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PERFECT ISOMETRIES AND THE ALPERIN-MCKAY CONJECTURE

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ABSTRACT. We give a brief survey of results and conjectures concerning the local determination of invariants of Brauer p-blocks of finite groups. We highlight the connections between the various conjectures, in particular those of Alperin-McKay and of Broué, and identify where further conjectures have to be made. We focus on the problem of generalising Broué's conjecture, and suggest a generalisation of the idea of a perfect isometry. Finally we present evidence that such a generalised perfect isometry should exist in certain cases.

CONTENTS

1. Introduction

The purpose of this survey, closely based on the author's series of lectures given at the Symposium, is to give motivation for the generalisation of a conjecture of Broué, and to present one possibility for such a generalisation. As such, we are quite selective in the material presented, giving only those results, examples and conjectures which illuminate our chosen path. Hence we apologise in advance for omitting Dade's conjectures and those related to it, as well as some of the excellent work which has been done on Broué's conjectures and on fusion systems.

One of the main parts of the modular representation theory of finite groups concerns *local determination*, which is the determination of invariants of a block of a group by examining so-called local subgroups, with respect to a fixed prime p. Many of the main results in the area may be phrased in this way, for example the Green correspondence and Brauer's first and second main theorems, as well as many conjectures, including the Alperin-McKay conjecture of the title. The Alperin-McKay conjecture predicts a straightforward equality between the number of *height zero* irreducible characters in a

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block and the number of such irreducible characters in a uniquely determined block of a certain subgroup. We describe how Broué's conjecture explains the Alperin-McKay conjecture in restricted cases, resulting in particular in a structured bijection of irreducible characters rather than just an equality of numbers.

In the last part of the survey we will discuss the generalisation of the weaker of Broué's conjectures, and propose such a generalisation.

2. Background in block theory

Excellent references for this section are [5], [10], [12] and [13].

Let G be a finite group and p a prime. In order to study the characteristic zero representations of G in relation to the prime p, we consider a p-modular system $(K, \mathcal{O}, k)$ relating fields K and k of characteristic zero and p respectively via a ring $\mathcal{O}$. The conditions which we take on $(K, \mathcal{O}, k)$ are not intended to be in anyway minimal. Briefly, we let $\mathcal{O}$ be a complete local discrete valuation ring containing a primitive $|G|^3$-root of unity, such that $k = \mathcal{O}/J(\mathcal{O})$ is algebraically closed with $char(k) = p$ and K is the field of fractions of $\mathcal{O}$ (we take $|G|^3$th roots of unity rather than $|G|$th roots because at some stage we may need to take a central extension of G of order dividing $|G|^3$).

Our approach will mostly be motivated by the study of characters, so our first task is to partition the set $\mathrm{Irr}(G)$ of irreducible characters (with respect to K) of G into blocks. The advantage of studying representations one block at a time is that representations associated to the same block share some properties which we can take advantage of.

2.1. Characters in blocks.

Decompose the group algebra $\mathcal{O}G$ into indecomposable two-sided ideals:

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_n.$$

This corresponds to a decomposition of $1 \in Z(\mathcal{O}G)$ into primitive idempotents of $Z(\mathcal{O}G)$, say $1 = e_1 + \cdots + e_n$, with $e_i\mathcal{O}G = B_i$.

Similarly we may decompose kG. For $a = \sum_{g \in G} a_g g \in \mathcal{O}G$, write

$$\overline{a} = \sum_{g \in G} (a_g + J(\mathcal{O}))g \in kG.$$

Then $\overline{1} = \overline{e}_1 + \cdots + \overline{e}_n$ is a decomposition into primitive idempotents of $Z(kG)$, and

$$kG = \overline{e}_1 kG \oplus \cdots \oplus \overline{e}_n kG$$

is a decomposition into indecomposable two-sided ideals. Write $\overline{B}_i = \overline{e}_i kG$. (Note that each such decomposition of $\overline{1}$ lifts to a decomposition 1 into primitive idempotents of $Z(\mathcal{O}G)$). Essential in this is our choice of $\mathcal{O}$ complete.

We call the B_i (and $\overline{B}_i$) *blocks* of G, and the e_i (and $\overline{e}_i$) block idempotents.

Now let M be an $\mathcal{O}G$-module. Then

$$M = B_1 M \oplus \cdots \oplus B_n M.$$

Hence if M is indecomposable, then $M = B_i M$ for some unique block B_i, and we say that M *belongs* to B_i. The same argument holds for kG-modules.

If $\chi \in \mathrm{Irr}(G)$ (the set of irreducible (K-)characters of G), then χ is afforded by some irreducible KG-module V. There is an indecomposable $\mathcal{O}G$-lattice M such that $V =$

$K \otimes_{\mathcal{O}} M$. We say that χ belongs to the block to which M belongs. This is independent of the choice of M.

So we can partition $\mathrm{Irr}(G)$ into sets $\mathrm{Irr}(B_i)$ of characters belonging to B_i.

Alternatively, $\chi \in \mathrm{Irr}(B_i)$ if $\chi(e_j) = \chi(1)$ for $j = i$ and $\chi(e_j) = 0$ otherwise, giving the same partition. In much of block theory there are several different ways of making any definition, and they are usually equivalent.

We can determine the block idempotents quite explicitly from the values of the characters in a block.

The primitive idempotent of $Z(KG)$ corresponding to χ is

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g.$$

Fixing a block B, with block idempotent e_B, we have

$$e_B = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi.$$

2.2. **Brauer characters and decomposition matrices.** Our aim here is to give the characters of the (projective) indecomposable summands of $\mathcal{O}G$ as a left $\mathcal{O}G$-module. To do this we use Brauer characters. These are a way of assigning class functions with values in K to simple kG-modules, in order that we may more directly compare the simple kG-modules to $\mathrm{Irr}(G)$.

