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O-MINIMAL STRUCTURES

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1. INTRODUCTION AND MOTIVATION

The notion of an o-minimal expansion of the ordered field of real numbers was invented by L van den Dries [vdD1] as a framework for investigating the model theory of the real exponential function $exp : \mathbb{R} \to \mathbb{R} : x \to e^x$, and thereby settle an old problem of Tarski. More on this later, but for the moment it is best motivated as being a candidate for Grothendieck's idea of "tame topology" as expounded in his Esquisse d'un Programme [Gr]. It seems to me that such a candidate should satisfy (at least) the following criteria.

(A) It should be a framework that is flexible enough to carry out many geometrical and topological constructions on real functions and on subsets of real euclidean spaces.

(B) But at the same time it should have built in restrictions so that we are *a priori* guaranteed that pathological phenomena can never arise. In particular, there should be a meaningful notion of dimension for all sets under consideration and any that can be constructed from these by use of the operations allowed under (A).

(C) One must be able to prove finiteness theorems that are uniform over fibred collections.

None of the standard restrictions on functions that arise in elementary real analysis satisfy both (A) and (B). For example, there exists a *continuous* function $G : (0, 1) \to (0, 1)^2$ which is surjective, thereby destroying any hope of a dimension theory for a framework that admits all continuous functions. Restricting to the smooth (i.e. C^∞)

environment fares no better. For every closed subset of any euclidean space, in particular the subset *graph*(G) of $\mathbb{R}^3$, is the set of zeros of some smooth function. So by the use of a few simple constructions that we would certainly wish to allow under (A), we soon arrive at dimension-destroying phenomena. The same is even true (though this is harder to prove) if we start from just those smooth functions that are everywhere real analytic (i.e. equal the sum of their Taylor series on a neighbourhood of every point), although, as we shall see, this class of functions is *locally* well-behaved and as such can serve as a model for the three criteria above.

Rather than enumerate analytic conditions on sets and functions sufficient to guarantee the criteria (A), (B) and (C) however, we shall give one succinct axiom, the o-minimality axiom, which *implies* them. Of course, this is a rather open-ended (and currently flourishing) project because of the large number of questions that one can ask under (C). One must also provide concrete examples of collections of sets and functions that satisfy the axiom and this too is an active area of research. In this talk I shall survey both aspects of the theory.

Our formulation of the o-minimality axiom makes use of definability theory from mathematical logic. We begin with a collection $\mathcal{F}$ of real valued functions of real variables (not necessarily all of the same number of arguments). We consider the ordered field structure on $\mathbb{R}$ augmented by the functions in $\mathcal{F}$. This gives us a *first-order structure* (or *model*) $\mathbb{R}_{\mathcal{F}} := \langle \mathbb{R}; +, \cdot, -, <, \mathcal{F} \rangle$, and we denote the corresponding first-order logical language by $L(\mathcal{F})$. We then call the structure $\mathbb{R}_{\mathcal{F}}$ *o-minimal* if whenever $\phi(x)$ is an $L(\mathcal{F})$-formula (with parameters) then the subset of $\mathbb{R}$ defined by $\phi(x)$ is a finite union of open intervals and points (i.e. it is the union of finitely many connected sets). I shall elucidate what is meant by an $L(\mathcal{F})$*-formula* and by the subset of $\mathbb{R}$ (and, more generally, of $\mathbb{R}^n$) *defined* by such a formula in the next two sections. However, I should emphasize at this stage that such a formula not only defines a subset , denoted $\phi(\mathbb{R}_{\mathcal{F}})$, of $\mathbb{R}^n$, but also a subset $\phi(\mathcal{R})$ of $\mathcal{R}^n$ where $\mathcal{R}$ is any ordered ring augmented by a collection of functions, $\mathcal{F}^*$ say, such that $\mathcal{F}$ and $\mathcal{F}^*$ are in correspondence via a bijection that preserves the number of places (arity) of the functions. One can, and should, define the notion o-minimality for such structures $\langle \mathcal{R}; \mathcal{F}^* \rangle$ and it was at (rather more than) this level of generality that the true foundations of the subject were laid by Pillay and Steinhorn in [P-S], shortly after van den Dries' work on the real field. Indeed, it turned out that the solution to Tarski's problem on the real exponential function (the case $\mathcal{F} = \{exp\}$ in the above notation) relied heavily on the Pillay-Steinhorn theory of o-minimality for structures based on ordered fields other than the reals. This having been said, I shall concentrate in this lecture on the real case, alluding only occasionally to the more general situation, and leave the reader to adapt the definitions and theorems to the setting of o-minimal expansions of arbitrary ordered fields.

