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Cylindrical Lévy processes in Banach spaces

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Abstract

Cylindrical probability measures are finitely additive measures on Banach spaces that have sigma-additive projections to Euclidean spaces of all dimensions. They are naturally associated to notions of weak (cylindrical) random variable and hence weak (cylindrical) stochastic processes. In this paper we focus on cylindrical Lévy processes. These have (weak) Lévy-Itô decompositions and an associated Lévy-Khintchine formula. If the process is weakly square integrable, its covariance operator can be used to construct a reproducing kernel Hilbert space in which the process has a decomposition as an infinite series built from a sequence of uncorrelated bona fide one-dimensional Lévy processes. This series is used to define cylindrical stochastic integrals from which cylindrical Ornstein-Uhlenbeck processes may be constructed as unique solutions of the associated Cauchy problem. We demonstrate that such processes are cylindrical Markov processes and study their (cylindrical) invariant measures.

Keywords and phrases: cylindrical probability measure, cylindrical Lévy process, reproducing kernel Hilbert space, Cauchy problem, cylindrical Ornstein-Uhlenbeck process, cylindrical invariant measure.

MSC 2000: *primary* 60B11, *secondary* 60G51, 60H05, 28C20.

1 Introduction

Probability theory in Banach spaces has been extensively studied since the 1960s and there are several monographs dedicated to various themes within the subject - see e.g.

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Heyer [5], Linde [7], Vakhania et al [16], Ledoux and Talagrand [6]. In general, the theory is more complicated than in Euclidean space (or even in an infinite-dimensional Hilbert space) and much of this additional complexity arises from the interaction between probabilistic ideas and Banach space geometry. The theory of type and cotype Banach spaces (see e.g. Schwartz [15]) is a well-known example of this phenomenon.

From the outset of work in this area, there was already interest in cylindrical probability measures (cpms), i.e. finitely additive set functions whose "projections" to Euclidean space are always bona fide probability measures. These arise naturally in trying to generalise a mean zero normal distribution to an infinite-dimensional Banach space. It is clear that the covariance Q should be a bounded linear operator that is positive and symmetric but conversely it is not the case that all such operators give rise to a sigma-additive probability measure. Indeed in a Hilbert space, it is necessary and sufficient for Q to be trace-class (see e.g. Schwartz [15], p.28) and if we drop this requirement (and one very natural example is when Q is the identity operator) then we get a cpm.

Cpms give rise to cylindrical stochastic processes and these appear naturally as the driving noise in stochastic partial differential equations (SPDEs). An introduction to this theme from the point of view of cylindrical Wiener processes can be found in Da Prato and Zabczyk [9]. In recent years there has been increasing interest in SPDEs driven by Lévy processes and Peszat and Zabczyk [8] is a monograph treatment of this topic. Some specific examples of cylindrical Lévy processes appear in this work and Priola and Zabczyk [10] makes an in-depth study of a specific class of SPDEs driven by cylindrical stable processes. In Brzeźniak and Zabczyk [3] the authors study the path-regularity of an Ornstein-Uhlenbeck process driven by a cylindrical Lévy process obtained by subordinating a cylindrical Wiener process.

The purpose of this paper is to begin a systematic study of cylindrical Lévy processes in Banach spaces with particular emphasis on stochastic integration and applications to SPDEs. It can be seen as a successor to an earlier paper by one of us (see Riedle [11]) in which some aspects of this programme were carried out for cylindrical Wiener processes.

The organisation of the paper is as follows. In section 2 we review key concepts of cylindrical proabability, introduce the cylindrical version of infinite divisibility and obtain the corresponding Lévy-Khintchine formula. In section 3 we introduce cylindrical Lévy processes and describe their Lévy-Itô decomposition. An impediment to developing the theory along standard lines is that the noise terms in this formula depend non-linearly on vectors in the dual space to our Banach space. In particular this makes the "large jumps" term difficult to handle. To overcome these problems we restrict ourself to the case where the cylindrical Lévy process is square-integrable with a well-behaved covariance operator. This enables us to develop the theory along similar lines to that used for cylindrical Wiener processes as in Riedle [11] and to find a series representation for the cylindrical Lévy process in a reproducing kernel Hilbert space that is determined by the covariance operator. This is described in section 4 of this paper where we also utilise this series expansion to define stochastic integrals of suitable predictable processes.

Finally, in section 5 we consider SPDEs driven by additive cylindrical Lévy noise. In the more familiar context of SPDEs driven by legitimate Lévy processes in Hilbert space, it is well known that the weak solution of this equation is an Ornstein-Uhlenbeck process and the investigation of these processes has received a lot of attention in the literature (see e.g. Chojnowska-Michalik [4], Applebaum [2] and references therein). In our case we require that the initial condition is a cylindrical random variable and so we are able to construct cylindrical Ornstein-Uhlenbeck processes as weak solutions to our SPDE. We study the Markov property (in the cylindrical sense) of the solution and also find conditions for there to be a unique invariant cylindrical measure. Finally, we give a condition under which the Ornstein-Uhlenbeck process is "radonified", i.e. it is a stochastic process in the usual sense.

