$

where $K_1 = \eta(E)^{-2} [16e^\beta + 2]$. Adding over all $J \in \mathcal{D}_n''$, note that each element of \mathcal{D}_n on the right-hand side appears at most $2K_1$ times, whence

$$|\mathbb{X}_n|^{-2} \sum_{J \in \mathcal{D}_n''} |\mathbb{X}_n(J, s)|^2 \leq 2K_1^2 \tau_n(s).$$

Together with (3.7) this yields (3.1), as desired.

Conclusion of the proof. Recall the definition of $\mathcal{C}_\nu(r)$ given in the introduction. Let

$$\phi_n(s) = \mathcal{C}_{\eta_s}(a^n).$$

It is clear that the correlation dimension of η_s exists if and only if the limit $L = \lim_{n \rightarrow \infty} \log \phi_n(s)/n$ exists, in which case $D(\eta_s) = L/\log(a)$.

Let us rewrite ϕ_n as

$$\begin{aligned}\phi_n(s) &= \int \eta_s(\Pi_s(y) - a^n, \Pi_s(y) + a^n) d\eta(y) \\ &= \int \eta(\Pi_s^{-1}(\Pi_s(y) - a^n, \Pi_s(y) + a^n) \cap Q) d\eta(y) \\ &=: \int \eta(\mathcal{T}(s, y)) d\eta(y).\end{aligned}$$

(Recall that Q is the unit square; Q could be replaced by any bounded convex set containing E). Note that a fixed line ℓ intersects at most $C2^n$ rectangles $Q(\xi)$ with $\xi \in \Xi_n$, whence, by using (2.1) and letting $n \rightarrow \infty$, we see that $\eta(\ell) = 0$. Therefore

$$\lim_{t \rightarrow s} \mathbf{1}_{\mathcal{T}(t, y)}(x) = \mathbf{1}_{\mathcal{T}(s, y)}(x) \quad \mu - \text{a.e.},$$

whence, by the dominated convergence theorem,

$$\lim_{t \rightarrow s} \eta(\mathcal{T}(t, y)) = \eta(\mathcal{T}(s, y)).$$

Applying the dominated convergence theorem again we find that the functions $\{\phi_n\}$ are continuous.

It follows from the proof of [11, Theorem 18.2] that there exists $K_2 > 0$ such that

$$K_2^{-1} \tau_n(s) \leq \phi_n(s) \leq K_2 \tau_n(s).$$

Therefore we obtain from (3.1) that

$$\phi_{m+n}(s) \leq K_3 \phi_n(s) \phi_m(\mathbf{R}^n(s)),$$

where $K_3 = A K_2^3$. Hence, if we let $\bar{\phi}_n = K_3 \phi_n$, we have

$$\log \bar{\phi}_{m+n}(s) \leq \log \bar{\phi}_n(s) + \log \bar{\phi}_m(\mathbf{R}^n(s)).$$

We have shown that $\{\log \bar{\phi}_n\}$ is a continuous subadditive cocycle over $([0, \beta), \mathbf{R})$. By Theorem 2.3, and taking into account the negative factor $1/\log(a)$, for almost every $s \in [0, \beta)$ we have

$$\lim_{n \rightarrow \infty} \frac{\log(\phi_n(s))}{n \log(a)} = \sup_n \frac{\int \log \phi_n(\zeta) d\mathcal{L}(\zeta)}{n \log(a)} =: \tilde{D}. \quad (3.8)$$

Moreover, for all $s \in [0, \beta)$ we have the inequality

$$\liminf_{n \rightarrow \infty} \frac{\log(\phi_n(s))}{n \log(a)} \geq \tilde{D}. \quad (3.9)$$

Recall that $\underline{D}(\nu)$ is the supremum of all α such that the α -energy $I_\alpha(\nu)$ is finite; see (1.2). It follows from the potential-theoretic proof

of Marstrand's projection theorem (see e.g. [8, Chapter 9]) that

$$D(\eta_s) = D(\eta),$$

for almost every $s \in \mathbb{R}$. Thus $\tilde{D} = D(\eta) = \dim_H(E)$.