Let S be a simple kG-module, and let $\rho : G \to GL_t(k)$ be an associated representation.

Let $g \in G_{p'}$, the set of p-regular elements of G, and let m be the p'-part of the exponent of G. Then $\rho(g)$ has eigenvalues which are m-roots of unity. Let ω be a primitive m-th root of unity in K, and note that the groups of m-th roots of unity of K and of k are isomorphic. Say $\omega \to \overline{\omega}$ under such an isomorphism.

$Trace(\rho(g))$ is a sum of m-th roots of unity, say $\sum_i \overline{\omega}^{r_i}$. Define $\varphi(g) = \sum_i \omega^{r_i}$. We call φ the irreducible Brauer character associated to S. This is a class function defined on p-regular elements.

We assign φ to the same block as S, and write $\mathrm{IBr}(B)$ for the set of irreducible Brauer characters belonging to B.

The number of distinct irreducible Brauer characters equals the number of p-regular conjugacy classes of G. Further, the irreducible Brauer characters $\mathrm{IBr}(G)$ span the space of class functions defined on p-regular conjugacy classes of G.

If χ is a character of G, write $\chi_{p'}$ for the restriction of χ to the p-regular conjugacy classes. In fact $\chi_{p'}$ is a non-negative integer linear combination of irreducible Brauer characters. If $\chi \in \mathrm{Irr}(G)$, then write

$$\chi = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\phi},$$

where the $d_{\chi\phi}$ are non-negative integers.

The $d_{\chi\varphi}$ are called the *decomposition numbers* of G, and we call the matrix $D = (d_{\chi\varphi})$ the *decomposition matrix* of G.

If χ and φ are in different blocks, then $d_{\chi\varphi} = 0$, and so we can define the decomposition matrix D_B of a block B.

We obtain the Cartan matrix C by $C = D^T D$, where D^T denotes the transpose of D. Recall that C is the matrix recording the occurrence of the simple kG-modules as composition factors of the projective covers of all the simple kG-modules. Again we may take the Cartan matrix C_B of a block B.

We may now write down the character of a projective indecomposable $\mathcal{O}G$-module P.

Now $P/J(\mathcal{O})P$ is a projective indecomposable kG-module, which is the projective cover of a simple kG-module, say S. Let φ be the irreducible Brauer character associated to S. Then P has character

$$\Phi = \Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi.$$

Two important facts concerning the characters Φ are that

(i) $\Phi(g) = 0$ whenever g is p-singular (i.e., g has order divisible by p);

(ii) if χ is a character of G such that $\chi(g) = 0$ for all p-singular g, then χ is a $\mathbb{Z}$-linear combination of characters Φ_φ for $\varphi \in \mathrm{IBr}(G)$.

2.3. **Defect groups.** We present one (of several equivalent) characterisations of the defect groups of a block. As we mentioned earlier, we will usually consider local determination of invariants of blocks from invariants of normalisers of p-subgroups. The defect groups of a block are a G-conjugacy class of p-subgroups associated to B, and local determination will occur via normalisers of subgroups of the defect groups.

Let S be an indecomposable kG-module and let $H \leq G$. We say S is H-projective if there is a kH-module T such that $S | \mathrm{Ind}_H^G(T)$.

Now if $P \in \mathrm{Syl}_p(G)$, then S is P-projective. Further, if $J \leq H$ and S is J-projective, then S is H-projective.

Hence there are subgroups Q which are minimal such that S is Q-projective, and these must be p-groups. Call these the *vertices* of S.

Using the Mackey decomposition, we can see that the vertices of S form a G-conjugacy class of p-subgroups.

Let B be a block. Define the *defect groups* of B to be the p-subgroups of G maximal amongst the vertices of the simple modules in B.

The defect groups of B form a conjugacy class of p-subgroups of G. If D is a defect group of B and $|D| = p^d$, then we say B has *defect d*. The defect is related to the degrees of the irreducible characters in B (in fact we may also determine the defect groups themselves from the irreducible characters, but we do not describe that here).

For $\chi \in \mathrm{Irr}(G)$, define the defect of χ to be the integer $d(\chi)$ such that $|G|_p = p^{d(\chi)}\chi(1)_p$. Then

$$d = max\{d(\chi) : \chi \in \mathrm{Irr}(B)\}.$$

Examples

(a) We call the block containing the trivial character the *principal block*. The Sylow p-subgroups are the defect groups.

(b) The blocks of defect zero, where the trivial group is the only defect group, are of particular importance. These are simple algebras, and we do not expect to obtain any information about them from local subgroups. A block B has defect zero if and only if $k(B) = 1$. The unique irreducible character χ in a block of defect zero satisfies $\chi(1)_p = |G|_p$.

2.4. **Brauer correspondence.** If we are to compare blocks of G with blocks of subgroups of G, then we need a way of naturally associating them.

Let P be a p-subgroup of G and let $H \leq G$ such that $C_G(P) \leq H \leq N_G(P)$. Define $Br_P : Z(kG) \to Z(kH)$ by defining $Br_P(\hat{C}) = 0$ if $C \cap C_G(P) = \emptyset$ and $Br_P(\hat{C}) = \sum_{g \in C \cap C_G(P)} g$ otherwise, where C is any conjugacy class of G and $\hat{C} = \sum_{g \in C} g$. Then Br_P is an algebra homomorphism, called the *Brauer homomorphism*.

Let b be a block of H, with defect group Q, and suppose that $C_G(Q) \leq H \leq N_G(Q)$. Then there is a unique block B of G such that $Br_Q(\overline{e}_B)\overline{e}_b = \overline{e}_b$. We write $b^G = B$, and call B the *Brauer correspondent* of b in G. We also sometimes call B the *induced block*.