Notation and Terminology: $\mathbb{R}_+ := [0, \infty)$. The Borel σ-algebra of a topological space T is denoted by $\mathcal{B}(T)$. By a Lévy process in a Banach space we will always mean a stochastic process starting at zero (almost surely) that has stationary and independent increments and is stochastically continuous. We do not require that almost all paths are necessarily càdlàg i.e. right continuous with left limits.

2 Cylindrical measures

Let U be a Banach space with dual U^*. The dual pairing is denoted by $\langle u, a\rangle$ for $u \in U$ and $a \in U^*$. For each $n \in \mathbb{N}$, let U^{*n} denote the set of all n-tuples of vectors from U^*. It is a real vector space under pointwise addition and scalar multiplication and a Banach space with respect to the "Euclidean-type" norm $\left\|\mathrm{a}_{(\mathrm{n})}\right\|^2 := \sum_{k=1}^n \|a_k\|^2$, where $\mathrm{a}_{(\mathrm{n})} = (a_1, \ldots, a_n) \in U^{*n}$. Clearly U^{*n} is separable if U^* is. For each $\mathrm{a}_{(\mathrm{n})} = (a_1, \ldots, a_n) \in U^{*n}$ we define a linear map

$$\pi_{a_1,\ldots,a_n} : U \to \mathbb{R}^n, \qquad \pi_{a_1,\ldots,a_n}(u) = (\langle u, a_1\rangle, \ldots, \langle u, a_n\rangle).$$

We often use the notation $\pi_{\mathrm{a}_{(\mathrm{n})}} := \pi_{a_1,\ldots,a_n}$ and in particular when $a_n = a \in U^*$, we will write $\pi(a) = a$. It is easily verified that for each $\mathrm{a}_{(\mathrm{n})} = (a_1, \ldots, a_n) \in U^{*n}$ the map $\pi_{\mathrm{a}_{(\mathrm{n})}}$ is bounded with $\left\|\pi_{\mathrm{a}_{(\mathrm{n})}}\right\| \leqslant \left\|\mathrm{a}_{(\mathrm{n})}\right\|$.
The Borel σ-algebra in U is denoted by $\mathcal{B}(U)$. Let Γ be a subset of U^*. Sets of the form

$$\begin{aligned} Z(a_1, \ldots, a_n; B) &:= \{u \in U : (\langle u, a_1\rangle, \ldots, \langle u, a_n\rangle) \in B\} \\ &= \pi^{-1}_{a_1,\ldots,a_n}(B), \end{aligned}$$

where $a_1, \ldots, a_n \in \Gamma$ and $B \in \mathcal{B}(\mathbb{R}^n)$ are called *cylindrical sets* . The set of all cylindrical sets is denoted by $\mathcal{Z}(U, \Gamma)$ and it is an algebra. The generated σ-algebra is denoted by $\mathcal{C}(U, \Gamma)$ and it is called the *cylindrical σ-algebra with respect to* (U, Γ). If $\Gamma = U^*$ we write $\mathcal{Z}(U) := \mathcal{Z}(U, \Gamma)$ and $\mathcal{C}(U) := \mathcal{C}(U, \Gamma)$.
From now on we will assume that U is separable and note that in this case, the Borel σ-algebra $\mathcal{B}(U)$ and the cylindrical σ-algebra $\mathcal{C}(U)$ coincide.
The following lemma shows that for a finite subset $\Gamma \subseteq U^*$ the algebra $\mathcal{Z}(U, \Gamma)$ is a σ-algebra and it gives a generator in terms of a generator of the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$, where we recall that a generator of a σ-algebra $\mathfrak{C}$ in a space X is a set E in the power set of X such that the smallest σ-algebra containing E is $\mathfrak{C}$.

Lemma 2.1. *If $\Gamma = \{a_1, \ldots, a_n\} \subseteq U^*$ is finite we have*

$$\mathcal{C}(U, \Gamma) = \mathcal{Z}(U, \Gamma) = \sigma(\{Z(a_1, \ldots, a_n; B) : B \in \mathcal{F}\}),$$

where $\mathcal{F}$ is an arbitrary generator of $\mathcal{B}(\mathbb{R}^n)$.