Lipschitz maps do not increase upper correlation dimension; this can be easily checked from the definition. Therefore, using (2.2),

$$\overline{D}(\eta_s) \leq \overline{D}(\eta) = \dim_H(E),$$

for all $s \in [0, \beta)$. But from (3.9) and the fact that $\tilde{D} = \dim_H(E)$ we also get

$$\underline{D}(\eta_s) \geq \dim_H(E)$$

for all $s \in [0, \beta)$. Thus for all $s \in [0, \beta)$ we have $\overline{D}(\eta_s) = \underline{D}(\eta_s) = \dim_H(E)$. This completes the proof of the theorem. \square

4. THE CASE $d_a + d_b > 1$

Recall that a real number θ is a **Pisot number** if θ is an algebraic integer, $\theta > 1$ and all the algebraic conjugates of θ have modulus strictly smaller than 1. The main result of this section is the following:

Theorem 4.1. *Suppose that $0 < a < b < 1/2$, $\log b / \log a$ is irrational, and $1/a$ and $1/b$ are both Pisot numbers. Then there exists a dense G_δ , and therefore uncountable, set $B \subset (0, \infty)$, such that if $\lambda \in B$, then $\nu_{a,b}^\lambda$ is a singular measure.*

The proof of Theorem 4.1 will be given at the end of this section. Let $F(\cdot)$ denote the Fourier transform of a measure, defined by

$$F(\mu)(\xi) = \int e^{ix\xi} d\mu(x).$$

By elementary properties of the Fourier transform, which are still valid for Fourier transforms of measures, we have

$$F(\nu_{a,b}^\lambda)(\xi) = F(\mu_a)(\xi) F(\mu_b)(\lambda\xi). \quad (4.1)$$

Salem proved that $F(\mu_a)(\xi) \not\rightarrow 0$ as $\xi \rightarrow \infty$ if and only if $1/a$ is a Pisot number; see e.g. [12] for a proof, as well as further background on Pisot numbers. Thus, if either $1/a$ or $1/b$ is not Pisot, then

$$\lim_{\xi \rightarrow \infty} F(\nu_{a,b}^\lambda)(\xi) = 0 \quad \text{for all } \lambda \in \mathbb{R} \setminus \{0\}.$$

In the proof of Theorem 4.1 we will show a converse of this: if $1/a$ and $1/b$ are both Pisot (and $0 < a, b < 1/2$) then, for $\lambda \in B$,

$$F(\nu_{a,b}^\lambda)(\xi) \not\rightarrow 0 \text{ as } \xi \rightarrow \infty. \quad (4.2)$$

Theorem 4.1 provides a counterexample to the principle that for dynamically defined sets and measures, the set of exceptions to the projection theorems should be determined by natural algebraic relations.

An open question, due to H. Furstenberg, is whether this principle is valid for orthogonal projections of simple self-similar sets like the one-dimensional Sierpiński gasket, defined as

$$\mathcal{S} = \left\{ \sum_{j=0}^{\infty} 3^{-j} \omega_j : \omega_j \in \{(0, 0), (1, 0), (0, 1)\} \right\}.$$

The conjecture in this case is that Hausdorff dimension is preserved for orthogonal projections in directions with irrational slope.

By taking $a = 1/4$ and $b = 1/3$, Theorem 4.1 gives the first example we know for which the principle described above is known to fail. Here the set of exceptions is uncountable, and since $\log 4 / \log 3$ is irrational, there is no exact overlap for $\lambda \neq 0$. On the other hand, Theorem 1.1 is one of the few cases in which the principle has been proved to hold.

By Theorem 1.1, $\nu_{1/3, 1/4}^{\lambda}$ has correlation, Hausdorff and packing dimension equal to 1 for all nonzero λ . Thus for $\lambda \in B$ there is a loss of absolute continuity, but not a dimension drop. We remark that for certain 1-dimensional measures in \mathbb{R}^2 (Hausdorff measures restricted to purely nonrectifiable sets of positive finite one-dimensional Hausdorff measure) their projections onto almost every line are one-dimensional but singular, due to a classical theorem of Marstrand. The crucial difference is that the measure we are projecting, $\mu_{1/3} \times \mu_{1/4}$, has dimension strictly greater than 1. The same considerations apply to all $0 < a, b < 1/2$ such that $1/a$ and $1/b$ are both Pisot, $\log b / \log a$ is irrational, and $\dim_H(C_a) + \dim_H(C_b) > 1$.

Note also that Theorem 4.1 does not contradict the result of Senge and Strauss [14] mentioned in the introduction, since Senge and Strauss only deal with the case $\lambda = 1$. It is still possible that $\mu_a * \mu_b$ is absolutely continuous whenever $d_a + d_b > 1$ and $\log b / \log a \notin \mathbb{Q}$, but Theorem 4.1 precludes using the ideas in the proof of Theorem 1.1 to prove such a result.

