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2005

MIMS EPrint: 2005.50

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ISSN 1749-9097
Krichever-Novikov continuous basis for plane algebraic curves.

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December 2005
A continuous KN basis is a family of functions \( \Phi(P,u) \) on an algebraic curve \( V \), that is \( P \in V \), numbered by a continuous parameter \( u \). It is assumed that \( \Phi(P,u) \) is smooth in \( u \). KN basis is characterized by the property

\[
\Phi(P,u)\Phi(P,v) = L\Phi(P,u + v)
\]

where \( L \) is a linear differential operator in \( u \), not depending on the point \( P \). KN basis is the basis of Fourier-Laurent transform on the curve \( V \).

We shall start with basic definitions. Then we focus on the construction and the properties of differential operator \( L \). We demonstrate a connection of the multiplicative property of the KN basis of \( V \) with the addition law on the Jacobian of \( V \).
References
Buchstaber, V. M. and Leykin, D. V.


P.G.Grinevich and S.P.Novikov (Topological Charge of the real periodic finite-gap Sine-Gordon solutions: Dedicated to the Memory of J.K.Moser, Commun. Pure Appl. Math., 56(7), 2003, 956-978) proposed an analog of the Fourier-Laurent integral transform on the Riemann surfaces. They use a continuous analog of the discrete Krichever-Novikov bases, which were introduced and studied for the needs of the quantum string theory in the late 80-s.

"Let us consider the following set of data: a nonsingular Riemann surface $\Gamma$ of the genus $g$ with marked point $\infty \in \Gamma$ and selected local parameter near this point $z = k^{-1}$, $z(\infty) = 0$. We construct a function $\psi^0 = \psi(P, x)$ holomorphic on $\Gamma \setminus \infty$ and exponential near the infinite point:

$$\psi(z, x) = k^g \exp\{kx\}(1 + \sum_{i>0} \eta_i(x)k^{-i}).$$"
“Problem. Which multiplicative properties have the basic functions $\psi(P, x) = \psi_x(P)$ depending on $x$ as parameter?”

“Theorem. Let $x, y \neq 0$. There exists a differential operator $L$ in the variable $x$ of the order $g$ with coefficients dependent on the both variables $x, y$ such that the following Almost Graded Commutative Associative Ring Structure is defined by the formula

$$\psi(P, x)\psi(P, y) = L\psi(P, x + y)$$

$$L = \partial_x^g + [\eta(x) + \eta(y) - \eta(x + y)]\partial_x^{g-1} + \ldots$$”

When $g = 1$ one has: $L = \partial_x - (\zeta(x) + \zeta(y) - \zeta(x + y))$.

In [4] we construct the operator $L$ for curves of higher genera. The algorithm is based on reduction of this problem to an effective description of the addition law on Jacobi variety.
Group of covariant shifts

Def. Covariant shift

\[ W_{\alpha, \beta, c}(f(u)) := \exp \{ \pi i (\langle 2u + \alpha, \beta \rangle + c) \} f(u + \alpha), \]

where \( u, \alpha, \beta \in \mathbb{C}^g \); \( c \in \mathbb{C} \), \( i^2 = -1 \) and \( \langle \cdot, \cdot \rangle \) is Euclidean scalar product.

Group of covariant shifts := \( S \)

\[ W_{\alpha_2, \beta_2, c_2} W_{\alpha_1, \beta_1, c_1} = W_{\alpha_1 + \alpha_2, \beta_1 + \beta_2, c_1 + c_2 + \langle \beta_1, \alpha_2 \rangle - \langle \alpha_1, \beta_2 \rangle} \]
Representations of lattices

We use representations
\[ \mathbb{Z}^g \times \mathbb{Z}^g \rightarrow S \]
defined by the formula

\[ (n, n') \mapsto (\alpha, \beta, c) = ((n, n')\Omega, \phi(n, n')) , \]

Where:

1. \( \Omega \in \text{Sp}(2g, \mathbb{C}) : \)
   \[ \Omega^t J \Omega = J, \quad J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} . \]

2. \( \phi : \mathbb{Z}^{2g} \rightarrow \mathbb{Z}_2 \) is an Arf function:
   for all \( Q_1 \) and \( Q_2 \) in \( \mathbb{Z}^{2g} \) Arf identity holds
   \[ \phi(Q_1 + Q_2) = \phi(Q_1) + \phi(Q_2) + Q_1 J Q_2^t \mod 2; \]