2. THE SEMI-ALGEBRAIC CASE

2.1. Formulas and the sets they define

In this section I shall describe the logical formalism for the case $\mathcal{F} = \emptyset$, i.e. where the structure is just that of the ordered field of real numbers $\bar{\mathbb{R}} := \langle \mathbb{R}; <, +, \cdot, -, 0, 1\rangle$. The corresponding language $L(\emptyset)$, which we denote from now on by just L, consists of formal symbols for variables X_i (for $i = 1, 2, \ldots$) together with some formal system of notation for polynomials in these variables (with integer coefficients). It must also contain a symbol for the ordering and some *logical* symbols as will be explained in (v) and (vi) below.

The fact that we are concentrating on one particular structure here allows us to make several shortcuts in the description of logical concepts. In particular one can, in fact, dispense with the formal language and the notion of L-formula altogether and simply specify the definable sets by the following inductive procedure:

(i) For $p(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots X_n]$, the sets $\{\mathbf{a} \in \mathbb{R}^n : p(\mathbf{a}) = 0\}$ and $\{\mathbf{a} \in \mathbb{R}^n : p(\mathbf{a}) > 0\}$ are both definable subsets of $\mathbb{R}^n$;

(ii) If A and B are definable subsets of $\mathbb{R}^n$ then so are $A \cap B$, $A \cup B$ and $\mathbb{R}^n \setminus A$;

(iii) If A is a definable subset of $\mathbb{R}^n$, then $\pi_n[A]$ is a definable subset of $\mathbb{R}^{n-1}$, where $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1} : \langle x_1, \ldots, x_n\rangle \mapsto \langle x_1, \ldots, x_{n-1}\rangle$ is the projection map onto the first n-1 coordinates.

It is, however, very difficult, even in our present limited situation, to do any model theory without the notion of L-formula, and almost impossible to give examples. So I give the definition. An *L-formula* is a formal string of symbols that codes the inductive construction of a definable set as follows:

(iv) Expressions of the form $p(X_1, \ldots, X_n) = 0$ and $p(X_1, \ldots, X_n) > 0$ (for $p(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots X_n]$) are L-formulas. These are known as *atomic L-formulas*. If ϕ is such a formula then $\phi(\bar{\mathbb{R}})$ denotes the corresponding subset of $\mathbb{R}^n$ as given in (i). We say that $\phi(\bar{\mathbb{R}})$ is the subset of $\mathbb{R}^n$ *defined by the L-formula ϕ.*

(v) If ϕ, ψ are L-formulas, with $\phi(\bar{\mathbb{R}}) = A \subseteq \mathbb{R}^n$ and $\psi(\bar{\mathbb{R}}) = B \subseteq \mathbb{R}^n$, then the expressions $(\phi \wedge \psi)$, $(\phi \vee \psi)$ and $\neg\phi$ are also L-formulas. Then $(\phi \wedge \psi)(\bar{\mathbb{R}}) := A \cap B$, $(\phi \vee \psi)(\bar{\mathbb{R}}) := A \cup B$ and $\neg\phi(\bar{\mathbb{R}}) := \mathbb{R}^n \setminus A$.

(vi) If ϕ is an L-formula, with $\phi(\bar{\mathbb{R}}) = A \subseteq \mathbb{R}^n$, then the expression $\exists X_n \phi$ is also an L-formula and we set $\exists X_n \phi(\bar{\mathbb{R}}) := \pi_n[A] \subseteq \mathbb{R}^{n-1}$. (The symbol "$\exists$" is called the *existential quantifier.*)

2.2. Examples

(1) Let α be the expression $\exists X_3(X_3^2+X_1\cdot X_3+X_2=0)$. Then α is an L-formula and $\alpha(\bar{\mathbb{R}})$ consists of all pairs $\langle b,c\rangle\in\mathbb{R}^2$ such that the quadratic equation $x^2+bx+c=0$ has a real solution. (Actually, to be perfectly precise, α is not an L-formula because the parentheses should not be there. But I prefer to err on the side of clarity.)