Proof. Because for any $a_{i_1}, \ldots, a_{i_k} \in \Gamma$, $k \in \{1, \ldots, n\}$, and $B \in \mathcal{B}(\mathbb{R}^k)$ we have

$$Z(a_{i_1}, \ldots, a_{i_k}; B) = Z(a_1, \ldots, a_n; \tilde{B})$$

by extending B suitably to $\tilde{B} \in \mathcal{B}(\mathbb{R}^n)$ it follows that

$$\begin{aligned}\mathcal{Z}(U,\Gamma) &= \{Z(a_{i_1},\dots,a_{i_k};B) : a_{i_1},\dots,a_{i_k} \in \Gamma, B \in \mathcal{B}(\mathbb{R}^k), k \in \{1,\dots,n\}\}\\ &= \{Z(a_1,\dots,a_n;\tilde{B}) : \tilde{B} \in \mathcal{B}(\mathbb{R}^n)\}\\ &= \{\pi^{-1}_{a_1,\dots,a_n}(\tilde{B}) : \tilde{B} \in \mathcal{B}(\mathbb{R}^n)\}\\ &= \pi^{-1}_{a_1,\dots,a_n}(\mathcal{B}(\mathbb{R}^n)).\end{aligned}$$

The last family of sets is known to be a σ-algebra which verifies that $\mathcal{C}(U,\Gamma) = \mathcal{Z}(U,\Gamma)$. Moreover, we have for every generator $\mathcal{F}$ of $\mathcal{B}(\mathbb{R}^n)$ that

$$\pi^{-1}_{a_1,\dots,a_n}(\mathcal{B}(\mathbb{R}^n)) = \pi^{-1}_{a_1,\dots,a_n}(\sigma(\mathcal{F})) = \sigma(\pi^{-1}_{a_1,\dots,a_n}(\mathcal{F})),$$

which completes the proof. □

A function $\mu : \mathcal{Z}(U) \to [0,\infty]$ is called a *cylindrical measure on* $\mathcal{Z}(U)$, if for each finite subset $\Gamma \subseteq U^*$ the restriction of μ to the σ-algebra $\mathcal{C}(U,\Gamma)$ is a measure. A cylindrical measure is called finite if $\mu(U) < \infty$ and a cylindrical probability measure if $\mu(U) = 1$. For every function $f : U \to \mathbb{C}$ which is measurable with respect to $\mathcal{C}(U,\Gamma)$ for a finite subset $\Gamma \subseteq U^*$ the integral $\int f(u)\,\mu(du)$ is well defined as a complex valued Lebesgue integral if it exists. In particular, the characteristic function $\varphi_\mu : U^* \to \mathbb{C}$ of a finite cylindrical measure μ is defined by

$$\varphi_\mu(a) := \int_U e^{i\langle u,a\rangle}\,\mu(du) \qquad \text{for all } a \in U^*.$$

For each $\mathrm{a}_{(\mathrm{n})} = (a_1,\dots,a_n) \in U^{*n}$ we obtain an image measure $\mu \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}$ on $\mathcal{B}(\mathbb{R}^n)$. Its characteristic function $\varphi_{\mu\circ\pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}}$ is determined by that of μ:

$$\varphi_{\mu\circ\pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}}(\beta) = \varphi_\mu(\beta_1 a_1 + \dots + \beta_n a_n) \tag{2.1}$$

for all $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{R}^n$.
If μ_1 and μ_2 are cylindrical probability measures on U their convolution is the cylindrical probability measure defined by

$$(\mu_1 * \mu_2)(A) = \int_U 1_A(x+y)\mu_1(dx)\mu_2(dy),$$

for each $A \in \mathcal{Z}(U)$. Indeed if $A = \pi^{-1}_{a_{(n)}}(B)$ for some $n \in \mathbb{N}, a_{(n)} \in U^{*n}, B \in \mathcal{B}(\mathbb{R}^n)$, then it is easily verified that

$$(\mu_1 * \mu_2)(A) = (\mu_1 \circ \pi^{-1}_{a_{(n)}}) * (\mu_2 \circ \pi^{-1}_{a_{(n)}})(B). \tag{2.2}$$

A standard calculation yields $\varphi_{\mu_1 * \mu_2} = \varphi_{\mu_1}\varphi_{\mu_2}$. For more information about convolution of cylindrical probability measures, see [13]. The n-times convolution of a cylindrical probability measure μ with itself is denoted by μ^{*n}.