To define an Arf function \( \phi \) fix
\( (\ell, \ell') \in \mathbb{Z}^{2g} \) then
\[ \phi(n, n') = \langle n + \ell, n' + \ell' \rangle - \langle \ell, \ell' \rangle \mod 2. \]
Explicit formula of a representation

Write $\Omega$ in block form

$$
\Omega = \begin{pmatrix}
\Omega_{1,1} & \Omega_{1,2} \\
\Omega_{2,1} & \Omega_{2,2}
\end{pmatrix};
$$

then we have a representation:

$$W_{\Omega}^{\ell,\ell'}(n, n') := W_{\alpha, \beta, c},$$

where

$$
\alpha = (n\Omega_{1,1} + n'\Omega_{2,1}) \\
\beta = (n\Omega_{1,2} + n'\Omega_{2,2}) \\
c = \langle n + \ell, n' + \ell' \rangle - \langle \ell, \ell' \rangle
$$
Construction of Sigma-function

Let \( \Omega \in \text{Sp}(2g, \mathbb{C}) \) and \( |\Omega_{1,1}| \neq 0 \).

Set

\[
G_\Omega(u) = \exp\left\{ -\frac{\pi i}{2} u \kappa u^t \right\}, \quad \kappa = \Omega^{-1}_{1,1} \Omega_{1,2}.
\]

**Def.** \( \sigma(u, \Omega; \ell, \ell') := \sum_{(n,n') \in \mathbb{Z}^{2g}} W^{\ell, \ell'}_{\Omega}(n, n') G_\Omega(u) \)

**Theorem.** \( \sigma(u, \Omega; \ell, \ell') \) is entire function of \( u \in \mathbb{C}^g \) iff \( \text{Im} \, \tau \) is positive definite.

where

\[
\tau = \Omega_{2,1} \Omega_{1,1}^{-1}, \quad \Rightarrow \quad \Omega_{2,2} = \tau \Omega_{1,1} \kappa + (\Omega_{1,1}^t)^{-1}
\]
Families of Sigma-functions

**Theorem.** Fix $\Omega$ and $(\ell, \ell')$.

If

$$W_{\Omega}^{\ell, \ell'}(k, k') F(u) = F(u), \quad \forall (k, k') \in \mathbb{Z}^{2g}$$

then

$$F(u) = \text{const} \cdot \sigma(u, \Omega; \ell, \ell').$$

Fix a map

$$\Omega : \mathbb{C}^q \to \text{Sp}(2g, \mathbb{C}), \quad |\Omega_{1,1}(\lambda)| \neq 0$$

then we have

**Family of Sigma-Functions:**

$$\sigma(u, \lambda; \ell, \ell') := \frac{\sigma(u, \Omega(\lambda); \ell, \ell')}{\sqrt{|\Omega_{1,1}(\lambda)|}}, \quad \lambda \in \mathbb{C}^q.$$
The Heat operators

Fix an arbitrary smooth vector field

\[
L = \sum_{j=1}^{q} v_j(\lambda) \frac{\partial}{\partial \lambda_j}.
\]

Introduce the second order operator

\[
H_{\Omega} = \sum_{r,s=1}^{g} \left( \alpha_{r,s} \partial_{r,s} + 2\alpha_{g+r,s} u_r \partial_s + \alpha_{g+r,g+s} u_{r} u_s \right),
\]

where \((\alpha_{r,s}) = L(\Omega^t)J\Omega\) and \(\partial_{r,s} = \frac{\partial^2}{\partial u_r \partial u_s}\)

\[
\delta_{\Omega}(\lambda) = \frac{1}{2} \text{sk-tr}(L(\Omega^t)J\Omega).
\]

\[
\text{sk-tr}M := \sum_{i=1}^{g} m_{i,g+1-i} \text{ for } M = (m_{i,j})
\]

Lemma. For any constant \(K \in \text{Sp}(2g, \mathbb{C})\)

\[
H_{\Omega} = H_{K\Omega}, \quad \delta_{\Omega}(\lambda) = \delta_{K\Omega}(\lambda).
\]
Lemma.

\[
(2L + H_\Omega + \delta_\Omega(\lambda)) \frac{G_\Omega(u)}{\sqrt{|\Omega_{1,1}(\lambda)|}} = 0.
\]

Lemma. For all \((k, k') \in \mathbb{Z}^{2g}\) the covariant shift \(W_{\Omega(\lambda)}(k, k')\) and the heat operator \(2L + H_\Omega + \delta_\Omega(\lambda)\) commute as the operators on the space of smooth functions of \(u\) and \(\lambda\).