(2) Let β be the expression $\exists X_5\exists X_6\exists X_7\exists X_8(((X_1\cdot X_5+X_2\cdot X_7-1=0\ \wedge\ X_1\cdot X_6+X_2\cdot X_8=0)\wedge X_3\cdot X_5+X_4\cdot X_7=0)\wedge X_3\cdot X_6+X_4\cdot X_8-1=0)$. Then β is an L-formula and $\beta(\bar{\mathbb{R}})$ consists of all quadruples $\langle a,b,c,d\rangle\in\mathbb{R}^4$ such that the matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ has an inverse.

The idea behind the notion of L-formula should now be clear. Let ϕ be an L-formula with $\phi(\bar{\mathbb{R}})\subseteq\mathbb{R}^n$ and let $a_1,\ldots,a_n\in\mathbb{R}^n$. Then "reading" ϕ using the dictionary $\wedge$ ="and", $\vee$ = "or", $\neg$ ="not", $\exists X_i$ = "there exists $X_i\in\mathbb{R}$ such that" and replacing the variables $X_1,\ldots,X_n$ by $a_1,\ldots,a_n$, we arrive at a statement of (mathematical) English expressing "$\langle a_1,\ldots,a_n\rangle\in\phi(\bar{\mathbb{R}})$". We therefore often write ϕ as $\phi(X_1,\ldots,X_n)$ to emphasize the fact that it should be read as "the n-tuple $\langle X_1,\ldots,X_n\rangle$ has the property expressed by ϕ".

It is in this way that the formula ϕ also defines a subset $\phi(\mathcal{R})$ of $\mathcal{R}^n$ for any ordered ring $\mathcal{R}$. More rigourously, just follow those construction steps 2.1(i)-2.1(iii) coded by ϕ, but replace $\mathbb{R}$ everywhere by (the underlying set of the) ring $\mathcal{R}$ and interpret the formal polynomials $p(X_1,\ldots,X_n)$ in 2.1(i) by using the addition and multiplication of the ring $\mathcal{R}$. The careful reader might now question whether such a set $\phi(\mathcal{R})$ is well defined, that is, whether a given formula ϕ *uniquely* determines such a construction procedure. It does, and the proof of this result (known as the Unique Readability Theorem) and of many other syntactic properties of formulas (such as the conditions under which variables may be permuted or expressions substituted for variables) occupy endless pages in many introductory texts on logic. The student encountering such texts for the first time needs to be patient: very little happens for a long time. In this lecture I am, of course, neglecting such tiresome details. My aim is to convey, as quickly and efficiently as possible, the role played by logical definablility in the foundations of o-minimality and especially in how the three criteria for a tame topology set out in section 1 are justified. So I proceed by presenting more examples. They are intended to familiarise the reader with the flexibility of logical definability and thereby justify criterion (A) for the class of L-definable sets and functions.

2.3. More examples (and exercises)

(1) Fix $n\geq 1$. Then the set of $\langle a_1,\ldots,a_n\rangle\in\mathbb{R}^n$ such that the polynomial $x^n+a_1x^{n-1}+\cdots+a_n$ is positive definite, is L-definable. The defining formula is $\forall X_{n+1}(X_{n+1}^n+X_1\cdot X_{n+1}^{n-1}+\cdots+X_n>0)$, where the *universal* quantifier "$\forall X_{n+1}$" is an abbreviation for "$\neg\exists\neg X_{n+1}$" and therefore may be read as "for all X_{n+1}". Another

abbreviation that will prove useful in the sequel is the symbol $\rightarrow$ (read as "implies"): $(\phi \rightarrow \psi)$ is an abbreviation for $(\neg\phi \vee \psi)$.

(2) The class of L-definable sets is closed under many standard topological operations. For example, if $S \subseteq \mathbb{R}^n$ is L-definable then so is $\bar{S}$, the closure of S in the ambient space $\mathbb{R}^n$. To see this, let ϕ be an L-formula defining the set S, i.e. $S = \phi(\mathbb{R})$. We must find an L-formula $\psi(X_1, \ldots, X_n)$ so that $\psi(\bar{\mathbb{R}}) = \bar{S}$. To do this we simply use the naive definition of closure: $\mathbf{a} \in \bar{S}$ if and only if $\forall\epsilon(\epsilon > 0 \rightarrow \exists\mathbf{y}(\mathbf{y} \in S \ \wedge \ ||\mathbf{y} - \mathbf{a}||^2 < \epsilon)$.