For notational reasons it will be convenient to rescale and translate μ_t so that the convex hull of its support becomes $[-1/(1-t), 1/(1-t)]$. Since we are rescaling the supports of μ_a and μ_b by a different factor, this change also induces a rescaling of the parameter λ , but this does not affect the statement of Theorem 4.1.

The advantage of this change of coordinates is that μ_t becomes the distribution of the random sum

$$\sum_{j=0}^{\infty} \pm t^j,$$

where $\mathbf{P}(+) = \mathbf{P}(-) = 1/2$ and all the choices are independent. This in turn yields the well-known expression for the Fourier transform of μ_t as an infinite product:

$$\begin{aligned} F(\mu_t)(\xi) &= \prod_{j=0}^{\infty} \frac{F(\delta_{t^j})(\xi) + F(\delta_{-t^j})(\xi)}{2} \\ &= \prod_{j=0}^{\infty} \frac{e^{it^j\xi} + e^{-it^j\xi}}{2} \\ &= \prod_{j=0}^{\infty} \cos(t^j\xi), \end{aligned}$$

where δ_x denotes the unit Dirac mass at x . These infinite products have been studied intensively, see e.g. [12].

The following known lemma describes the set B of parameters λ for which we will prove that $\nu_{a,b}^\lambda$ is singular. Since we are not aware of a suitable reference, and the proof is short, we include it for the convenience of the reader.

Lemma 4.2. *Let $0 < a, b < 1$ be numbers such that $\log b / \log a$ is irrational, and let $\varepsilon > 0$ be arbitrary. Let*

$$B = \{\lambda > 0 : |\lambda a^{-n} - b^{-m}| < \varepsilon \text{ for infinitely many pairs } (n, m) \in \mathbb{N}^2\}.$$

Then B is a dense G_δ subset of $(0, \infty)$. In particular, it is uncountable.

Proof. For each $N \in \mathbb{N}$, let

$$B_N = \{\lambda > 0 : |\lambda a^{-n} - b^{-m}| < \varepsilon \text{ for some } n, m \geq N\}.$$

It is clear that B_N is open. We claim that it is also dense in $(0, \infty)$. Indeed, let $I = (c, d) \subset (0, \infty)$ be any nonempty interval. Then I meets B_N whenever $a^n b^{-m} \in I$ for some $n, m \geq N$. By taking logarithms this is equivalent to

$$\frac{\log b}{|\log a|} m - n \in \left(\frac{\log c}{|\log a|}, \frac{\log d}{|\log a|} \right).$$

But one can find arbitrarily large integers n, m satisfying this by the irrationality of $\log b / \log a$.

Noting that $B = \bigcap_{N \geq 1} B_N$ and applying Baire's Theorem concludes the proof of the lemma. \square

Proof of Theorem 4.1. We start by recalling some basic facts about Pisot numbers. Let $\theta > 2$ be Pisot, and write $\theta_1, \dots, \theta_r$ for the algebraic conjugates of θ . Since $\theta^n + \sum_{i=1}^r \theta_i^n$ is an integer for all $n \in \mathbb{N}$, we find that

$$\text{dist}(\theta^n, \mathbb{Z}) \leq r\gamma^n, \quad (4.3)$$

for all natural numbers n , where

$$\gamma := \max_{i=1, \dots, r} |\theta_i| < 1.$$

Moreover, since θ is an algebraic integer, so is θ^n for all $n \in \mathbb{N}$; in particular, θ^n cannot be of the form $k + 1/2$ with k an integer. From these two facts we obtain that the infinite product

$$F(\mu_{1/\theta})(\pi) = \prod_{j=0}^{\infty} \cos(\pi\theta^j)$$

is absolutely convergent. Likewise, since $\theta > 2$, then the infinite product $\prod_{j=0}^{\infty} \cos(\pi\theta^{-j})$ is also absolutely convergent.

The above observations also imply that the number

$$\varepsilon := \frac{1}{2} \text{dist} \left(\{b^{-n} : n \in \mathbb{N}\}, \mathbb{Z} + \frac{1}{2} \right) \quad (4.4)$$

is strictly positive.