If we fix a p-subgroup D of G, then the Brauer correspondence gives a bijection between blocks of G with defect group D and blocks of $N_G(D)$ with defect group D (this is Brauer's first main theorem of block theory). Further (Brauer's third main theorem), the principal blocks correspond under this bijection.

Note that this is a slightly simplified definition, but one which suffices for our purposes. The Brauer correspondence may be defined in greater generality than this. Note also that there are several different definitions, which are not necessarily equivalent unless we put restrictions on H similar to those above.

3. Local determination

We would like to obtain information about representations of a block B of a group G from information about subgroups. So we have two questions: "what sort of subgroups?", and "what sort of information?"

What sort of subgroups?

One immediate restriction on our choice of subgroup is that we would like the Brauer correspondence to be defined, i.e., we would like there to exist blocks of subgroups with Brauer correspondent B, and we would like the blocks of our subgroups to have a Brauer correspondent. We saw in the previous section conditions for the existence of Brauer correspondents.

Further, Clifford's theorem tells us that $O_p(G) \leq \ker(S)$ for every simple kG-module S. This and other results tells us that in some respects we the presence of normal p-subgroups allows us to obtain information from smaller groups still.

Existing results and conjectures involve the extraction of information from *local subgroups*, which may mean:

- normalisers $N_G(Q)$ of p-subgroups Q
- stablisers (under conjugation) of chains of p-subgroups
- centralizers $C_G(Q)$ of p-subgroups
- subgroups H with $O_p(H) \neq 1$ (see the very nice paper by Thevenaz [16])
- slightly altered versions of the above (e.g., normalizers of subpairs)

What sort of information?

This ranges from numerical information, e.g.,

- $k(B) = |\operatorname{Irr}(B)|$
- $l(B) = |\operatorname{IBr}(B)|$
- $k_0(B) = |\{\chi \in \operatorname{Irr}(B) : d(\chi) = d(B)\}|$ (irreducible characters of *height* zero) elementary divisors of the Cartan matrix

to categorical information, e.g.,

- the derived category $\mathcal{D}^b(B)$
- stable module category $\underline{mod}(B)$

There are many examples of theorems in local determination, but we concentrate here on the conjectures.

Throughout, let B be a block of G with defect group D. One of the earliest conjectures is

Conjecture 1 (Alperin-McKay). *Let b be the unique block of $N_G(D)$ with Brauer correspondent B. Then*

$$k_0(B) = k_0(b).$$

Here, local determination is particularly straightforward. However, we do not always expect to obtain our information from just one source. For example, Alperin's weight conjecture gives $l(B)$ in terms of information from many $N_G(Q)$, for p-subgroups Q. Let $\mathcal{P}_0(G)$ be the set of p-subgroups of G.

Conjecture 2 (Alperin's weight conjecture).

$$l(B) = \sum_{Q \in \mathcal{P}_0(G)} f_0^{(B)}(N_G(Q)/Q),$$

where $f_0^{(B)}(N_G(Q)/Q)$ is the number of Q-projective simple $kN_G(Q)$-modules in blocks with Brauer correspondent B.

The Knörr-Robinson reformulation of Alperin's weight conjecture is even more complicated in terms of the number of local subgroups used, and gives $k(B)$ in terms of an alternating sum over stabilisers of chains of p-subgroups.

Note that in Alperin's conjecture, using properties of the Brauer correspondence it suffices to consider subgroups Q contained in a defect group for B.

If D is abelian, then Alperin's weight conjecture predicts that $l(B) = l(b)$, and the Knörr-Robinson reformulation predicts that $k(B) = k(b)$, i.e., local determination comes from one subgroup. To see this for Alperin's weight conjecture, consider $Q \leq D$, and let S be a simple $kN_G(Q)$-module in a block b with Brauer correspondent B. Let R be a defect group of b with $R \leq D$ (replacing D by a conjugate containing Q if necessary). Then $Q \leq R$ and by [9] we have $R = C_R(Q) \leq Q \leq R$. So b has defect group Q, and so by Brauer's first main theorem b^G has defect group Q. But $b^G = B$, so $D = Q$ after all. But every simple $kN_G(D)$-module is D-projective, so we are done.

Alperin's weight conjecture (and its reformulations by Knörr and Robinson are just two of a wide array of conjectures concerning ever more detailed numerical invariants. These would take a long time to state, so we have only presented those which are relevant to out story.

4. Broué's conjecture

We would like to understand the numerical conjectures introduced in the last section more deeply, for example as consequences of results about the module categories. However, we also saw that in general local determination of numerical invariants involves comparing a number of groups at once. Since we know best how to compare two categories, we start by looking at situations where we expect local determination (of a block B of G) to use just one block of one subgroup. We saw that one such case is where the defect group D is abelian.

Throughout this section, let b be the unique block of $N_G(D)$ with Brauer correspondent B. An excellent reference for this section is [10].

Conjecture 3 (Broué)**.** *Suppose D is abelian. Then the derived categories $\mathcal{D}^b(B)$ and $\mathcal{D}^b(b)$ are equivalent (as triangulated categories).*

Remark 4. Actually, more recent versions of Broué's conjecture state that we should have a splendid equivalence (also known as a Rickard equivalence). This places additional restrictions on the tilting complex giving the derived equivalence, which amongst other things ensure that we also have a family of compatible derived equivalences between various subgroups.

Broué's conjecture is very hard to verify for a given block, but it is known in many cases. For reasons of space we do not attempt to list these here.