To turn this into an L-formula we must perform some of the syntactic operations referred to above. We increase the subscripts of *all* the variables in ϕ by $n+1$. The result of doing this may be written, by the convention described above, as $\phi(X_{n+2}, \ldots, X_{2n+1})$. (So this formula expresses "$\langle X_{n+2}, \ldots, X_{2n+1}\rangle \in S$".) The required formula ψ is: $\forall X_{n+1}(X_{n+1} > 0 \rightarrow \exists X_{n+2}, \ldots, \exists X_{2n+1}(\phi(X_{n+2}, \ldots, X_{2n+1}) \wedge (X_1 - X_{n+2})^2 < X_{n+1} \wedge \ldots \wedge (X_n - X_{2n+1})^2 < X_{n+1}))$. (For polynomials p,q we often prefer to write $p < q$ for $q - p > 0$.) I leave it to the reader to translate the usual definitions of, say, the interior of S and of the boundary of S by use of the L-dictionary and hence show that these sets are also L-definable.

(3) We say that a function $F : S \rightarrow \mathbb{R}$, where S is a non-empty subset of $\mathbb{R}^n$, is L-definable if its graph $\{\langle\mathbf{x}, y\rangle \in S \times \mathbb{R} : F(\mathbf{x}) = y\}$ is an L-definable subset of $\mathbb{R}^{n+1}$. Suppose that S is open in $\mathbb{R}^n$. Then by translating the usual $\epsilon - \delta$ definition one sees that the set X of points in S at which F is differentiable is an L-definable set and that each partial derivative of F is an L-definable function on X. We may repeat this process (on F restricted to the interior of X) to see that all partial derivatives of F, of all orders, are L-definable functions (on their appropriate domains which, as we shall see later, are always non-empty).

2.4. The o-minimality of $\bar{\mathbb{R}}$

So far we have only introduced the *parameter-free* concept of an L-formula and of the sets and functions that they define. Now consider a definable subset, A say, of $\mathbb{R}^{n+m}$. For each $\mathbf{a} \in \mathbb{R}^m$ let $A_\mathbf{a}$ denote the fibre $\{\mathbf{b} \in \mathbb{R}^\mathbf{n} : \langle\mathbf{b}, \mathbf{a}\rangle \in \mathbf{A}\}$. We say that the subset $A_\mathbf{a}$ of $\mathbb{R}^n$ is *definable with parameters* (or, with parameters $\mathbf{a}$ if we need to be precise). We shall also call the collection $\{A_\mathbf{a} : \mathbf{a} \in \mathbb{R}^m\}$ a *definable collection* of subsets of $\mathbb{R}^n$. It is rather easy to see that if we change, in 2.1(i) and 2.1(iv), the polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$ to $\mathbb{R}[X_1, \ldots, X_n]$, then the resulting class of formulas (called L-formulas *with parameters*) define exactly those sets definable with parameters in the sense described above. We have now made precise the definition of o-minimality (for the structure $\bar{\mathbb{R}}$) stated in section 1: every subset of $\mathbb{R}$ definable by an L-formula with parameters is a finite union of open intervals and points. The fact that this is indeed the case follows from Tarski's famous quantifier elimination theorem [T] (also known as the Tarski-Seidenberg algorithm).

THEOREM 2.1. — *Every L-formula is equivalent (over $\bar{\mathbb{R}}$) to one containing no occurrences of quantifiers. That is, for each n, every definable subset of $\mathbb{R}^n$ can be obtained from sets of type 2.1(i) by applications of the boolean operations 2.1(ii).*

