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# A SCHANUEL PROPERTY FOR EXPONENTIALLY TRANSCENDENTAL POWERS

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ABSTRACT. We prove the analogue of Schanuel's conjecture for raising to the power of an exponentially transcendental real number. All but countably many real numbers are exponentially transcendental. We also give a more general result for several powers in a context which encompasses the complex case.

## 1. INTRODUCTION

We prove a Schanuel property for raising to a real power:

**Theorem 1.1.** *Let  $\lambda \in \mathbb{R}$  be exponentially transcendental, let  $\bar{y} \in (\mathbb{R}_{>0})^n$ , and suppose  $\bar{y}$  is multiplicatively independent. Then*

$$\text{td}(\bar{y}, \bar{y}^\lambda / \lambda) \geq n.$$

Here and later,  $\text{td}(X/Y)$  denotes the transcendence degree of the field extension  $\mathbb{Q}(X, Y)/\mathbb{Q}(Y)$  (for  $X, Y$  subsets of the ambient field, in this case  $\mathbb{R}$ ). To say that  $\bar{y}$  is multiplicatively independent means that if  $m_1, \dots, m_n \in \mathbb{Z}$  and  $\prod y_i^{m_i} = 1$  then  $m_i = 0$  for each  $i$ . The usual exponential function  $\exp : \mathbb{R} \rightarrow \mathbb{R}$  makes the reals into an *exponential field*, formally a field of characteristic zero equipped with a homomorphism from its additive to multiplicative groups. In any exponential field  $\langle F; +, \cdot, \exp \rangle$ , we say that an element  $x \in F$  is *exponentially algebraic* in  $F$  iff there is  $n \in \mathbb{N}$ ,  $\bar{x} = (x_1, \dots, x_n) \in F^n$ , and exponential polynomials  $f_1, \dots, f_n \in \mathbb{Z}[\bar{X}, e^{\bar{X}}]$  such that  $x = x_1$ ,  $f_i(\bar{x}, e^{\bar{x}}) = 0$  for each  $i = 1, \dots, n$ , and the determinant of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{pmatrix}$$

is nonzero at  $\bar{x}$ . If  $x$  is not exponentially algebraic in  $F$  we say it is *exponentially transcendental* in  $F$ . More generally, for a subset  $A$  of  $F$ , we can define the notion of  $x$  being *exponentially algebraic over  $A$*  with the same definition except that the  $f_i$  can have coefficients from  $A$ . Observe that the non-vanishing of the Jacobian in the reals means that  $\bar{x}$  is an isolated zero of the system of equations, and hence all but countably many real numbers are exponentially transcendental. Thus a consequence of theorem 1.1 is that the numbers  $\lambda, \lambda^\lambda, \lambda^{\lambda^2}, \lambda^{\lambda^3}, \dots$  are algebraically independent for all but countably many  $\lambda$ , although, unfortunately, one does not know any explicit  $\lambda$  for which this is true.

This paper contains a complete proof of theorem 1.1, assuming only some knowledge of o-minimality from the reader (and using a theorem of Ax). The paper [Kir08] of the second author develops the theory of exponential algebraicity in an arbitrary exponential field, and, using that, we can prove a more general theorem.

**Theorem 1.2.** *Let  $F$  be any exponential field, let  $\lambda \in F$  be exponentially transcendental, and let  $\bar{x} \in F^n$  be such that  $\exp(\bar{x})$  is multiplicatively independent. Then*

$$\text{td}(\exp(\bar{x}), \exp(\lambda\bar{x})/\lambda) \geq n.$$

Theorem 1.1 follows from 1.2 by taking  $x_i = \log y_i$ .

We define the exponential algebraic closure  $\text{ecl}(A)$  of a subset  $A$  of  $F$  to be the set of  $x \in F$  which are exponentially algebraic over  $A$ . In [Kir08] it is shown that  $\text{ecl}$  is a pregeometry in any exponential field, and hence we have notions of dimension and independence. We also prove a general Schanuel property for raising to several independent powers, which uses a slightly subtle notion of relative linear dimension. For any subfield  $K$  of  $F$ , we can think of  $F$  as a  $K$ -vector space. For subsets  $X, Y$  of  $F$ , consider the  $K$ -linear subspaces  $\langle XY \rangle_K$  and  $\langle Y \rangle_K$  of  $F$  generated by  $X \cup Y$  and  $Y$  respectively. We define  $\text{ldim}_K(X/Y)$  to be the  $K$ -linear dimension of the quotient  $K$ -vector space  $\langle XY \rangle_K / \langle Y \rangle_K$ .