We relate Broué's conjecture to numerical conjectures such as Alperin-McKay's

Suppose that $\mathcal{D}^b(B)$ and $\mathcal{D}^b(b)$ are equivalent as triangulated categories (with no restrictions on D). Then $\mathcal{D}^b(\overline{B})$ and $\mathcal{D}^b(\overline{b})$ are also equivalent as triangulated categories . We have:

- $mod(B)$ and $mod(b)$ (and $mod(\overline{B})$ and $mod(\overline{b})$) have isomorphic Grothendieck groups (see [K-Z,6.3.3])
- B and b (and $\overline{B}$ and $\overline{b}$) have isomorphic centres (see [K-Z,6.3.2])

In particular,

- $k(B) = k(b)$, $l(B) = l(b)$
- also $k_0(B) = k_0(b)$, although this takes more work to prove.

Hence, in the abelian defect group case, Broué's conjecture gives the Alperin, Knörr-Robinson, Alperin-McKay (and Dade) conjectures.

Now suppose that D is non-abelian. Then in general we do not have $k(B) = k(b)$, $l(B) = l(b)$ (although we do expect $k_0(B) = k_0(b)$). Hence there cannot be a derived equivalence in general.

Even when we do have equality of numerical invariants, e.g., $k(B) = k(b)$, $l(B) = l(b)$, etc., there is sometimes no derived equivalence:

We say that D is a *trivial intersection* (TI) subgroup of G if for each $g \in G - N_G(D)$, we have $D^g \cap D = 1$. If B is a block with TI defect group D, then Alperin's conjecture states that $l(B) = l(b)$, and the Knörr-Robinson reformulation states that $k(B) = k(b)$. Actually

Theorem 5 (An-Eaton [2]). *Suppose B is a block with TI defect groups. Then Alperin's, Alperin-McKay's (and Dade's, Isaacs-Navarro's, Uno's) conjectures all hold for B.*

The principal 2-block of $Sz(8) = {}^2B_2(8)$ has TI defect groups. However, it has long been known that B and b are not derived equivalent in this case. This was first observed by Thompson, but see also Cliff [4], which shows that $Z(B)$ and $Z(b)$ are not isomorphic, and Robinson [15], which we will discuss later.

To summarise , we have numerical conjectures which may be applied to *all* blocks, and in a very restricted case (abelian defect groups) we have a deep structural explanation for them, albeit a conjectural one!

A big problem is how to explain the numerical coincidences in general.

One approach would be to attempt to generalise Broué's conjecture directly, e.g., to generalise the concept of a derived equivalence. Alternatively, we could use invariants of derived categories lying somewhere between the simplest numerical ones (number of irreducible characters, etc.) and the derived equivalence class of a category.

So we try to formulate conjectures implying those of Alperin, Alperin-McKay, Dade's, etc., which hold in some non-abelian defect cases. This should give evidence for possible generalisations of Broué's conjecture.

We begin by looking at some consequences of Broué's conjecture in more depth.

5. Perfect isometries

Excellent references for this section are [3], [7] and [10].

For a block (or sum of blocks) B of a group G, denote by

$$\mathcal{R}(G, B)$$

the additive group of characters generated by $\mathrm{Irr}(B)$. We may identify this with the Grothendieck group of $mod(K \otimes_{\mathcal{O}} B)$. We may consider $\mathcal{R}(G, B)$ as lying in $CF(G, B, K) \subset CF(G, K)$, the space of K-valued class functions spanned by $\mathrm{Irr}(G, B)$.

Let b be a block of another group H. Note that $B \otimes b^\circ$ is a block of $G \times H^\circ$, where b°, H° denote the opposite algebra, group respectively

Given

$$\mu \in \mathcal{R}(G \times H^\circ, B \otimes b^\circ),$$

we define maps

$$I_\mu : CF(H, b, K) \to CF(G, B, K)$$

$$R_\mu : CF(G, B, K) \to CF(H, b, K)$$

where I_μ and R_μ are adjoint linear maps with respect to the usual scalar product on characters, as follows:

Let $\alpha \in CF(H, b, K)$, $\beta \in CF(G, B, K)$, $h \in H$, $g \in G$. Define

$$I_\mu(\alpha)(g) = \frac{1}{|H|} \sum_{h \in H} \mu(g, h^{-1}) \alpha(h),$$

$$R_\mu(\beta)(h) = \frac{1}{|G|}\sum_{g\in G}\mu(g^{-1},h)\beta(g).$$

Actually, if we have a linear map $I : \mathcal{R}(H,b) \to \mathcal{R}(G,B)$ then defining

$$\mu = \sum_{\theta\in \mathrm{Irr}(H,b)} I(\theta)\theta$$

gives $I_\mu = I$. This follows from the orthogonality relations for ordinary characters.

Let $\mu \in \mathcal{R}(G\times H^\circ, B\otimes b^\circ)$. So far, the maps R_μ and I_μ induced by μ tell us nothing which relates the structures of B and b. They become more interesting when we require that μ is *perfect*. Before defining perfect characters, we motivate them.

Suppose that $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.

Then this equivalence may be induced by a bounded complex

$$M : \cdots \to M_{-r} \to M_{-r+1} \to \cdots \to M_s \to \cdots$$

of B-b-bimodules such that each M_r is projective as a B-module and as a b-module (see [10]).

Let μ_r be the character afforded by $K\otimes_{\mathcal{O}} M_r$. Then the generalised character

$$\mu = \sum_r (-1)^r\mu_r$$

gives an isometry I_μ. In particular, if Broué's conjecture holds, then we get an isometry $CF(N_G(D),b,K) \to CF(G,B,K)$ related to the complex inducing the equivalence of categories.

A complex M of B-b-bilmodules whose terms are projective as B-modules and as b-modules is called a *perfect complex*, and the definition of a perfect generalised character is related to this.