Theorem. The family \(\sigma(u, \lambda; \ell, \ell')\) solves the equation

\[
(2L + H_\Omega + \delta_\Omega(\lambda))\sigma(u, \lambda; \ell, \ell') = 0
\]
Example. \( q = 1, \ g = 1, \ \ L = \partial_\chi \)

\[
\Omega(\chi) = \begin{pmatrix} \omega & \omega \kappa \\ \tau \omega & \tau \omega \kappa + 1/\omega \end{pmatrix}
\]

\( \omega \neq 0, \ \text{Im} \tau > 0, \ \kappa \) are smooth functions in \( \chi \).

\[
H_\Omega = a_{1,1} \partial_u^2 + 2a_{1,2}u \partial_u + a_{2,2}u^2
\]

\[
\delta_\Omega(\chi) = a_{1,2}
\]

where

\[
a_{1,1} = -\omega^2 \partial_\chi \tau,
\]

\[
a_{1,2} = \frac{\partial_\chi \omega}{\omega} - \omega^2 \kappa \partial_\chi \tau,
\]

\[
a_{2,2} = \partial_\chi \kappa - \omega^2 \kappa^2 \partial_\chi \tau + 2\kappa \frac{\partial_\chi \omega}{\omega}
\]
Example. $q = 1, g$ is arbitrary

$L = \partial_{\lambda}$

\[\Omega(\lambda) = \begin{pmatrix} \omega & \omega \kappa \\ \tau \omega & \tau \omega \kappa + (\omega^t)^{-1} \end{pmatrix}\]

where $|\omega| \neq 0$, $\tau^t = \tau$, Im$\tau$ is positive definite, $\kappa^t = \kappa$ are smooth $(g \times g)$-matrix functions in $\lambda$.

\[H_\Omega = (\partial_u)^t A_{1,1} \partial_u + 2u^t A_{2,1} \partial_u + u^t A_{2,2} u\]

\[\delta_\Omega(\lambda) = \text{tr} A_{2,1}\]

where

\[A_{1,1} = -\omega^t (\partial_{\lambda} \tau) \omega,\]

\[A_{2,1} = \omega^{-1} \partial_{\lambda} \omega + \kappa A_{1,1},\]

\[A_{2,2} = \partial_{\lambda} \kappa + A_{2,1} \kappa + \kappa A_{2,1}^t - \kappa A_{1,1} \kappa.\]
Abelian Sigma-functions and Heat Equations in Non-holonomic frame.

Let $s > n > 1$, $\gcd(n, s) = 1$.

$$f(x, y, \lambda) = y^n - x^s - \sum_{j=0}^{n-2} \sum_{i=0}^{s-2} \lambda_{ns-in-js} x^i y^j,$$

the number $m := \#\{\lambda_k | k < 0\}$ is called modality.

Set: $\lambda_k = 0, \quad k < 0$.

**Def.** The family of $(n, s)$-curves:

$$V = V_\lambda = \{(x, y) \in \mathbb{C}^2, \lambda \in \mathbb{C}^{2g-m} | f(x, y, \lambda) = 0\}$$

genus of a generic curve $V_\lambda$ is $g = \frac{(n-1)(s-1)}{2}$.

**Example.** Hyperelliptic curves are $(2, 2g+1)$-curves, $m = 0$:

$$f(x, y, \lambda) = y^2 - x^{2g+1} - \sum_{i=0}^{2g-1} \lambda_{2(2g-i+1)} x^i$$
The principal Arf function of \((n, s)\)-curve

**Def.** *Weierstrass sequence* \((w_1, \ldots, w_g)\) is the ordered set \(\mathbb{N} \backslash \mathcal{M}\), where \(\mathcal{M} = \{an + bs\}, a, b \in \mathbb{N} \cup 0\).