**Theorem 1.3.** *Let  $F$  be any exponential field, let  $\ker$  be the kernel of its exponential map, let  $C$  be an  $\text{ecl}$ -closed subfield of  $F$ , and let  $\bar{\lambda}$  be an  $m$ -tuple which is exponentially algebraically independent over  $C$ . Then for any tuple  $\bar{z}$  from  $F$ :*

$$\text{td}(\exp(\bar{z})/C, \bar{\lambda}) + \text{ldim}_{\mathbb{Q}(\bar{\lambda})}(\bar{z}/\ker) - \text{ldim}_{\mathbb{Q}}(\bar{z}/\ker) \geq 0.$$

The reader who is interested only in the real case may ignore all the references to [Kir08]. On the other hand, the reader who is unfamiliar with o-minimality may prefer to ignore that part of this paper and instead refer to the algebraic proof of proposition 2.1 in [Kir08].

## 2. A SCHANUEL PROPERTY FOR EXPONENTIATION

We need the following relative Schanuel property for exponentiation itself.

**Proposition 2.1.** *Let  $F$  be an exponential field and let  $\bar{\lambda} \in F^m$  be exponentially algebraically independent. Let  $B \subseteq F$  be such that  $B \cup \bar{\lambda}$  is a basis for  $F$  with respect to the pregeometry  $\text{ecl}$ . Let  $C = \text{ecl}(B)$ . Then for any  $\bar{z} \in F^n$ ,*

$$\text{td}(\bar{\lambda}, \bar{z}, \exp(\bar{\lambda}), \exp(\bar{z})/C) - \text{ldim}_{\mathbb{Q}}(\bar{\lambda}, \bar{z}/C) \geq m.$$

*Proof.* Theorem 1.2 of [Kir08] states that  $\text{td}(\bar{\lambda}, \bar{z}, \exp(\bar{\lambda}), \exp(\bar{z})/C) - \text{ldim}_{\mathbb{Q}}(\bar{\lambda}, \bar{z}/C)$  is at least the dimension of the  $(m+n)$ -tuple  $(\bar{\lambda}, \bar{z})$  over  $C$  with respect to the pregeometry  $\text{ecl}$ . Since  $\bar{\lambda}$  is  $\text{ecl}$ -independent over  $C$  by assumption, this dimension is at least  $m$ .  $\square$

We give a more direct proof of proposition 2.1 in the real case. Firstly, by theorem 4.2 of [JW08], a real number  $x$  is in the exponential algebraic closure  $\text{ecl}(A)$  of a subset  $A$  of  $\mathbb{R}$  iff it lies in the definable closure of  $A$  in the structure  $\mathbb{R}_{\text{exp}} = \langle \mathbb{R}; +, \cdot, \exp \rangle$ . Definable closure is always a pregeometry in an o-minimal field, so  $\text{ecl}$  is a pregeometry on  $\mathbb{R}_{\text{exp}}$ .

For each  $i = 1, \dots, m$ , let  $K_i = \text{ecl}(B \cup \bar{\lambda} \setminus \lambda_i)$ , so  $C = \bigcap_{i=1}^m K_i$ . Then for each  $i$ ,  $\lambda_i \notin K_i$ , but for each  $a \in \mathbb{R}$  there is a function  $\theta : \mathbb{R} \rightarrow \mathbb{R}$ , definable in  $\mathbb{R}_{\text{exp}}$  with parameters from  $K_i$ , such that  $\theta(\lambda_i) = a$ . By o-minimality of  $\mathbb{R}_{\text{exp}}$ ,  $\theta$  is differentiable at all but finitely many  $x \in \mathbb{R}$ , and hence this exceptional set is contained in  $K_i$ . Thus  $\theta$  is differentiable on an open interval containing  $\lambda_i$ . Suppose that  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is another such function with  $\psi(\lambda_i) = a$ . Again by o-minimality, the boundary of the set  $\{x \in \mathbb{R} \mid \psi(x) = \theta(x)\}$  is finite and contained in  $K_i$ , so  $\theta$  and  $\psi$  agree on an open interval containing  $\lambda_i$ . It follows that there is a well-defined function  $\partial_i : \mathbb{R} \rightarrow \mathbb{R}$  which sends  $a$  to  $\frac{d\theta}{dx}(\lambda_i)$ , where  $\theta$  is any function definable in  $\mathbb{R}_{\text{exp}}$  with parameters from  $K_i$  such that  $\theta(\lambda_i) = a$ . It is straightforward to check that  $\partial_i$  is a derivation on the field  $\mathbb{R}$ , with field of constants  $K_i$ . Furthermore, we

also clearly have that  $\partial_i(\exp(a)) = \partial_i(a) \exp(a)$  for any  $a \in \mathbb{R}$ , and that  $\partial_i(p_j) = \delta_{ij}$ , the Kronecker delta.