Denote by $CF_{p'}(G,B,K)$ the subspace of class functions $\alpha \in CF(G,B,K)$ such that if $g \in G - G_{p'}$, then $\alpha(g) = 0$.

Definition 6. $\mu \in \mathcal{R}(G\times H^\circ, B\otimes b^\circ)$ is *perfect* if

(a) I_μ gives a map $CF(H,b,\mathcal{O}) \to CF(G,B,\mathcal{O})$ and R_μ gives a map $CF(G,B,\mathcal{O}) \to CF(H,b,\mathcal{O})$

(b) I_μ gives a map $CF_{p'}(H,b,\mathcal{O}) \to CF_{p'}(G,B,\mathcal{O})$ and R_μ gives a map $CF_{p'}(G,B,\mathcal{O}) \to CF_{p'}(H,b,\mathcal{O})$.

Proposition 7 (Broué)**.** $\mu \in \mathcal{R}(G\times H^\circ, B\otimes b^\circ)$ *is* perfect *if and only if*

(a') for all $(g,h)\in G\times H$, we have $\mu(g,h)/|C_G(g)| \in \mathcal{O}$ and $\mu(g,h)/|C_H(h)| \in \mathcal{O}$,

(b') if $\mu(g,h)\neq 0$, then both g and h are p-singular or both g and h are p-regular.

Remark 8. Suppose that a character μ of $G\times H^\circ$ is afforded by an $\mathcal{O}(G\times H^\circ)$-module which is projective as G- and H°-modules. Then μ is perfect.

Example 9. Suppose $H\le G$, and let μ be the character of the KG-KH-bimodule KG. Then μ is perfect. Here I_μ is induction and R_μ is restriction of characters. Explicitly,

$$\mu(g,h) = \sum_{\chi\in\mathrm{Irr}(G)}\sum_{\theta\in\mathrm{Irr}(H)} (Res^G_H(\chi),\theta)\chi(g)\theta(h).$$

Similarly, $K \otimes B$ may be considered as a $K \otimes B$-$K \otimes b$-bimodule in this way, to give blockwise induction and restriction, which means induction and restriction, but only taking only components in B or b.

We define a *perfect isometry* to be a map I_μ which is an isometry, such that μ is perfect. The inverse map is R_μ. This gives a 'bijection with signs' between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$.

Broué conjectures that:

Conjecture 10 (Broué's isometry conjecture)**.** *Let B be a block with abelian defect group D, and let b be the unique block of $N_G(D)$ with $b^G = B$. Then there is a perfect isometry*

$$I_\mu : CF(N_G(D), b, K) \to CF(G, B, K).$$

Remark 11. When discussing Conjecture 3, we mentioned splendid equivalences, which give families of derived equivalences. This is in part motivated by a stronger form of the above conjecture, which predicts an *isotypy.* This is a family of compatible perfect isometries. However, we will not discuss these in detail here, although they are very important to the subject. Actually, in some sense they aid the search for a perfect isometry between B and b. We should further remark however, that perfect isometries arising from stable equivalences in the TI defect group situation (in a similar way to property (P+) later) automatically give isotypys.

5.1. **Invariants preserved by perfect isometries.** Suppose that I_μ is a perfect isometry. Define

$$I_\mu^0 : Z(KHe_b) \to Z(KGe_B)$$

by

$$I_\mu^0(a) = \left(\frac{1}{H} \sum_{g \in G} \sum_{h \in H} \mu(g^{-1}, h) a_h \right) g,$$

where $a = \sum_{h \in H} a_h h$.

Since μ is perfect, this also defines an invertible $\mathcal{O}$-linear map

$$Z(\mathcal{O}He_b) \to Z(\mathcal{O}Ge_B).$$

Write R_μ^0 for the analogous map $Z(KGe_B) \to Z(KHe_b)$. Then $a \to I_\mu^0(aR_\mu^0(e_B))$ defines an algebra isomorphism

$$Z(\mathcal{O}He_b) \to Z(\mathcal{O}Ge_B).$$

The calculations used to show the algebra isomorphism can also be used to show that for each $\theta \in \mathrm{Irr}(b)$,

$$\frac{|G|/I_\mu(\theta)(1)}{|H|/\theta(1)} \in \mathcal{O}$$

and is invertible in $\mathcal{O}$. Hence I_μ preserves the defects of the ordinary irreducible characters. Since $d(B) = max\{d(\chi) : \chi \in \mathrm{Irr}(B)\}$, this means that the defect of a block is preserved. (It is not known - to the authors knowledge - that a perfect isometry, or even Morita equivalence preserves the isomorphism class of a defect group, although neither is the author aware of a counterexample).

It is also the case that, modulo p,

$$\frac{|G|/I_\mu(\theta)(1)}{|H|/\theta(1)}$$

is independent (up to sign) of the choice of $\theta \in \mathrm{Irr}(b)$.

Now suppose further that $H = N_G(D)$, that B (and so b) is the principal block, and that $I_\mu(1_H) = \pm 1_G$. Then D is a Sylow p-subgroup, and

$$[G : N_G(D)] \equiv 1 \ mod\ p.$$

Then

$$I_\mu(\theta)(1) \equiv \pm\theta(1) \ mod\ p.$$

This is a motivation for the following strengthening of the Alperin-McKay conjecture (although we do not claim that it was the original motivation).

Let B be a block of a group G with defect group D. Let b be the unique block of $N_G(D)$ with $b^G = B$. Let r be an integer. Write

$$\mathrm{Irr}(B,[r]) = \{\chi \in \mathrm{Irr}(B) : \frac{|G|}{\chi(1)_p} \equiv \pm r \ mod\ p\}$$

and $k(B,[r]) = |\,\mathrm{Irr}(B,[r])|$.