Definition 2.2. *A cylindrical probability measure μ on $\mathcal{Z}(U)$ is called* infinitely divisible *if for all $n \in \mathbb{N}$ there exists a cylindrical probability measure $\mu^{1/n}$ such that $\mu = \left(\mu^{1/n}\right)^{*n}$.*

It follow that a cylindrical probability measure μ with characteristic function φ_μ is infinitely divisible if and only if for all $n \in \mathbb{N}$ there exists a characteristic function $\varphi_{\mu^{1/n}}$ of a cylindrical probability measure $\mu^{1/n}$ such that

$$\varphi_\mu(a) = \big(\varphi_{\mu^{1/n}}(a)\big)^n \qquad \text{for all } a \in U^*.$$

The relation (2.1) implies that for every $\mathrm{a}_{(\mathrm{n})} \in U^{*n}$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ we have

$$\begin{aligned}\varphi_{\mu\circ\pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}}(\beta) &= \varphi_\mu(\beta_1 a_1 + \dots + \beta_n a_n)\\ &= \big(\varphi_{\mu^{1/n}}(\beta_1 a_1 + \dots + \beta_n a_n)\big)^n\\ &= \left(\varphi_{\mu^{1/n}\circ\pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}}(\beta)\right)^n.\end{aligned}$$

Thus, every image measure $\mu \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}$ of an infinitely divisible cylindrical measure μ is an infinitely divisible probability measure on $\mathcal{B}(\mathbb{R}^n)$.

Remark 2.3. A probability measure μ on $\mathcal{B}(U)$ is called infinitely divisible if for each $n \in \mathbb{N}$ there exists a measure $\mu^{1/n}$ on $\mathcal{B}(U)$ such that $\mu = (\mu^{1/n})^{*n}$ (see e.g. Linde [7], section 5.1). Consequently, every infinitely divisible probability measure on $\mathcal{B}(U)$ is also an infinitely divisible cylindrical probability measure on $\mathcal{Z}(U)$.

Because $\mu\circ a^{-1}$ is an infinitely divisible probability measure on $\mathcal{B}(\mathbb{R})$ the Lévy-Khintchine formula in $\mathbb{R}$ implies that for every $a \in U^*$ there exist some constants $\beta_a \in \mathbb{R}$ and $\sigma_a \in \mathbb{R}_+$ and a Lévy measure ν_a on $\mathcal{B}(\mathbb{R})$ such that

$$\varphi_\mu(a) = \varphi_{\mu\circ a^{-1}}(1) = \exp\left(i\beta_a - \tfrac{1}{2}\sigma_a^2 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{i\gamma} - 1 - i\gamma\, \mathbb{1}_{B_1}(\gamma)\big)\, \nu_a(d\gamma)\right), \tag{2.3}$$

where $B_1 := \{\beta \in \mathbb{R} : |\beta| \leqslant 1\}$. A priori all parameters in the characteristics of the image measure $\mu \circ a^{-1}$ depend on the functional $a \in U^*$. The following result sharpens this representation.

Theorem 2.4. *Let μ be a cylindrical probability measure on $\mathcal{Z}(U)$. If μ is infinitely divisible then there exists a cylindrical measure ν on $\mathcal{Z}(U)$ such that the representation* (2.3) *is satisfied with*

$$\nu_a = \nu \circ a^{-1} \qquad \textit{for all } a \in U^*.$$

Proof. Fix $\mathrm{a}_{(\mathrm{n})} = (a_1, \dots, a_n) \in U^{*n}$ and let $\nu_{a_1,\dots,a_n}$ denote the Lévy measure on $\mathcal{B}(\mathbb{R}^n)$ of the infinitely divisible measure $\mu \circ \pi^{-1}_{a_1,\dots,a_n}$. Define the family of cylindrical sets

$$\mathcal{G} := \{Z(a_1, \dots, a_n; B) : a_1, \dots, a_n \in U^*, n \in \mathbb{N}, B \in \mathcal{F}_{\mathrm{a}_{(\mathrm{n})}}\},$$

where

$$\mathcal{F}_{\mathrm{a}_{(\mathrm{n})}} := \{(\alpha, \beta] \subseteq \mathbb{R}^n : \nu_{a_1,\dots,a_n}(\partial(\alpha, \beta]) = 0,\, 0 \notin [\alpha, \beta]\}$$

and $\partial(\alpha, \beta]$ denotes the boundary of the n-dimensional interval

$$(\alpha, \beta] := \{v = (v_1, \dots, v_n) \in \mathbb{R}^n : \alpha_i < v_i \leqslant \beta_i,\ i = 1, \dots, n\}$$

for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, \beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$.

Our proof relies on the relation

$$\lim_{t_k\to 0}\frac{1}{t_k}\int_{\mathbb{R}^n}\mathbb{1}_B(\gamma)\,(\mu\circ\pi^{-1}_{a_1,\dots,a_n})^{*t_k}(d\gamma)=\int_{\mathbb{R}^n}\mathbb{1}_B(\gamma)\,\nu_{a_1,\dots,a_n}(d\gamma). \tag{2.4}$$

for all sets $B\in\mathcal{F}_{\mathrm{a}_{(\mathrm{n})}}$. This can be deduced from Corollary 2.8.9. in [14] which states that