Assign \(w(\xi) = \sum_i \xi^w_i\) then

\[
w(\xi) = \frac{1}{1 - \xi} - \frac{1 - \xi^{ns}}{(1 - \xi^n)(1 - \xi^s)}.
\]

Now \(g = w(1) = \frac{(n-1)(s-1)}{2}\).

Let us define \((\pi_1, \ldots, \pi_g)\) by the formula

\[
\pi_k = w_{g-k+1} - (g - k), \quad k = 1, \ldots, g.
\]

\((w_g = 2g - 1 \text{ and } \pi_1 = g.\)

**Def.** *The principal Arf function* is defined by

\[
\ell = (1, \ldots, 1) \quad \text{and} \quad \ell' = (\pi_1, \ldots, \pi_g).
\]

Let us fix this value of \((\ell, \ell')\) for the rest of the talk.

**Example.** For hyperelliptic family \((n, s) = (2, 2g + 1)\)

\[
\ell' = (g, g - 1, \ldots, 1).
\]
Meromorphic Abelian integrals
We do the hyperelliptic case for simplicity. Set:
\[ du_j(x, y) = \frac{x^{j-1}dx}{2y}, \quad j = 1, \ldots, g, \]
\[ dr_j(x, y) = -\left(x\partial_x\pi_+\left(\frac{f(x, y, \lambda)}{x^{2j}}\right)\right)du_j(x, y), \]
where \( \pi_+(\cdot) \) truncates the negative powers of \( x \).

**Def.** Universal Abelian cover of \( V \) is the space \( W \) of pairs \( ((x, y); [\gamma]) \), where \( (x, y) \in V; [\gamma] \) is an equivalence class of paths from \( \infty \in V \) to \( (x, y) \).

We say that \( \gamma_1 \) and \( \gamma_2 \) are in \( [\gamma] \) if the contour \( \gamma_1 \circ \gamma_2^{-1} \) is homologous to zero.

**Def.** Abelian maps: \( A : W \to \mathbb{C}^g \), \( A^* : W \to \mathbb{C}^g \),
\[ A_j(x, y; [\gamma]) = \int_{\gamma} du_j(x, y), \quad j = 1, \ldots, g. \]
\[ A^*_j(x, y; [\gamma]) = \int_{\gamma} dr_j(x, y). \]
The period map $\Omega : \mathbb{C}^{2g-m} \to \text{Sp}(2g, \mathbb{C})$

Let the contours $\gamma_1, \ldots, \gamma_{2g}$ give a basis in $H_1(V, \mathbb{Z})$, such that the intersection matrix is $J$:

$$J_{a,b} = \gamma_a \circ \gamma_b = \text{sign}(b-a) \delta_{g,|b-a|}, \quad a, b = 1, \ldots, 2g.$$ 

Set for $i, j = 1, \ldots, g$, 

$$\omega_{i,j} = \frac{1}{2} \oint_{\gamma_j} du_i(x, y), \quad \omega'_{i,j} = \frac{1}{2} \oint_{\gamma_{g+j}} du_i(x, y),$$

$$\eta_{i,j} = -\frac{1}{2} \oint_{\gamma_j} dr_i(x, y), \quad \eta'_{i,j} = -\frac{1}{2} \oint_{\gamma_{g+j}} dr_i(x, y),$$

then for $\Omega(\lambda) = \begin{pmatrix} \omega & \eta \\ \omega' & \eta' \end{pmatrix}$, where $\omega = (\omega_{i,j})$, etc.,

we have Legendre relation

$$\Omega(\lambda) J \Omega(\lambda)^t = \frac{\pi i}{2} J$$

Note: For nonsingular $V_\lambda$ the choice of basis contours provides $|\omega| \neq 0$ and positive definiteness of $\text{Im} \omega' \omega^{-1}$.

Note: $K = \ell \omega + \ell' \omega'$ is the vector of Riemann constants.
Frame tangent to Discriminant and Heat operators

Denote by $\Delta(\lambda)$ the discriminant of $f(x, y, \lambda)$:

$$\Delta(\lambda) = 0 \iff \exists (x, y) \ f = f_x = f_y = 0$$

Consider the space $\mathcal{T}$ of polynomial vector fields tangent to

$$\{\lambda \in \mathbb{C}^{2g-m} | \Delta(\lambda) = 0\}.$$

$L \in \mathcal{T}$ implies

$$L \Delta(\lambda) = \varphi(\lambda) \Delta(\lambda), \quad \varphi(\lambda) \in \mathbb{C}[\lambda].$$

$\mathcal{T}$ has the basis $\{L_1, \ldots, L_{2g}\}$ over $\mathbb{C}[\lambda]$. The basis gives a non-holonomic frame which defines a nontrivial polynomial Lie algebra (the structure is described in [1]).