Conjecture 12 (Isaacs-Navarro). *For each integer r, we have $k_0(B,[r]) = k_0(b,[r])$.*

So in the above situation, for the principal block, the Isaacs-Navarro conjecture is a consequence of a perfect isometry.

Remark 13. (a) Uno has announced a generalisation of the Isaacs-Navarro conjecture to arbitary character defects, which is also a strengthening of Dade's conjecture.

(b) Just as with the other numerical conjectures, when the defect group is TI, a straightforward equality is predicted, with all information coming from just one local subgroup, $N_G(D)$. I regard this as evidence that there should be a generalisation of a perfect isometry which at least holds in the TI defect group case.

Other invariants which are preserved by perfect isometries are $l(B)$ and the elementary divisors of the Cartan matrix.

Further evidence that the TI defect group case should be similar to the abelian defect group case is the following consequence of Theorem 5 (see [6]):

Proposition 14. *Suppose that B has TI defect group D, and let b be the unique block of $N_G(D)$ with Brauer correspondent B. Then the Cartan matrices of B and b have the same elementary divisors.*

5.2. **Existence and non-existence of perfect isometries.** As mentioned earlier, Cliff in [4] has proved that if G is $Sz(8)$ and B is the principal 2-block, then $Z(B)$ is not isomorphic to $Z(b)$, where b is the Brauer correspondent of B in $N_G(D)$. Hence there can be no perfect isometry in this case.

Robinson in [15] gives general conditions for the non-existence of a perfect isometry, based on the block having many irreducible characters constant on p-singular conjugacy classes when $N_G(D)/O_{p'}(N_G(D))$ is a Frobenius group. Such a condition can be checked easily for, e.g., the Suzuki groups.

As before, we do not attempt to list the cases for which Broué isometry conjecture is known. However, we draw the reader's attention to what may be considered the high point of work on the conjecture, which is [7], where it is proved that the conjecture holds for the principal block for $p = 2$.

6. Generalising perfect isometries

If we believe the numerical conjectures, then in general we expect local determination to be complicated, because we expect information to come from a number of subgroups simultaneously, as in Alperin's weight conjecture. However, in some cases, the numerical conjectures suggest that we may find information from just one subgroup. An example is when $N_G(D)$ controls fusion in D, which includes the case D is abelian. This also includes the case that D is TI.

We present here an observation of some very compelling behaviour in the TI defect group case, which leads to a generalisation of Conjecture 10. Most of the results in this section are taken from [6].

Throughout, let B be a block of G with defect group D, and let b be the unique block of $N_G(D)$ with Brauer correspondent B.

Definition 15. We say that B satisfies property (P) if there is perfect $\mu \in \mathcal{R}(G \times N_G(D)^\circ, B \otimes b^\circ)$ such that for each $\theta \in Irr_0(N_G(D), b)$, the map

$$I_\mu : CF(N_G(D), b, K) \to CF(G, B, K)$$

induced by μ satisfies

$$I_\mu(\theta) = \epsilon\chi + \Delta$$

for some $\chi \in \mathrm{Irr}_0(G, B)$, where $\epsilon \in \{-1, 1\}$ and no constituent of Δ has height zero, and for each $\chi \in \mathrm{Irr}_0(G, B)$, the map

$$R_\mu : CF(G, B, K) \to CF(N_G(D), b, K)$$

satisfies

$$R_\mu(\chi) = \epsilon\theta + \Theta$$

for some $\theta \in \mathrm{Irr}_0(N_G(D), b)$ where $\epsilon \in \{-1, 1\}$ and no constituent of Θ has height zero.

Remark 16. (a) Property (P) gives rise to a bijection 'with signs' between $\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(b)$, just as a perfect isometry does.

(b) If I_μ is a perfect isometry, then μ gives (P).

(c) If D is abelian, then Brauer's abelian defect group conjecture predicts that $\mathrm{Irr}_0(B) = \mathrm{Irr}(B)$, and we already know that $\mathrm{Irr}_0(b) = \mathrm{Irr}(b)$. Hence if Brauer's conjecture is true and D is abelian, then I_μ is a perfect isometry if and only if μ gives (P).

(P) holds for every example of a block with TI defect groups so far checked. Unfortunately, (P) does not hold for all blocks, for example:

Proposition 17. *Suppose that B is the principal block of $G = PSL_3(2)$. Then no choice of μ can give (P).*

Proof. $D \cong D_8$ and $N_G(D) = D$. By checking the short list of possibilities, we cannot have $\mu(1, h) = 0$ for each nontrivial $h \in D$. □

Note that $G = PSL_3(2)$ has non-TI Sylow 2-subgroups. Further, $D = N_G(D)$ does not control fusion in D.

However, we will see in the first example that often something stronger than (P) actually holds.

Now suppose that $H = N_G(D)$ and that $b^G = B$. Consider 'blockwise induction and restriction', which is given by $\Phi = \sum_{\chi\in\mathrm{Irr}(B)}\sum_{\theta\in\mathrm{Irr}(b)}(\mathrm{Res}^G_{N_G(D)}(\chi),\theta)\chi\theta$. The map I_Φ is then 'induction, taking terms in B,' and R_Φ is 'restriction, taking terms in b.' Just as with induction/restriction, this is a perfect character.

Definition 18. We say that B satisfies (P+) if there is $\mu \in \mathcal{R}(G\times N_G(D)^\circ, B\otimes b^\circ)$ giving (P) of the form $\mu = \Phi + \sum_{s,t} a_{s,t}\Gamma_s\Phi_t$, where each $a_{s,t}$ is an integer and Γ_s resp. Φ_t is the character of a projective indecomposable module of B resp. b.

A generalised character μ of this form is necessarily perfect.