Subsets of $\mathbb{R}^n$ that can be expressed as boolean combinations of zero-sets and positivity sets of real polynomials are called *semi-algebraic* and their study, *semi-algebraic geometry.* The essential point of Tarski's theorem is that the class of all semi-algebraic sets is closed under projection maps. The theorem can, in some cases be seen as a manifestation of facts of elementary algebra. For example, the formulas α, β of 2.2 are equivalent to $(X_1^2 - 4 \cdot X_2 > 0 \ \vee \ X_1^2 - 4 \cdot X_2 = 0)$ and $\neg(X_1 \cdot X_3 - X_2 \cdot X_4 = 0)$ respectively. To see that the sets of examples 2.3(ii) are semi-algebraic (for semi-algebraic S) is, however, more challenging. (One should perhaps mention here that there exist polynomials $p(\mathbf{X})$, irreducible ones even, such that the closure of the positivity set $\{\mathbf{a} \in \mathbb{R}^n : p(\mathbf{a}) > 0\}$ of $p(\mathbf{X})$ is definitely not the set $\{\mathbf{a} \in \mathbb{R}^n : p(\mathbf{a}) \geq 0\}$.) However, such considerations are not relevant to our present concerns. Our only interest here in Tarski's theorem is that it implies the o-minimality of the structure $\bar{\mathbb{R}}$. For clearly the zero-set and the positivity set of any univariate, real polynomial are both finite unions of open intervals and points, and the class of such sets is closed under the boolean operations. Hence *all* L-definable subsets of $\mathbb{R}$ have this form. (Tarski himself explicitly observed this consequence of his theorem but he did not pursue it.) Van den Dries' key insight was that the most fruitful way to generalize semi-algebraic geometry to transcendental analytic situations was not to focus on quantifier elimination theorems (which, he observed, rarely hold), but rather on the o-minimality axiom for sets definable by *arbitrary* formulas.

3. THE GENERAL CASE

We return to the situation of section 1: $\mathcal{F}$ is some collection of real functions, $\mathbb{R}_{\mathcal{F}}$ denotes the structure $\langle \mathbb{R}; +, \cdot, -, <, \mathcal{F} \rangle$ and $L(\mathcal{F})$ its language. The definition of an $L(\mathcal{F})$-*definable* subset of $\mathbb{R}^n$ follows the same inductive procedure as in the semi-algebraic case, the only difference being that in 2.1(i) we replace $\mathbb{Z}[X_1, \ldots, X_n]$ by the compositional closure of the collection of functions $\{+, \cdot, -, 0, 1, X_1, \ldots, X_n, \mathcal{F}\}$. We proceed similarly for the definition of an $L(\mathcal{F})$-formula. (Strictly speaking we should introduce a fixed *symbol* F_f (of specified number of places) for each function $f \in \mathcal{F}$ and a system of notation for the compositional closure.) Also, the notion of a set being $L(\mathcal{F})$-definable *with parameters*, and of a definable *collection* of sets, is just as in 2.4 except that we replace $\mathbb{R}[X_1, \ldots, X_n]$ by the compositional closure of $\{+, \cdot, -, \mathbb{R}, X_1, \ldots, X_n, \mathcal{F}\}$, where $\mathbb{R}$ is here being regarded as the set of all constant functions. We have thus made precise the formulation of the o-minimality axiom given in section 1. The claim of this talk is that if $\mathcal{F}$ is such that $\mathbb{R}_{\mathcal{F}}$ *is* o-minimal, then the collection of $L(\mathcal{F})$-definable sets and functions (with or without parameters) satisfies the three criteria for a framework for tame topology.

Regarding (A), let us just observe for the moment that examples 2.3(2) and (3) apply with "$L(\mathcal{F})$-definable" in place of "L-definable": one simply constructs the formula for, say, $\bar{S}$ out of one for S. In the next section I shall present more of the general theory of o-minimality as justification for the claim above, but first let me present one of the most important examples of an o-minimal structure. It was (re)discovered in this context by van den Dries [vdD2] as a consequence of a theorem of Gabrielov [Ga].