We use the basis vector fields $\{L_1, \ldots, L_{2g}\}$ and the period map $\Omega(\lambda)$ to construct $2g$ heat operators.
Examples of basis fields $L_i$

$g = 1$. We have $\Delta(\lambda) = 4\lambda_4^3 + 27\lambda_6^2$

$$L_0 = 4\lambda_4 \partial_4 + 6\lambda_6 \partial_6,$$

$$L_2 = 6\lambda_6 \partial_4 - \frac{4}{3} \lambda_4^2 \partial_6.$$

Here $\partial_k = \frac{\partial}{\partial \lambda_k}$, $\deg \lambda_k = k$. Then $\deg L_j = j$.

$g = 2$. The symmetric matrix $T$ transforms the standard fields $\partial_4, \partial_6, \partial_8, \partial_{10}$ to the basis fields $L_0, L_2, L_4, L_6$

\[
T = \begin{pmatrix}
4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\
* & \frac{40\lambda_8 - 12\lambda_4^2}{5} & \frac{50\lambda_{10} - 8\lambda_4 \lambda_6}{5} & \frac{-4\lambda_4 \lambda_8}{5} \\
* & * & \frac{20\lambda_4 \lambda_8 - 12\lambda_6^2}{5} & \frac{30\lambda_4 \lambda_{10} - 6\lambda_6 \lambda_8}{5} \\
* & * & * & \frac{4\lambda_6 \lambda_{10} - 8\lambda_8^2}{5}
\end{pmatrix}
\]

Note: $\Delta(\lambda) = |T|$. The matrix $T$ plays an important role

in Singularity Theory as the convolution matrix.
Examples of operators $H_i$

Here $D_i = \frac{\partial}{\partial u_i}$; $\deg u_i = -i$ $\deg H_i = i$.

$g = 1$.

$$H_0 = u_1 D_1 - 1$$
$$6H_2 = 3D_1^2 - \lambda_4 u_1^2$$

$g = 2$.

$$H_0 = u_1 D_1 + 3u_3 D_3 - 3$$
$$10H_2 = 5D_1^2 + 10u_1 D_3 - 8\lambda_4 u_3 D_1 - 3\lambda_4 u_1^2 + (15\lambda_8 - 4\lambda_4^2) u_3^2$$

$$5H_4 = 5D_1 D_3 + 5\lambda_4 u_3 D_3 - 6\lambda_6 u_3 D_1 - 5\lambda_4 - \lambda_6 u_1^2 + 5\lambda_8 u_1 u_3 + 3(5\lambda_{10} - \lambda_4 \lambda_6) u_3^2$$

$$10H_6 = 5D_3^2 - 6\lambda_8 u_3 D_1 - 5\lambda_6 - \lambda_8 u_1^2 + 20\lambda_{10} u_1 u_3 - 3\lambda_4 \lambda_8 u_3^2$$
Abelian Sigma-function.
Our construction gives the following result. 

**Theorem.** The heat equations

\[ L_i \sigma(u, \lambda) = H_i \sigma(u, \lambda), \]

for \( i \in \{nk + js\}, 0 \leq j < n - 1, 0 \leq k < s - 1, \) uniquely define the Abelian \( \sigma \)-function of \((n, s)\)-curve.

(1) It has the translation property

\[ \sigma(u + A[\chi]) = \sigma(u) \exp \left\{ -\langle A^*[\chi], u + \frac{1}{2}A[\chi] \rangle + \pi i (\langle k + \ell, k' + \ell' \rangle - \langle \ell, \ell' \rangle) \right\}, \]

where \([\chi] = \sum_{j=1}^{g} (k_j \gamma_j + k'_j \gamma_{g+j}).\)

(2) It is an entire function on \( \mathbb{C}^g \times \mathbb{C}^{2g-m}. \)

Its power series in \( u \) and \( \lambda \) has rational coefficients.