Remark 19. If B satisfies property (P+), then it is immediate that Conjecture 12 holds for that block.

We give the following example in full as an illustration. Note that the block we use does not have TI defect groups, but (P+) holds anyway.

Example 20. Let $G = S_5$ and $p = 2$. Let B be the principal block of G, so B has defect group $D \cong D_8$, a Sylow 2-subgroup of G. We have $N_G(D) = D$. Now the irreducible characters of G are $\chi_1 \dots, \chi_7$, with degrees $1, 1, 4, 4, 5, 5, 6$ respectively. We have $\mathrm{Irr}(B) = \{\chi_1, \chi_2, \chi_5, \chi_6, \chi_7\}$. The irreducible characters of $N_G(P)$ are $\theta_1, \dots, \theta_5$, with degrees $1, 1, 1, 1, 2$ respectively. We will need the following characters of projective indecomposable modules: $\Gamma = \chi_5 + \chi_6 + \chi_7$ and $\Phi_1 = \theta_1 + \cdots + \theta_4 + 2\theta_5$.

The restrictions of the irreducible characters of B to $N_G(D)$ are as follows:

$$\begin{array}{rclcl}
\mathrm{Res}^G_{N_G(D)}(\chi_1) & = & \theta_1 & = & \theta_1 \\
\mathrm{Res}^G_{N_G(D)}(\chi_2) & = & \theta_3 & = & \theta_3 \\
\mathrm{Res}^G_{N_G(D)}(\chi_5) & = & \theta_1+\theta_2+\theta_3+\theta_5 & = & \Phi_1-\theta_4-\theta_5 \\
\mathrm{Res}^G_{N_G(D)}(\chi_6) & = & \theta_1+\theta_3+\theta_4+\theta_5 & = & \Phi_1-\theta_2-\theta_5 \\
\mathrm{Res}^G_{N_G(D)}(\chi_7) & = & \theta_2+\theta_4+2\theta_5 & = & \Phi_1-\theta_1-\theta_3
\end{array}$$

Hence $\mu = \Phi - \Gamma\Phi_1$ gives the bijection with signs

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_5 \\ \chi_6 \end{pmatrix} \leftrightarrow \begin{pmatrix} \theta_1 \\ \theta_3 \\ -\theta_4 \\ -\theta_2 \end{pmatrix}.$$

(P) is partially motivated by the following, from [14]:

Theorem 21 (Navarro). *Let G be a p-solvable group such that $N_G(P) = P$ for a Sylow p-subgroup P. Then*

(a) for each $\theta \in \mathrm{Irr}(P)$ with $\theta(1) = 1$, we have $\mathrm{Ind}^G_P(\theta) = \chi + \Delta$ where $\theta \in \mathrm{Irr}(G)$ with $p \not|\chi(1)$ and $p|\delta(1)$ for each irreducible constituent δ of Δ, and

(b) for each $\chi \in \mathrm{Irr}(G)$ with $p \not|\chi(1)$, we have $\mathrm{Res}^G_P(\chi) = \theta + \Theta$, where $\theta \in \mathrm{Irr}(P)$ with $\theta(1) = 1$ and $p|\gamma(1)$ for each irreducible constituent γ of Θ.

This means that for the principal block of a p-solvable group with $N_G(P) = P$, the character for Φ for induction/restriction gives property (P+).

6.1. **Controlled blocks.** The principal block is a *controlled block* if, for P a Sylow p-subgroup of G, if $Q \leq P$ and $g \in G$ such that $Q^g \in P$, then $g = cn$ for some $c \in C_G(G)$ and $h \in N_G(P)$. There are examples of controlled blocks which do not have TI defect groups. E.g., the principal 3-blocks of J_2 and J_3, and also the principal 5-block of Co_3.

(P+) holds for J_2 and J_3, but not for Co_3. However it is not clear whether (P) holds for Co_3.

6.2. **Conjectures.** We feel confident that the following holds:

Conjecture 22. *Let B be a block with TI defect groups. Then (P+) holds for B.*

We speculate that, if $N_G(D)$ controls fusion in D, then (P) holds.

6.3. **Checking the conjectures.** The following is an important example, since it is the original example of a block with TI defect groups such that the conclusions of Broué's conjecture fail.

Example 23. Let $G = {}^2B_2(8)$ and $p = 2$. Let B be the principal block and $P \in \mathrm{Syl}_p(G)$.

The irreducible characters of $N_G(P)$ are $\theta_1, \ldots, \theta_{10}$, with degrees $1, 1, 1, 1, 1, 1, 1, 7, 14, 14$ respectively. These all lie in the principal block b. The irreducible characters of G are $\chi_1, \ldots, \chi_{11}$, with degrees $1, 14, 14, 35, 35, 35, 64, 65, 65, 65, 91$ respectively. All but χ_7 lie in B.

The characters of the projective indecomposable modules of $N_G(P)$ are $\Phi_i = \theta_i + \theta_8 + 2\theta_9 + 2\theta_{10}$, for $1 \leq i \leq 7$. The characters of the relevant projective indecomposable modules of B are

$\Gamma_2 = \chi_2 + \chi_3 + \chi_4 + 2\chi_5 + \chi_6 + 2\chi_8 + 2\chi_9 + 3\chi_{10} + 3\chi_{11}$,
$\Gamma_3 = \chi_2 + \chi_3 + \chi_4 + \chi_5 + 2\chi_6 + 3\chi_8 + 2\chi_9 + 2\chi_{10} + 3\chi_{11}$,
$\Gamma_4 = \chi_2 + \chi_3 + 2\chi_4 + \chi_5 + \chi_6 + 2\chi_8 + 3\chi_9 + 2\chi_{10} + 3\chi_{11}$,
$\Gamma_5 = \chi_5 + \chi_8 + \chi_{10} + \chi_{11}$,
$\Gamma_6 = \chi_6 + \chi_8 + \chi_9 + \chi_{11}$,
$\Gamma_7 = \chi_4 + \chi_9 + \chi_{10} + \chi_{11}$.