3.1. The globally subanalytic sets

We consider the collection $\mathcal{F}_{an}$ of all those functions $f : [-1,1]^n \to \mathbb{R}$ (for all $n \geq 1$) that are the restrictions to $[-1,1]^n$ of a real analytic function with domain some open subset $U \subseteq \mathbb{R}^n$ with $[-1,1]^n \subseteq U$. The structure $\mathbb{R}_{\mathcal{F}_{an}}$ is usually denoted just $\mathbb{R}_{an}$. (Strictly speaking the functions should be total in order to fit in with our previous account, so we set $f(\mathbf{a}) = 0$ if $\mathbf{a} \in \mathbb{R}^n \setminus [-1,1]^n$.) Then $\mathbb{R}_{an}$ is o-minimal ([D-vdD], [Ga]). The $L(\mathcal{F}_{an})$-definable sets are closely related to the much studied and widely used subanalytic sets. In fact the bounded $L(\mathcal{F}_{an})$-definable sets are precisely the bounded subanalytic sets. However, the set of integers, for example, is a subanalytic (in fact, semi-analytic) subset of $\mathbb{R}$ which is obviously not definable in any o-minimal structure. The precise characterization is this: a subset A of $\mathbb{R}^n$ is $L(\mathcal{F}_{an})$-definable if and only if $\theta[A]$ is a subanalytic subset of $\mathbb{R}^n$ for some semi-algebraic homeomorphism $\theta : \mathbb{R}^n \to (-1,1)^n$.

Thus o-minimality is a common generalization of both semi-algebraic and subanalytic geometry and, indeed, most of the topological and geometrical finiteness theorems that were originally established separately have now been proved for o-minimal structures in general. I shall now discuss such theorems and then present more examples of o-minimal structures in section 5.

4. SOME GENERAL THEORY FOR O-MINIMAL STRUCTURES

Let us fix an o-minimal structure $\mathbb{R}_{\mathcal{F}}$. The proofs of all the results listed below may be found in [vdD2]. They exemplify criterion (A) and, especially, criterion (C) for tame topology.

4.1. Connectivity

For each $n \geq 1$, every $L(\mathcal{F})$-definable subset of $\mathbb{R}^n$ is a finite union of connected sets (each of which is also $L(\mathcal{F})$-definable). For $n = 1$ this is just the definition of o-minimality. In fact, there is also a generalization of 'points' and 'open intervals' to higher dimensions, giving rise to a cylindrical cell-decomposition theorem for $L(\mathcal{F})$-definable subsets of $\mathbb{R}^n$. This in turn implies a uniformity, as required for criterion (C), in fibred collections: if $\mathcal{S}$ is a definable collection of subsets of $\mathbb{R}^n$, then there exists a positive integer N such that each set in $\mathcal{S}$ is the union of at most N connected sets.

4.2. Dimension

Indeed, for each $p \geq 1$, any $L(\mathcal{F})$-definable subset A of $\mathbb{R}^n$ is a finite union of connected C^p submanifolds of $\mathbb{R}^n$ and this leads to a well behaved notion $dim(A)$, the *dimension* of A. Once more, there is a uniform bound on the number of such submanifolds required for sets in a definable collection. Further, the (integer valued) function $\mathbf{a} \mapsto dim(A_{\mathbf{a}})$ is $L(\mathcal{F})$-definable, whenever $\{A_{\mathbf{a}} : \mathbf{a} \in \mathbb{R}^m\}$ is a definable collection of subsets of $\mathbb{R}^n$.

4.3. Differentiability

Let $f : U \to \mathbb{R}$ be an $L(\mathcal{F})$-definable function, where U is a non-empty, open subset of $\mathbb{R}^n$. Then for each $p \geq 1$, there exists an $L(\mathcal{F})$-definable, open set $V \subseteq U$ with $dim(U \setminus V) < n$ such that $f|V$ is of class $C^{(p)}$ (cf 2.3(3)). In all *known* o-minimal structures we may even take $p = \infty$ here, but this seems unlikely to be true in general.

4.4. Homeomorphism types

Let $\mathcal{S} := \{A_{\mathbf{a}} : \mathbf{a} \in \mathbb{R}^m\}$ be a definable collection of subsets of $\mathbb{R}^n$. Then there exists a *finite* subset Δ of $\mathbb{R}^m$ such that every set $A_{\mathbf{a}}$ is homeomorphic to some $A_{\mathbf{c}}$ with $\mathbf{c} \in \Delta$ (and the homeomorphisms are themselves also (uniformly) $L(\mathcal{F})$-definable).

Thus, for example, with $\mathcal{F} = \emptyset$ and fixed $n, d \geq 1$, we may take $\mathcal{S}$ to be the collection of all zero sets (or positivity sets) of polynomials in n variables and of total degree at most d (so we take m to be the number of monomials of degree at most d in n variables). The conclusion is that there exists a positive integer $N = N(n, d)$ such there are at most N homeomorphism types of such sets.