(3) The grading \( \deg x = n, \deg y = s \) and \( \deg \lambda_k = k \) gives \( \deg f(x, y, \lambda) = ns \) and the grading of \( u \) s.t.

\[ \deg \sigma(u, \lambda) = - \sum_{j=1}^{g} \ell'_{j} = -\frac{(n^2 - 1)(s^2 - 1)}{24}. \]
Krichever-Novikov continuous basis

**Def.**

\[
\psi(x, y; [\gamma]) := \exp \left\{ - \int_{[\gamma]} \langle A^*((x', y'), [\gamma']), dA(x', y') \rangle \right\}
\]

\(\psi(x, y; [\gamma])\) is the unique entire function \(W \rightarrow \mathbb{C}\) with:

1. Single essentially singular point \(\infty \in V\)
   \[
   \psi \sim \xi^g (1 + O(\xi)).
   \]
2. No zeros and poles in \(V \setminus \infty\).

\(\xi\) is local parameter at \(\infty\), \(\deg \xi = -1\).

**Def.**

\[
\Psi(u, (x, y)) := \frac{\sigma(A(x, y; [\gamma]) - u)}{\psi(x, y; [\gamma]) \sigma(u)} \exp \left\langle A^*(x, y; [\gamma]), u \right\rangle
\]

\(\Psi\) is single-valued function \(\mathbb{C}^g \times V \rightarrow \mathbb{C}\).

If \(g = 1\), \(\Psi(u, (\varphi(\xi), \varphi'(\xi))) = \frac{\sigma(\xi - u)}{\sigma(\xi) \sigma(u)} \exp \{u \zeta(\xi)\}\).

This gives a solution of Lamè equation

\[
\partial_u^2 \Psi(u, (x, y)) - 2\varphi(u) \Psi(u, (x, y)) = x \Psi(u, (x, y))
\]
Fix $u \in \mathbb{C}^g$. Then $\Psi(u,(x,y))$ is the **unique single-valued function** on $V$ with:

(1) $g$ zeros on $V$ at $A^{-1}(u)$.
(2) Single essentially singular point $\infty \in V$

$$\Psi \sim \xi^{-g} \exp\{p(\xi^{-1};u,\lambda)\}(1 + O(\xi)),$$

where $p(t;u,\lambda) = p_1(u,\lambda)t + \cdots + p_{2g-1}(u,\lambda)t^{2g-1}$ is fixed by the choice of $f(x,y,\lambda)$.

$p_k(u,\lambda)$ is homogeneous polynomial $\deg p_k(u,\lambda) = -k$. In general case

$$p_1(u_1,0,\ldots,0,\lambda) = u_1,$$

$$p_j(u_1,0,\ldots,0,\lambda) = 0, \quad j > 1.$$

Note: $\Psi(u,(x,y))$ is the **Baker-Akhiezer Function** corresponding to the degenerate set of Krichever data.
**Example.** For hyperelliptic curves

\[ p(t; u, 0) = \sum_{i=1}^{g} u_{2i-1} t^{2i-1}. \]

- \( g = 1, \quad p(t; u, \lambda) = u_1 t, \)
- \( g = 2, \quad p(t; u, \lambda) = u_1 t + u_3 t^3, \)
- \( g = 3, \quad p(t; u, \lambda) = \left( u_1 + \frac{1}{2} \lambda u_5 \right) t + u_3 t^3 + u_5 t^5. \)

Also, the equation

\[ \partial_{u_1}^2 \Phi - 2 \varphi_{1,1}(u) \Phi = x \Phi, \]

where

\[ \varphi_{1,1}(u) = -\frac{\partial^2}{\partial u_1^2} \log \sigma(u) \]

has solutions

\[ \Phi_{\pm} = \Psi(\pm u, (x, y)), \]

\[ \begin{vmatrix} \partial_{u_1} \Phi & \partial_{u_1} \Phi_- \\ \Phi_+ & \Phi_- \end{vmatrix} = 2y. \]
In our notation the \( \psi(P, t) \) of Grinevich and Novikov is \( \Psi(te_1, (x, y)) \), where \( e_1 \) is the 1-st ort in \( \mathbb{C}^g \).