We give the restrictions of the χ_i below, along with constituents of the images in R_μ in $\mathrm{Irr}_0(b)$ (which we write as R^0_μ), where

$$\mu = \Phi - (\Gamma_4 - \Gamma_5 - \Gamma_6 - \Gamma_7)\Phi_2 - (\Gamma_2 - \Gamma_5 - \Gamma_6 - \Gamma_7)\Phi_3 - (\Gamma_3 - \Gamma_5 - \Gamma_6 - \Gamma_7)\Phi_4,$$

and Φ is as before.

χ_i	$\mathrm{Res}^G_{N_G(P)}(\chi_i)$	$R^0_\mu(\chi_i)$
χ_1	θ_1	θ_1
χ_2	$\theta_9 = \Phi_2 + \Phi_3 + \Phi_4 - \theta_2 - \theta_3 - \theta_4 - 3\theta_8 - 5\theta_9 - 6\theta_{10}$	$-\theta_2 - \theta_3 - \theta_4 - 3\theta_8$
χ_3	$\theta_{10} = \Phi_2 + \Phi_3 + \Phi_4 - \theta_2 - \theta_3 - \theta_4 - 3\theta_8 - 6\theta_9 - 5\theta_{10}$	$-\theta_2 - \theta_3 - \theta_4 - 3\theta_8$
χ_4	$\theta_8 + \theta_9 + \theta_{10} = -\theta_2 - \theta_9 - \theta_{10} + \Phi_2$	$-\theta_2$
χ_5	$\theta_8 + \theta_9 + \theta_{10} = -\theta_3 - \theta_9 - \theta_{10} + \Phi_3$	$-\theta_3$
χ_6	$\theta_8 + \theta_9 + \theta_{10} = -\theta_4 - \theta_9 - \theta_{10} + \Phi_4$	$-\theta_4$
χ_8	$\theta_4 + \theta_5 + \theta_8 + 2\theta_9 + 2\theta_{10} = \theta_5 + \Phi_4$	θ_5
χ_9	$\theta_2 + \theta_7 + \theta_8 + 2\theta_9 + 2\theta_{10} = \theta_7 + \Phi_2$	θ_7
χ_{10}	$\theta_3 + \theta_6 + \theta_8 + 2\theta_9 + 2\theta_{10} = \theta_6 + \Phi_3$	θ_6
χ_{11}	$\theta_8 + 3\theta_9 + 3\theta_{10}$	θ_8

We are able to verify that (P+) holds when B is the principal 2-block of any ${}^2B_2(2^{2m+1})$, and when B is any p-block of $SU_3(p^m)$. We are also able to prove the following:

Theorem 24. *Let p be 5 or a prime such that $3 \nmid (p+1)$. Let B be a block with TI non-abelian defect group D such that $|D| \leq p^5$. Then (P) holds for B.*

Remark 25. Further, (P+) holds if G is quasisimple.

Outline of proof: We use Clifford-theoretic methods similar to those in [2] to reduce to non-abelian simple groups, their automorphism groups and their covering groups. In [2] certain Morita equivalences are constructed to achieve a similar reduction, and we use the fact that Morita equivalences give perfect isometries.

It suffices to consider blocks with TI defect groups of central p'-extensions of automorphism groups of non-abelian simple groups.

These have been classified in [1], and it suffices to check the following cases:

(a) $D \cong 3^{1+2}_-$ and G is $\mathrm{Aut}({}^2G_2(3)') = {}^2G_2(3)$;

(b) $D \cong 5^{1+2}_+$ and G is $3.McL$, $\mathrm{Aut}(McL)$, $SU_3(5)$, $GU_3(5)$, $PSU_3(5).2$ or $PGU_3(5).2$, where the extension is by the unique field automorphism of order 2;

(c) $D \cong 5^{1+2}_-$ and G is $\mathrm{Aut}({}^2B_2(32))$;

(d) $D \cong p^{1+2}_+$ and G is $PSU_3(p)$ or $PSU_3(p).2$, where the extension is by the unique field automorphism of order 2 and $3 \nmid p+1$.

Finally, we have checked all of these cases.

6.4. **Other generalisations.** (I) The problem of generalising perfect isometries has also been studied by Jean-Baptiste Gramain in [8].

He uses the definition of a perfect isometry in Külshammer-Olsson-Robinson's paper [11] on generalised blocks of symmetric groups. The generalisation does not include Broué's conjecture on perfect isometries, but does give an isometry involving *all* irreducible characters. Gramain verifies the conjecture for various classes of blocks with TI defect groups. It is not clear whether a counterexample exists when the defect group is not TI.

(II) In the main part of this section, we have been attempting to generalise the idea of a perfect isometry by generalising from an isometry, whilst still considering perfect characters. We may also attempt to find isometries with strong structural properties so that we may generalise Broué's conjecture. In examples of blocks with TI defect groups tested, the following occurs:

There exists an isometry

$$I_\mu : CF(N_G(D), b, K) \to CF(G, B, K)$$

where μ satisfies

(*) Suppose $\mu(g, h) \neq 0$. Let g_p be the (uniquely defined) part of g, and h_p the p-part of h. Then either g_p and h_p are both conjugate to an element of the derived subgroup D', or neither are.

In the case that D is abelian this is one of the conditions for a perfect isometry.

However, there is little evidence for this phenomenon, and there is no analogue for the other condition for a perfect isometry. Also, there are counterexamples when D is not TI.

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