$$\lim_{t_k\to 0}\frac{1}{t_k}\int_{\mathbb{R}^n}f(\gamma)\,(\mu\circ\pi^{-1}_{a_1,\dots,a_n})^{*t_k}(d\gamma)=\int_{\mathbb{R}^n}f(\gamma)\,\nu_{a_1,\dots,a_n}(d\gamma) \tag{2.5}$$

for all bounded and continuous functions $f:\mathbb{R}^n\to\mathbb{R}$ which vanish on a neighborhood of 0. The relation (2.4) can be seen in the following way: let $B=(\alpha,\beta]$ be a set in $\mathcal{F}_{\mathrm{a}_{(\mathrm{n})}}$ for $\alpha,\beta\in\mathbb{R}^n$. Because $0\notin\bar{B}$ there exists $\varepsilon>0$ such that $0\notin[\alpha-\varepsilon,\beta+\varepsilon]$ where $\alpha-\varepsilon:=(\alpha_1-\varepsilon,\dots,\alpha_n-\varepsilon)$ and $\beta+\varepsilon:=(\beta_1+\varepsilon,\dots,\beta_n+\varepsilon)$. Define for $i=1,\dots,n$ the functions $g_i:\mathbb{R}\to[0,1]$ by

$$g_i(c)=\Big(1-\tfrac{(\alpha_i-c)}{\varepsilon}\Big)\,\mathbb{1}_{(\alpha_i-\varepsilon,\alpha_i]}(c)+\mathbb{1}_{(\alpha_i,\beta_i]}(c)+\Big(1-\tfrac{(c-\beta_i)}{\varepsilon}\Big)\,\mathbb{1}_{(\beta_i,\beta_i+\varepsilon]}(c),$$

and interpolate the function $\gamma\mapsto\mathbb{1}_{(\alpha,\beta]}(\gamma)$ for $\gamma=(\gamma_1,\dots,\gamma_n)$ by

$$f((\gamma_1,\dots,\gamma_n)):=g_1(\gamma_1)\cdot\ldots\cdot g_n(\gamma_n).$$

Because $\mathbb{1}_B\leqslant f\leqslant\mathbb{1}_{(\alpha-\varepsilon,\beta+\varepsilon]}$ we have

$$\frac{1}{t_k}\int_{\mathbb{R}^n}\mathbb{1}_B(\gamma)\,(\mu\circ\pi^{-1}_{a_1,\dots,a_n})^{*t_k}(d\gamma)\leqslant\frac{1}{t_k}\int_{\mathbb{R}^n}f(\gamma)\,(\mu\circ\pi^{-1}_{a_1,\dots,a_n})^{*t_k}(d\gamma)$$

and

$$\int_{\mathbb{R}^n}f(\gamma)\,\nu_{a_1,\dots,a_n}(d\gamma)\leqslant\int_{\mathbb{R}^n}\mathbb{1}_{(\alpha-\varepsilon,\beta+\varepsilon]}(\gamma)\,\nu_{a_1,\dots,a_n}(d\gamma)=\nu_{a_1,\dots,a_n}((\alpha-\varepsilon,\beta+\varepsilon]).$$

Since f is bounded, continuous and vanishes on a neighborhood of 0, it follows from (2.5) that

$$\limsup_{t_k\to 0}\frac{1}{t_k}\int_{\mathbb{R}^n}\mathbb{1}_B(\gamma)\,(\mu\circ\pi^{-1}_{a_1,\dots,a_n})^{*t_k}(d\gamma)\leqslant\nu_{a_1,\dots,a_n}((\alpha-\varepsilon,\beta+\varepsilon]). \tag{2.6}$$

By considering $(\alpha+\varepsilon,\beta-\varepsilon]\subseteq(\alpha,\beta]$ we obtain similarly that

$$\nu_{a_1,\dots,a_n}((\alpha+\varepsilon,\beta-\varepsilon])\leqslant\liminf_{t_k\to 0}\frac{1}{t_k}\int_{\mathbb{R}^n}\mathbb{1}_B(\gamma)\,(\mu\circ\pi^{-1}_{a_1,\dots,a_n})^{*t_k}(d\gamma). \tag{2.7}$$

Because $\nu_{a_1,\dots,a_n}(\partial B)=0$ the inequalities (2.6) and (2.7) imply (2.4).
Now we define a set function

$$\nu:\mathcal{Z}(U)\to[0,\infty],\qquad \nu(Z(a_1,\dots,a_n;B)):=\nu_{a_1,\dots,a_n}(B).$$