Now let B be the set given by Lemma 4.2, with ε defined in (4.4). Fix $\lambda \in B$ for the rest of the proof. Using (4.1), we see that the Fourier transform of $\nu_{a,b}^\lambda$ is given by

$$\Phi(\xi) = \prod_{j=0}^{\infty} \cos(a^j \xi) \prod_{j=0}^{\infty} \cos(\lambda b^j \xi) =: \Phi_1(\xi) \Phi_2(\xi).$$

Let $N, M \in \mathbb{N}$ be such that $|\lambda a^{-N} - b^{-M}| < \varepsilon$. We have

$$\begin{aligned} |\Phi_1(\pi a^{-N})| &= \prod_{j=0}^{\infty} |\cos(\pi a^{-N+j})| \\ &\geq \prod_{j=-\infty}^{\infty} |\cos(\pi a^{-j})| \\ &=: c_1 > 0, \end{aligned}$$

by our earlier observations.

Write $\sigma = \lambda a^{-N} - b^{-M}$, and note that by the definition of ε , and using that $|\sigma b^j| \leq \varepsilon$ for $j \geq 0$,

$$\text{dist} \left(b^{j-M} + \sigma b^j, \mathbb{Z} + \frac{1}{2} \right) \geq \varepsilon/2,$$

for all $j = 0, \dots, M$. Thus, using (4.3) and that $0 < b < 1$, we get that the products

$$\prod_{j=0}^{M-1} |\cos(\pi(b^{j-M} + \sigma b^j))|$$

are uniformly bounded below by some constant $c_2 > 0$ independent of M .

Using this, we estimate:

$$\begin{aligned} |\Phi_2(\pi a^{-N})| &= \prod_{j=0}^{\infty} |\cos(\pi(b^{-M} + \sigma)b^j)| \\ &= \prod_{j=0}^{M-1} |\cos(\pi(b^{j-M} + \sigma b^j))| \prod_{j=M}^{\infty} |\cos(\pi b^{j-M}(1 + \sigma b^M))| \\ &\geq c_2 |F(\mu_b)(\pi(1 + \sigma b^M))|. \end{aligned}$$

Recall that $F(\mu_b)(\pi) \neq 0$; thus $F(\mu_b)(\pi(1 + \sigma b^M))$ is bounded away from zero for sufficiently large M .

We have shown that Φ is bounded away from zero on the set

$$\{\pi a^{-N} : |a^{-N}\lambda - b^{-M}| < \varepsilon \text{ for some large } M \in \mathbb{N}\}.$$

Since, by assumption, the set above is infinite (hence unbounded), this shows that $\Phi(\xi) \not\rightarrow 0$ as $\xi \rightarrow \infty$, so $\nu_{a,b}^\lambda$ is not absolutely continuous. By The Jessen-Wintner law of pure types (see, e.g., [1, Theorem 3.26]), it follows that $\nu_{a,b}^\lambda$ is singular, as claimed. \square

5. REMARKS AND OPEN QUESTIONS

We finish the paper with some generalizations, remarks and open problems.

General self-similar sets. The sets C_a are among the simplest examples of self-similar sets. Recall that a set $C \subset \mathbb{R}$ is said to be self-similar if there are affine maps $f_i(x) = \lambda_i x + t_i$, $i = 1, \dots, m$, with $|\lambda_i| < 1$, such that

$$C = \bigcup_{i=1}^m f_i(C).$$

If all the λ_i coincide, we say that C is an homogeneous self-similar set. It is not hard to see that the proof of Theorem 1.1 extends, with minor modifications, to the restrictions of Hausdorff measure to pairs C, C' of homogeneous self-similar sets satisfying the strong separation condition. In the case of sets, it is possible to reduce the general self-similar case to the homogeneous one, by observing that any self-similar

set contains an homogeneous one of arbitrarily close dimension; see [9, Proposition 6]. However, for measures such reduction does not work, and we do not know if Theorem 1.1 is valid for measures on arbitrary self-similar measures.

Bernoulli convolutions. Notice that the definition of μ_t as the distribution of a random sum makes sense whenever $t \in (0, 1)$; if $t > 1/2$ then the support of μ_t becomes an interval. The family μ_t for $t \in (0, 1)$ is known as the family of **Bernoulli convolutions**. The proof of Theorem 4.1 applies, with minor modifications, also in the case where a or b are in $(1/2, 1)$. The case $a = 1/2$ (or $b = 1/2$) is exceptional, since $\mu_{1/2}$ is the restriction of Lebesgue measure to its supporting interval. Pisot numbers also play a prominent rôle in the study of Bernoulli convolutions: the only parameters $t \in (1/2, 1)$ for which μ_t is known to be singular are reciprocal of Pisot numbers; on the other hand, Solomyak proved that μ_t is absolutely continuous for almost every $t \in (1/2, 1)$. The reader is referred to [15] for a proof of these facts and further background on Bernoulli convolutions.