5. MORE EXAMPLES OF O-MINIMAL STRUCTURES

5.1. Quasi-analytic classes

Let $\bar{M} = \langle M_0, M_1, \ldots \rangle$ be an increasing sequence of real numbers with $M_0 \geq 1$. For $n \geq 1$, let $\mathcal{C}_n(\bar{M})$ denote the collection of all C^∞ functions $f : [-1,1]^n \to \mathbb{R}$ satisfying $|f^{(\alpha)}(\mathbf{x})| \leq c^{|\alpha|} \cdot M_{|\alpha|}$ for all $\mathbf{x} \in [-1,1]^n$ and for all multi-indices $\alpha \in \mathbb{N}^n$, where $c > 0$ is a constant that may depend on f, but not on α or $\mathbf{x}$. (For $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \ldots + \alpha_n$ and $f^{(\alpha)}$ denotes the α'th partial derivative of f with respect to $\mathbf{x}$.)

We now say that the sequence $\bar{M}$ *determines a quasi-analytic class* if for all $n \geq 1$, all $f \in \mathcal{C}_n(\bar{M})$ and all $\mathbf{a} \in [-1,1]^n$, either f is identically zero or else there exists $\alpha \in \mathbb{N}^n$ such that $f^{(\alpha)}(\bar{a}) \neq 0$. In other words, the map sending a function in $\mathcal{C}_n(\bar{M})$ to its formal Taylor series at a point $\mathbf{a} \in [-1,1]^n$ is injective.

Theorem 5.1. — *(Rolin-Speissegger-Wilkie* [RSW]*) Suppose that the sequence $\bar{M}$ determines a quasi-analytic class and set $\mathcal{F} = \bigcup\{f \in \mathcal{C}_n(\bar{M}) : n \geq 1\}$ (with functions being set to* 0 *outside the unit box). Then $\mathbb{R}_{\mathcal{F}}$ is o-minimal.*

In the case $M_p = p!$ each $\mathcal{C}_n(\bar{M})$ consists of precisely the functions that have real analytic continuations to some open set containing the box $[-1, 1]^n$, and then $\mathbb{R}_{\mathcal{F}} = \mathbb{R}_{an}$. However, there are larger quasianalytic classes. For by a theorem of Denjoy and Carleman, $\bar{M}$ determines a quasianalytic class if and only if the series $\Sigma_{p=0}^{p=\infty} \frac{M_p}{M_{p+1}}$ diverges, and this can be used in conjunction with theorem 5.1 to construct new o-minimal structures. However, such structures are more for theoretical interest than practical use in that they can illustrate the limitations of the general theory. For example, one can construct a sequence $\bar{M}$ satisfying the Denjoy-Carleman condition such that there exists a function $f \in \mathcal{C}_1(\bar{M})$ which is nowhere analytic. (Every example of an o-minimal structures constructed prior to [RSW] had the property that every definable function was piecewise analytic.) Also, there exist two sequences $\bar{M}$ and $\bar{K}$, both satisfying the Denjoy Carleman condition, such that the structure $\mathbb{R}_{\mathcal{C}_1(\bar{M}) \cup \mathcal{C}_1(\bar{K})}$ is not o-minimal. So there is no maximum o-minimal structure.

5.2. Tarski's problem on the real exponential function

Tarski asked whether his work on the ordered field of real numbers could be extended to the structure $\mathbb{R}_{\{exp\}}$. (Of course, $exp : \mathbb{R} \to \mathbb{R}$ is real analytic, but it is not definable in the structure $\mathbb{R}_{an}$ because its graph is not *globally* analytic.) In fact, he was interested in questions of effectivity in the sense that his own procedure for producing a quantifier-free formula equivalent to a given arbitrary formula of L was completely effective. We still do not have any complete answers in this direction for the structure $\mathbb{R}_{\{exp\}}$, although we do now know enough about the $L(\{exp\})$-definable sets to reduce the problem to a purely number-theoretic one (see [MW]). This arose out of the work in my paper [W] where I showed that any $L(\{exp\})$-definable set is a projection of a set definable by a quantifier-free formula. Since Khovanski had already shown ([K]) that that such sets have only finitely many connected components, we also obtain the following result.