First, we show that ν is well defined. For $Z(a_1,\dots,a_n;B)\in\mathcal{G}$ equation (2.4) allows us to conclude that

$$\begin{aligned}\nu(Z(a_1,\dots,a_n;B))&=\lim_{t_k\to 0}\frac{1}{t_k}\int_{\mathbb{R}^n}\mathbb{1}_B(\gamma)\,(\mu\circ\pi^{-1}_{a_1,\dots,a_n})^{*t_k}(d\gamma)\\&=\lim_{t_k\to 0}\frac{1}{t_k}\int_{\mathbb{R}^n}\mathbb{1}_B(\gamma)\,(\mu^{*t_k}\circ\pi^{-1}_{a_1,\dots,a_n})(d\gamma)\\&=\lim_{t_k\to 0}\frac{1}{t_k}\int_U\mathbb{1}_B(\pi_{a_1,\dots,a_n}(u))\,\mu^{*t_k}(du)\\&=\lim_{t_k\to 0}\frac{1}{t_k}\mu^{*t_k}(Z(a_1,\dots,a_n;B)).\end{aligned}$$

It follows that for two sets in $\mathcal{G}$ with $Z(a_1,\dots,a_n;B) = Z(b_1,\dots,b_m;C)$ that

$$\nu(Z(a_1,\dots,a_n;B)) = \nu(Z(b_1,\dots,b_m;C)),$$

which verifies that ν is well defined on $\mathcal{G}$.
Having shown that ν is well-defined on $\mathcal{G}$ for fixed $\mathrm{a}_{(\mathrm{n})} = (a_1,\dots,a_n) \in U^{*n}$ we now demonstrate that it's restriction to the σ-algebra $\mathcal{Z}(U,\{a_1,\dots,a_n\})$ is a measure so that it yields a cylindrical measure on $\mathcal{Z}(U)$.
Define a set of n-dimensional intervals by

$$\mathcal{H} := \{(\alpha,\beta] \subseteq \mathbb{R}^n : 0 \notin [\alpha,\beta]\}.$$

Because $\nu_{a_1,\dots,a_n}$ is a σ-finite measure the set

$$\mathcal{H}\backslash\mathcal{F}_{\mathrm{a}_{(\mathrm{n})}} = \{(\alpha,\beta] \in \mathcal{H} : \nu_{a_1,\dots,a_n}(\partial(\alpha,\beta]) \neq 0\}$$

is countable. Thus, the set $\mathcal{F}_{\mathrm{a}_{(\mathrm{n})}}$ generates the same σ-algebra as $\mathcal{H}$ because the countably missing sets in $\mathcal{F}_{\mathrm{a}_{(\mathrm{n})}}$ can easily be approximated by sets in $\mathcal{F}_{\mathrm{a}_{(\mathrm{n})}}$. But $\mathcal{H}$ is known to be a generator of the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ and so Lemma 2.1 yields that

$$\mathcal{G}_{\mathrm{a}_{(\mathrm{n})}} := \{Z(a_1,\dots,a_n;B) : B \in \mathcal{F}_{\mathrm{a}_{(\mathrm{n})}}\}$$

generates $\mathcal{Z}(U,\{a_1,\dots,a_n\})$.
Furthermore, $\mathcal{G}_{\mathrm{a}_{(\mathrm{n})}}$ is a semi-ring because $\mathcal{F}_{\mathrm{a}_{(\mathrm{n})}}$ is a semi-ring. Secondly, ν restricted to $\mathcal{G}_{\mathrm{a}_{(\mathrm{n})}}$ is well defined and is a pre-measure. For, if $\{Z_k := Z_k(a_1,\dots,a_n;B_k) : k \in \mathbb{N}\}$ are a countable collection of disjoint sets in $\mathcal{G}_{\mathrm{a}_{(\mathrm{n})}}$ with $\cup Z_k \in \mathcal{G}_{\mathrm{a}_{(\mathrm{n})}}$ then the Borel sets B_k are disjoint and it follows that

$$\begin{aligned}\nu\left(\bigcup_{k\geqslant 1} Z_k\right) &= \nu\left(\bigcup_{k\geqslant 1} \pi^{-1}_{a_1,\dots,a_n}(B_k)\right) = \nu\left(\pi^{-1}_{a_1,\dots,a_n}\left(\bigcup_{k\geqslant 1} B_k\right)\right)\\ &= \nu_{a_1,\dots,a_n}\left(\bigcup_{k\geqslant 1} B_k\right) = \sum_{k=1}^{\infty}\nu_{a_1,\dots,a_n}(B_k) = \sum_{k=1}^{\infty}\nu(Z_k).\end{aligned}$$

Thus, ν restricted to $\mathcal{G}_{\mathrm{a}_{(\mathrm{n})}}$ is a pre-measure and because it is σ-finite it can be extended uniquely to a measure on $\mathcal{Z}(U,\{a_1,\dots,a_n\})$ by Caratheodory's extension theorem, which verifies that ν is a cylindrical measure on $\mathcal{Z}(U)$. □

By the construction of the cylindrical measure ν in Theorem 2.4 it folllows that every image measure $\nu \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}$ is a Lévy measure on $\mathcal{B}(\mathbb{R}^n)$ for all $\mathrm{a}_{(\mathrm{n})} \in U^{*n}$. This motivates the following definition:

Definition 2.5. *A cylindrical measure ν on $\mathcal{Z}(U)$ is called a* cylindrical Lévy measure *if for all $a_1,\dots,a_n \in U^*$ and $n \in \mathbb{N}$ the measure $\nu \circ \pi^{-1}_{a_1,\dots,a_n}$ is a Lévy measure on $\mathcal{B}(\mathbb{R}^n)$.*

Remark 2.6. Let ν be a Lévy measure ν on $\mathcal{B}(U)$ (see [7] for a definition). Then, if Definition 2.5 is sensible ν should be also a *cylindrical* Lévy measure. That this is true, we explain in the following.
According to Proposition 5.4.5 in [7] the Lévy measure ν satisfies

$$\sup_{\|a\|\leqslant 1}\int_{\|u\|\leqslant 1} |\langle u,a\rangle|^2\,\nu(du) < \infty. \tag{2.8}$$

This result can be generalised to

$$\sup_{\|a\|\leqslant 1} \int_{\{u:|\langle u,a\rangle|\leqslant 1\}} |\langle u,a\rangle|^2 \, \nu(du) < \infty. \tag{2.9}$$

For, the result (2.8) relies on Proposition 5.4.1 in [7] which is based on Lemma 5.3.10 therein. In the latter the set $\{u : \|u\| \leqslant 1\}$ can be replaced by the larger set $\{u : |\langle u,a\rangle| \leqslant 1\}$ for $a \in U^*$ with $\|a\| \leqslant 1$ because in the proof the inequality (line -10, page 72 in [7])

$$1 - \cos t \geqslant \tfrac{t^2}{3} \qquad \text{for all } |t| \leqslant 1,$$

is applied for $t = \|u\|$ while we apply it for $t = |\langle u,a\rangle|$. Then we can follow the original proof in [7] to obtain (2.9). From (2.9) it is easy to derive

$$\sup_{\|a\|\leqslant M} \int_{\{u:|\langle u,a\rangle|\leqslant N\}} |\langle u,a\rangle|^2 \, \nu(du) < \infty \tag{2.10}$$

for all $M, N \geqslant 0$.
For arbitrary $\mathrm{a}_{(\mathrm{n})} = (a_1, \dots, a_n) \in U^{*n}$ and $B_n := \{\beta \in \mathbb{R}^n : |\beta| \leqslant 1\}$ we have that

$$\begin{aligned}\pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}(B_n) = \{u : \langle u,a_1\rangle^2 + \dots + \langle u,a_n\rangle^2 \leqslant 1\} &\subseteq \{u : \tfrac{1}{n}(\langle u,a_1\rangle + \dots + \langle u,a_n\rangle)^2 \leqslant 1\} \\ &= \{u : |\langle u,(a_1 + \dots + a_n)\rangle| \leqslant \sqrt{n}\,\} =: D,\end{aligned}$$

where we used the inequality $(\gamma_1 + \dots + \gamma_n)^2 \leqslant n(\gamma_1^2 + \dots + \gamma_n^2)$ for $\gamma_1, \dots, \gamma_n \in \mathbb{R}$. It follows from (2.10) that

$$\int_{B_n} |\beta|^2 \, (\nu \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}})(d\beta) = \sum_{k=1}^n \int_{\pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}(B_n)} |\langle u,a_k\rangle|^2 \, \nu(du) \leqslant \sum_{k=1}^n \int_D |\langle u,a_k\rangle|^2 \, \nu(du) < \infty.$$

As a result we obtain that ν is a cylindrical Lévy measure on $\mathcal{B}(\mathbb{R}^n)$.

In the next section we will sharpen the structure of the Lévy-Khintchine formula for infinitely divisible cylindrical measures. It is appropriate to state the result at this juncture:

Theorem 2.7. *Let μ be an infinitely divisible cylindrical probability measure. Then there exist a map $r : U^* \to \mathbb{R}$, a quadratic form $s : U^* \to \mathbb{R}$ and a cylindrical Lévy measure ν on $\mathcal{Z}(U)$ such that:*

$$\varphi_\mu(a) = \exp\left(ir(a) - \tfrac{1}{2}s(a) + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\gamma} - 1 - i\gamma\, \mathbb{1}_{B_1}(\gamma)\right)(\nu \circ a^{-1})(d\gamma) \right)$$

for all $a \in U^*$.

3 Cylindrical stochastic processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space that is equipped with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$.
Similarly to the correspondence between measures and random variables there is an analogous random object associated to cylindrical measures:

Definition 3.1. *A* cylindrical random variable Y in U *is a linear map*

$$Y : U^* \to L^0(\Omega, \mathcal{F}, P).$$

A cylindrical process X in U is a family $(X(t) : t \geqslant 0)$ of cylindrical random variables in U.