By Ax's theorem [Ax71, theorem 3],  $\text{td}(\bar{\lambda}, \bar{z}, \exp(\bar{\lambda}), \exp(\bar{z})/C) - \text{ldim}_{\mathbb{Q}}(\bar{\lambda}, \bar{z}/C)$  is at least the rank of the matrix

$$\begin{pmatrix} \partial_1 z_1 & \cdots & \partial_1 z_n & \partial_1 \lambda_1 & \cdots & \partial_1 \lambda_m \\ \vdots & & \vdots & \vdots & & \vdots \\ \partial_m z_1 & \cdots & \partial_m z_n & \partial_m \lambda_1 & \cdots & \partial_m \lambda_m \end{pmatrix}$$

which is  $m$  since the right half is just the  $m \times m$  identity matrix. That completes the proof of proposition 2.1 in the real case. The general case works the same way, but a different and much more involved argument is used in [Kir08] to produce the derivations  $\partial_i$  without using o-minimality.

### 3. LINEAR DISJOINTNESS

The other key ingredient in the proofs is the concept of linear disjointness. We briefly recall the definition and some basic properties.

**Definition 3.1.** Let  $F$  be a field, and let  $K, L$ , and  $E$  be subfields of  $F$  with  $E \subseteq K \cap L$ . Then  $K$  is *linearly disjoint from  $L$  over  $E$* , written  $K \perp_E L$ , iff every tuple  $\bar{k}$  of elements of  $K$  that is  $E$ -linearly independent is also  $L$ -linearly independent.

**Lemma 3.2.**

- (i)  $K \perp_E L$  iff  $L \perp_E K$
- (ii)  $K \perp_E L$  iff for any tuple  $\bar{l}$  from  $L$ ,  $\text{ldim}_K(\bar{l}) = \text{ldim}_E(\bar{l})$
- (iii) If  $\bar{k}$  is algebraically independent over  $L$ , then  $E(\bar{k}) \perp_E L$ .

*Proof.* (i) and (ii) are straightforward; (iii) is proposition VIII 3.3 of [Lan93].  $\square$

**Lemma 3.3.** Suppose  $K \perp_E L$ . Then for any tuple  $\bar{x}$  from  $F$  and any subset  $A \subseteq L$ ,

$$\text{ldim}_K(\bar{x}/L) - \text{ldim}_E(\bar{x}/L) \leq \text{ldim}_K(\bar{x}/A) - \text{ldim}_E(\bar{x}/A).$$

*Proof.* Let  $\bar{l} \in L$  be a finite tuple such that  $\text{ldim}_K(\bar{x}/\bar{l}A) = \text{ldim}_K(\bar{x}/L)$  and  $\text{ldim}_E(\bar{x}/\bar{l}A) = \text{ldim}_E(\bar{x}/L)$ .

Now:

$$\begin{aligned} \text{ldim}_K(\bar{x}/A) - \text{ldim}_K(\bar{x}/\bar{l}A) &= \text{ldim}_K(\bar{l}/A) - \text{ldim}_K(\bar{l}/\bar{x}A) && \text{(by the addition formula)} \\ &= \text{ldim}_E(\bar{l}/A) - \text{ldim}_K(\bar{l}/\bar{x}A) && \text{(by Lemma 3.2(ii))} \\ &\geq \text{ldim}_E(\bar{l}/A) - \text{ldim}_E(\bar{l}/\bar{x}A) \\ &= \text{ldim}_E(\bar{x}/A) - \text{ldim}_E(\bar{x}/\bar{l}A) && \text{(by the addition formula).} \end{aligned}$$

$\square$

### 4. PROOFS OF THE MAIN THEOREMS

*Proof of theorem 1.3.* By proposition 2.1, for any tuple  $\bar{z}$  from  $F$  we have:

$$\text{td}(\bar{z}, \exp(\bar{z}), \bar{\lambda}, \exp(\bar{\lambda})/C) - \text{ldim}_{\mathbb{Q}}(\bar{z}, \bar{\lambda}/C) \geq m.$$

Expanding using the addition formula gives

$$\begin{aligned} \text{td}(\bar{\lambda}/C) + \text{td}(\bar{z}/C, \bar{\lambda}) + \text{td}(\exp(\bar{z})/C, \bar{\lambda}, \bar{z}) \\ + \text{td}(\exp(\bar{\lambda})/C, \bar{\lambda}, \bar{z}, \exp(\bar{z})) - \text{ldim}_{\mathbb{Q}}(\bar{\lambda}/C, \bar{z}) - \text{ldim}_{\mathbb{Q}}(\bar{z}/C) \geq m. \end{aligned}$$