The relation defining the multiplicative structure of the base \( \{\Psi(te_1, (x, y))\} \) is a particular case of the relation

\[
\psi(u, (x, y))\psi(v, (x, y)) = L\psi(w, (x, y)) \bigg|_{w=u+v},
\]

where \( u, v \in \mathbb{C}^g \) and

\[
L = \sum_{j=0}^{g} a_j(u, v, w) \frac{\partial^{g-i}}{\partial w_1^{g-i}}, \quad \text{deg } L = g.
\]
We define the family of functions on $V$ with parameter $w \in \mathbb{C}^g$

$$G_k(w)(x, y) = \frac{\partial_{w_1}^k \psi(w, (x, y))}{\psi(w, (x, y))}, \quad k = 0, 1, \ldots$$

Each $G_k(w)(x, y)$ is a rational function on $V$. It has $g + k$ poles in $\{k \infty, A^{-1}(w)\}$. Its coefficients are Abelian functions on the Jacobi variety of $V$.

$$G_0(w)(x, y) = 1,$$

$$G_1(w)(x, y) = -(\zeta_1(A(x, y; [\gamma]) - w) + \zeta_1(w) + + \langle A^*(x, y; [\gamma]), e_1 \rangle),$$

where $\zeta_1(w) = \partial_{w_1} \log \sigma(w)$.

For $k > 1$ we have the recurrence

$$G_{k+1}(w)(x, y) = \partial_{w_1} G_k(w)(x, y) + G_1(w)(x, y) G_k(w)(x, y).$$

We express $G_{k+1}(w)(x, y)$ as rational functions of $(x, y)$. 
Example. In the hyperelliptic case

\[
G_1^w(x, y) = \frac{1}{2} \left( 2y + \sum_{i=1}^{g} \wp_{1,1}(g-i)(w)x^{g-i} \right),
\]

\[
G_2^w(x, y) = x + 2\wp_{1,1}(w)
\]

from the recurrence we have for \( k > 2 \)

\[
G_k^w(x, y) = a_k + b_k G_1^w(x, y)
\]

\[
a_{k+1} = \partial w_1 a_k + (x + 2\wp_{1,1}(w))b_k,
\]

\[
b_{k+1} = \partial w_1 b_k + a_k,
\]

Clearly, \( a_k \) and \( b_k \) are polynomials in \( x \).

\[
\wp_{i,j}(w) = -\frac{\partial^2 \log \sigma(w)}{\partial w_i \partial w_j},
\]

\[
\wp_{i,j,k}(w) = -\frac{\partial^3 \log \sigma(w)}{\partial w_i \partial w_j \partial w_k}
\]

where \( i, j, k \) are any odd integers between 0 and \( 2g \).
We prove that for all $u, v, w \in \mathbb{C}^g$

\[
\Psi(u, (x, y))\Psi(v, (x, y))\Psi(-u - v, (x, y)) = R_{3g}^{(u,v)}(x, y),
\]

\[
\Psi(w, (x, y))\Psi(-w, (x, y)) = R_{2g}^{(w)}(x, y).
\]

As function on $V$

$R_{3g}^{(u,v)}(x, y)$ has

3g-tuple pole at $\infty$ and 3g zeros at

\[\{A^{-1}(u), A^{-1}(v), A^{-1}(-u - v)\}\].

$R_{2g}^{(w)}(x, y)$ has

2g-tuple pole at $\infty$ and 2g zeros at

\[\{A^{-1}(w), A^{-1}(-w)\}\].

The functions $R_{3g}^{(u,v)}(x, y)$ and $R_{2g}^{(w)}(x, y)$

define addition and inverse operations on

Sym$^g(V)$. 
**Theorem.** The operator $L$ is defined by the equality

$$\frac{R_{3g}^{(u,v)}(x, y)}{R_{2g}^{(u+v)}(x, y)} = \sum_{i=0}^{g} a_i(u, v, u+v)G_i^{(u+v)}(x, y).$$

This reduces the problem to comparing the coefficients at monomials, after cancelation of the common denominator on both sides.

**Example.** For hyperelliptic curves we have

$$\alpha_0(u, v, w) = 1,$$

$$\alpha_1(u, v, w) = -\zeta_1(u) - \zeta_1(v) + \zeta_1(w),$$

$$2\alpha_2(u, v, w) = -\varphi_{1,1}(u) - \varphi_{1,1}(v) - 3\varphi_{1,1}(w) + \alpha_1(u, v, w)^2), \text{ etc.}$$