Theorem 5.2. — *The structure $\mathbb{R}_{\{exp\}}$ is o-minimal.*

Soon after this, van den Dries, Macintyre and Marker ([vdD-M-M]), by rather different methods, managed to add the full exponential function to $\mathbb{R}_{an}$. That is, they showed that the structure $\mathbb{R}_{\mathcal{F}^*}$ is o-minimal (and much more besides), where $\mathcal{F}^* := \mathcal{F}_{an} \cup \{exp\}$, and this is probably the most useful tame topology for the working mathematician. (See for example [S-V], where functions of the form $f(x_1, \ldots, x_n, logx_1, \ldots, logx_n)$ (where f is globally analytic) arise. Schmid and Vilonen must control the behaviour of the zero set, Z say, of such a function near a positive coordinate plane, P say. This may be studied by analysing the intersection $\bar{Z} \cap \bar{P}$ of their closures, which is $L(\mathbb{R}_{\mathcal{F}^*})$-definable (cf the remarks concerning 2.3(2) in section 3), and hence the representations described in 4.1 and 4.2 apply.)

6. NEW FIBRED COLLECTIONS OF POLYNOMIALS

In this short final section I present a result (due to Coste and van den Dries, and motivated by work of Risler) that concerns a uniformity in certain collections of semi-algebraic sets but does not seem to be provable by considering the real ordered field $\bar{\mathbb{R}}$ alone. Further, unlike the result discussed in 4.4, which may be formulated, and is in fact true, for the complex field (just split a complex polynomial into its real and imaginary parts and apply the real result to the sum of their squares) this result is definitely false there.

Fix positive integers n, d and consider the collection $\mathcal{P}_{n,d}$ of zero sets of polynomials in n variables that can be written as the sum of at most d monomials (of any degree). The result states that there is a bound $l = l(n, d)$ on the number of homeomorphism types of sets in $\mathcal{P}_{n,d}$. (For $n = 1$ this is a consequence of Descartes' Rule of Signs: one may take $l = 2d + 1$.)

Now of course, each set in $\mathcal{P}_{n,d}$ is an L-definable subset of $\mathbb{R}^n$. However, it follows easily from quantifier elimination that for $n > 1$, $\mathcal{P}_{n,d}$ is not contained in any definable collection of subsets of $\mathbb{R}^n$ *relative to the structure* $\bar{\mathbb{R}}$. So the argument of 4.4 seems not to apply. The idea of Coste and van den Dries is based on the observation that a monomial function becomes a definable function of both the n given variables *and their exponents* if we pass to the structure $\mathbb{R}_{\{exp\}}$.

To simplify the argument, I consider the collection $\{Z \cap Q^+ : Z \in \mathcal{P}_{n,d}\}$, where Q^+ denotes the positive quadrant of $\mathbb{R}^n$, rather than $\mathcal{P}_{n,d}$. We introduce $d + 1$ n-tuples of variables $\mathbf{t}$, $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(d)}$ and a d-tuple of variables $\mathbf{u}$ and consider the function

$$g(\mathbf{u}, \mathbf{y}, \mathbf{t}) := \Sigma_{i=1}^{d} u_i \, exp(\mathbf{y}^{(i)} \cdot \mathbf{t})$$

of the $d + nd + n$ variables displayed. (The $\cdot$ denotes scalar product.)

If $P(x_1, \ldots, x_n)$ is the sum of d monomials with real coefficients, then we may clearly find $\alpha \in \mathbb{R}^d$ and $\mathbf{k} \in \mathbb{N}^{nd}$ ($\subseteq \mathbb{R}^{nd}$) such that $P(exp(t_1), \ldots, exp(t_n)) = g(\alpha, \mathbf{k}, \mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$.

But note that $\{\{\langle exp(t_1), \ldots, exp(t_n)\rangle \in Q^+ : g(\mathbf{u}, \mathbf{y}, \mathbf{t}) = 0\} : \langle \mathbf{u}, \mathbf{y}\rangle \in \mathbb{R}^{d+nd}\}$ is a definable collection of subsets of $\mathbb{R}^n$ over the structure $\mathbb{R}_{\{exp\}}$. The result now follows as in 4.4 after noting that the exponential map on each coordinate induces a homeomorphism from $\mathbb{R}^n$ to Q^+.

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