Since  $\bar{\lambda}$  is algebraically independent over  $C$ , we have  $\text{td}(\bar{\lambda}/C) = m$ , and we deduce

$$(1) \quad \text{td}(\bar{z}/C, \bar{\lambda}) + \text{td}(\exp(\bar{z})/C, \bar{\lambda}) + \text{td}(\exp(\bar{\lambda})/C, \exp(\bar{z})) \\ - \text{ldim}_{\mathbb{Q}}(\bar{\lambda}/C, \bar{z}) - \text{ldim}_{\mathbb{Q}}(\bar{z}/C) \geq 0.$$

We also have:

$$(2) \quad \text{td}(\exp(\bar{\lambda})/C, \exp(\bar{z})) \leq \text{ldim}_{\mathbb{Q}}(\bar{\lambda}/C, \bar{z})$$

because if  $\lambda_1, \dots, \lambda_t$  form a  $\mathbb{Q}$ -linear basis for  $\bar{\lambda}$  over  $(C, \bar{z})$ , then for  $i > t$ ,  $\exp(\lambda_i)$  is in the algebraic closure of  $(C, \exp(\bar{z}), \exp(\lambda_1), \dots, \exp(\lambda_t))$ . A similar argument shows

$$(3) \quad \text{td}(\bar{z}/C, \bar{\lambda}) \leq \text{ldim}_{\mathbb{Q}(\bar{\lambda})}(\bar{z}/C)$$

since if  $z_i$  is in the  $\mathbb{Q}(\bar{\lambda})$ -linear span of  $(z_1, \dots, z_t, C)$  then  $z_i$  is in the algebraic closure of  $(C, \bar{\lambda}, z_1, \dots, z_t)$ .

Combining (1) with (2) and (3) gives

$$\text{td}(\exp(\bar{z})/C, \bar{\lambda}) + \text{ldim}_{\mathbb{Q}(\bar{\lambda})}(\bar{z}/C) - \text{ldim}_{\mathbb{Q}}(\bar{z}/C) \geq 0.$$

By lemma 3.2(iii),  $\mathbb{Q}(\bar{\lambda})$  is linearly disjoint from  $C$  over  $\mathbb{Q}$ . Also  $\ker \subseteq \text{ecl}(\emptyset) \subseteq C$ , so, by lemma 3.3,

$$\text{td}(\exp(\bar{z})/C, \bar{\lambda}) + \text{ldim}_{\mathbb{Q}(\bar{\lambda})}(\bar{z}/\ker) - \text{ldim}_{\mathbb{Q}}(\bar{z}/\ker) \geq 0$$

as required.  $\square$

*Proof of theorem 1.2.* By theorem 1.3, taking  $\bar{z} = (\bar{x}, \lambda\bar{x})$ ,

$$\text{td}(\exp(\bar{x}), \exp(\lambda\bar{x})/\lambda) \geq \text{ldim}_{\mathbb{Q}}(\bar{x}, \lambda\bar{x}/\ker) - \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}, \lambda\bar{x}/\ker) \\ = \text{ldim}_{\mathbb{Q}}(\bar{x}/\ker) + \text{ldim}_{\mathbb{Q}}(\lambda\bar{x}/\bar{x}, \ker) - \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}/\ker) \\ = n + \text{ldim}_{\mathbb{Q}}(\lambda\bar{x}/\bar{x}, \ker) - \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}/\ker).$$

Thus it suffices to prove that  $\text{ldim}_{\mathbb{Q}}(\lambda\bar{x}/\bar{x}, \ker) \geq \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}/\ker)$ . Let  $\bar{k}$  be a finite tuple from  $\ker$  such that  $\text{ldim}_{\mathbb{Q}}(\lambda\bar{x}/\bar{x}, \ker) = \text{ldim}_{\mathbb{Q}}(\lambda\bar{x}/\bar{x}, \bar{k})$  and  $\text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}/\ker) = \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}/\bar{k})$ .