The measure of $C_a + \lambda C_b$. In [9] it was asked whether $C_a + C_b$ has positive Lebesgue measure whenever $d_a + d_b > 1$ and $\log b / \log a \notin \mathbb{Q}$. Theorem 4.1 suggests that $C_a + \lambda C_b$ may have zero Lebesgue measure for some nonzero values of λ if $1/a$ and $1/b$ are Pisot numbers, but we do not have a proof. Even in this case, it could still be that $C_a + C_b$ has positive measure, but one would need a different method of proof.

Natural measures on $C_a + C_b$. Besides $\mu_a * \mu_b$, other possible natural measures on $C_a + C_b$ are the restrictions of Hausdorff and packing measures in the appropriate dimension. However, Eroğlu [2] proved that

$$\mathcal{H}^{d_a+d_b}(C_a + C_b) = 0,$$

for all a, b such that $d_a + d_b \leq 1$. We do not know whether $C_a + C_b$ has positive $d_a + d_b$ -dimensional packing measure when $\log b / \log a$ is irrational (it is easy to see that it is finite).

Sums of more than two central Cantor sets. Let a_1, \dots, a_k be a collection of real numbers in $(0, 1/2)$ which is linearly independent over \mathbb{Q} . Then

$$D(\mu_{a_1} * \dots * \mu_{a_k}) = \min(d_{a_1} + \dots + d_{a_k}, 1).$$

The proof of this is similar to that of Theorem 1.1. We sketch the main differences. The space \mathbb{X} becomes the family of all k -tuples of finite words with elements in $\{0, 1\}$. A family $\{\mathbb{X}_n\}$ of subsets of \mathbb{X} is then constructed, with the property that if $\xi \in \mathbb{X}_n$, then each parallelepiped

$Q(\xi)$, defined in the obvious way, has size

$$a_1^n \times a_1^n e^{\mathbf{R}_2^n(0)/\beta_2} \times \dots a_1^n e^{\mathbf{R}_k^n(0)/\beta_k},$$

where β_i are real numbers (the analogous of β), and $\mathbf{R}_2, \dots, \mathbf{R}_n$ are rotations of the circle. It follows from the hypothesis of linear independence that

$$\mathbf{R}(x_2, \dots, x_k) = (\mathbf{R}_2(x_2), \dots, \mathbf{R}_k(x_k))$$

is a uniquely ergodic transformation of the $(k-1)$ -dimensional torus. Instead of a single family $\{\mathbb{Y}_n\}$, now 2^{k-1} auxiliary families are required (one for each subset of $\{2, \dots, k\}$). Details are left to the reader.

L^q -dimensions. The L^q dimensions of a measure generalize the concept of correlation dimension. We define them only for $q > 1$ since that is the case we are interested in. Let

$$\mathcal{C}_\nu^q(r) = \int \nu(B(x, r))^{q-1} d\nu(x),$$

and set

$$D_q(\nu) = \liminf_{r \rightarrow 0} \frac{\mathcal{C}_\nu^q(r)}{\log(r)}.$$

Thus $D_2(\nu)$ equals the lower correlation dimension of ν . The L^q dimensions are of fundamental importance in multifractal analysis, see for example [3]. In general, there is no projection theorem for L^q -dimensions for $q > 2$: the L^q -dimension can drop for all orthogonal projections, even for very simple measures like arc length on the unit circle. However, the rest of the proof of Theorem 1.1 extends to L^q dimensions for any $q > 1$. In particular, the analogue of (3.1) holds for L^q -dimensions, with an almost identical proof. Applying Furman's Theorem and the subadditive ergodic theorem as in the proof of Theorem 1.1, we obtain the following result:

Theorem 5.1. *Let $q > 2$. If*

$$D_q(\mu_a * (\mu_b \circ S_{\lambda_0}^{-1})) > d$$

for some $\lambda_0 > 0$, then

$$D_q(\mu_a * (\mu_b \circ S_\lambda^{-1})) > d$$

for almost every $\lambda > 0$.

This is related to the investigations in [6].

Acknowledgement. We thank Esa Järvenpää for helpful comments.

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