Let  $A_0 := \langle \lambda\bar{x}, \bar{k} \rangle_{\mathbb{Q}}$ . Then  $\text{ldim}_{\mathbb{Q}}(\lambda\bar{x}, \bar{k}/\bar{x}, \lambda^{-1}\bar{k}) = \text{ldim}_{\mathbb{Q}}(A_0/A_0 \cap \lambda^{-1}A_0)$ . Inductively define  $A_{i+1} := A_i \cap \lambda^{-1}A_i$  for  $i \in \mathbb{N}$ . Suppose for some  $i$  that  $A_{i+1} = A_i$ . Then multiplication by  $\lambda$  induces a  $\mathbb{Q}$ -linear automorphism of  $A_i$ . It follows that for any  $f(\lambda) \in \mathbb{Q}[\lambda]$ , multiplication by  $f(\lambda)$  is a  $\mathbb{Q}$ -linear endomorphism of  $A_i$ . This endomorphism has trivial kernel because  $f(\lambda)$  is not a zero divisor of the field (unless  $f(\lambda) = 0$ ), and  $A_i$  is finite-dimensional, so it is invertible. Its inverse must be multiplication by  $f(\lambda)^{-1}$ , and hence  $A_i$  is a  $\mathbb{Q}(\lambda)$ -vector space. Since  $\lambda$  is transcendental,  $\text{ldim}_{\mathbb{Q}} \mathbb{Q}(\lambda)$  is infinite, so  $A_i = \{0\}$ . So  $\text{ldim}_{\mathbb{Q}} A_{i+1} < \text{ldim}_{\mathbb{Q}} A_i$  unless  $A_i = \{0\}$ . Thus for some  $N \in \mathbb{N}$  we have  $A_N = \{0\}$ .

For each  $i$  we have a chain of subspaces  $A_{i+1} \subseteq A_{i+1} + \lambda A_{i+1} \subseteq A_i$ , so

$$\text{ldim}_{\mathbb{Q}}(A_i/A_{i+1}) = \text{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) + \text{ldim}_{\mathbb{Q}}(A_{i+1} + \lambda A_{i+1}/A_{i+1}) \\ = \text{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) + \text{ldim}_{\mathbb{Q}}(\lambda A_{i+1}/A_{i+1} \cap \lambda A_{i+1}) \\ = \text{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) + \text{ldim}_{\mathbb{Q}}(\lambda A_{i+1}/\lambda A_{i+2}) \\ = \text{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) + \text{ldim}_{\mathbb{Q}}(A_{i+1}/A_{i+2}).$$

Thus inductively we obtain

$$\text{ldim}_{\mathbb{Q}}(A_0/A_1) = \sum_{i=0}^N \text{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}).$$

Now for each  $i$ ,

$$\text{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) \geq \text{ldim}_{\mathbb{Q}(\lambda)}(A_i/A_{i+1} + \lambda A_{i+1}) = \text{ldim}_{\mathbb{Q}(\lambda)}(A_i/A_{i+1})$$

hence

$$\text{ldim}_{\mathbb{Q}}(A_0/A_1) \geq \sum_{i=0}^N \text{ldim}_{\mathbb{Q}(\lambda)}(A_i/A_{i+1}) = \text{ldim}_{\mathbb{Q}(\lambda)}(A_0)$$

that is,

$$(4) \quad \text{ldim}_{\mathbb{Q}}(\lambda \bar{x}, \bar{k}/\bar{x}, \lambda^{-1} \bar{k}) \geq \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}, \lambda^{-1} \bar{k}).$$

But

$$(5) \quad \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}, \lambda^{-1} \bar{k}) = \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}, \bar{k}) = \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}/\bar{k}) + \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{k})$$

and

$$(6) \quad \begin{aligned} \text{ldim}_{\mathbb{Q}}(\lambda \bar{x}, \bar{k}/\bar{x}, \lambda^{-1} \bar{k}) &\leq \text{ldim}_{\mathbb{Q}}(\lambda \bar{x}, \bar{k}/\bar{x}) \\ &= \text{ldim}_{\mathbb{Q}}(\lambda \bar{x}/\bar{k}, \bar{x}) + \text{ldim}_{\mathbb{Q}}(\bar{k}/\bar{x}) \\ &\leq \text{ldim}_{\mathbb{Q}}(\lambda \bar{x}/\bar{k}, \bar{x}) + \text{ldim}_{\mathbb{Q}}(\bar{k}) \\ &= \text{ldim}_{\mathbb{Q}}(\lambda \bar{x}/\bar{k}, \bar{x}) + \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{k}) \end{aligned}$$

the last line holding by lemma 3.2(ii), since  $\mathbb{Q}(\lambda) \perp_{\mathbb{Q}} C$  and  $\bar{k} \subseteq \ker \subseteq C$ .

Putting together (4), (5), and (6) gives  $\text{ldim}_{\mathbb{Q}}(\lambda \bar{x}/\bar{x}, \ker) \geq \text{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}/\ker)$  as required.  $\square$

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