The characteristic function of a cylindrical random variable X is defined by

$$\varphi_X : U^* \to \mathbb{C}, \qquad \varphi_X(a) = E[\exp(iXa)].$$

The concepts of cylindrical measures and cylindrical random variables match perfectly. Indeed, if $Z = Z(a_1, \dots, a_n; B)$ is a cylindrical set for $a_1, \dots, a_n \in U^*$ and $B \in \mathcal{B}(\mathbb{R}^n)$ we obtain a cylindrical probability measure μ by the prescription

$$\mu(Z) := P((Xa_1, \dots, Xa_n) \in B). \tag{3.11}$$

We call μ the *cylindrical distribution of* X and the characteristic functions φ_μ and φ_X of μ and X coincide. Conversely for every cylindrical measure μ on $\mathcal{Z}(U)$ there exists a probability space $(\Omega, \mathcal{F}, P)$ and a cylindrical random variable $X : U^* \to L^0(\Omega, \mathcal{F}, P)$ such that μ is the cylindrical distribution of X, see [16, VI.3.2].
By some abuse of notation we define for a cylindrical process $X = (X(t) : t \geqslant 0)$:

$$X(t) : U^{*n} \to L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n), \qquad X(t)(a_1, \dots, a_n) := (X(t)a_1, \dots, X(t)a_n).$$

In this way, one obtains for fixed $(a_1, \dots, a_n) \in U^{*n}$ an n-dimensional stochastic process

$$(X(t)(a_1, \dots, a_n) : t \geqslant 0).$$

It follows from (3.11) that its marginal distribution is given by the image measure of the cylindrical distribution μ_t of $X(t)$:

$$P_{X(t)(a_1, \dots, a_n)} = \mu_t \circ \pi^{-1}_{a_1, \dots, a_n} \tag{3.12}$$

for all $a_1, \dots, a_n \in U^{*n}$. Combining (3.12) with (2.1) shows that

$$\varphi_{X(t)(a_1, \dots, a_n)}(\beta_1, \dots, \beta_n) = \varphi_{X(t)(\beta_1 a_1 + \dots + \beta_n a_n)}(1) \tag{3.13}$$

for all $\beta_1, \dots, \beta_n \in \mathbb{R}^n$ and $a_1, \dots, a_n \in U^*$.

We give now the proof of Theorem 2.7.

Proof. (of Theorem 2.7).
Because of (2.3), i.e.

$$\varphi_\mu(a) = \varphi_{\mu \circ a^{-1}}(1) = \exp\left(i\beta_a - \tfrac{1}{2}\sigma_a^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\gamma} - 1 - i\gamma\, \mathbb{1}_{B_1}(\gamma)\right) \nu_a(d\gamma) \right),$$

we have to show that $\varphi_{\mu \circ a^{-1}}$ is in the claimed form. Theorem 2.4 implies that there exists a cylindrical Lévy measure ν such that $\nu_a = \nu \circ a^{-1}$ for each $a \in U^*$. By defining $r(a) := \beta_a$ it remains to show that the function

$$s : U^* \to \mathbb{R}_+, \qquad s(a) := \sigma_a^2$$

is a quadratic form. Let X be a cylindrical random variable with distribution μ. By the Lévy-Itô decomposition in $\mathbb{R}$ (see e.g. Chapter 2 in [1]) it follows that

$$Xa = r(a) + \sigma_a W_a + \int_{0<|\beta|<1} \beta\, \tilde{N}_a(d\beta) + \int_{|\beta|\geqslant 1} \beta\, N_a(d\beta) \qquad P\text{-a.s.}, \tag{3.14}$$

where W_a is a real valued centred Gaussian random variable with $EW_a^2 = 1$, N_a is an independent Poisson random measure on $\mathbb{R}\backslash\{0\}$ and $\tilde{N}_a$ is the compensated Poisson random measure.
By applying (3.14) to Xa, Xb and $X(a+b)$ for arbitrary $a, b \in U^*$ we obtain

$$\sigma_{a+b} W_{a+b} = \sigma_a W_a + \sigma_b W_b \quad P\text{-a.s.} \tag{3.15}$$

Similarly, for $\beta \in \mathbb{R}$ we have

$$\sigma_{\beta a} W_{\beta a} = \beta \sigma_a W_a \quad P\text{-a.s.} \tag{3.16}$$

By squaring both sides of (3.16) and then taking expectations it follows that the function s satisfies $s(\beta a) = \beta^2 s(a)$. Similarly, one derives from (3.15) that $\sigma_{a+b}^2 = \sigma_a^2 + \sigma_b^2 + 2\rho(a,b)$, where $\rho(a,b) := Cov(\sigma_a W_a, \sigma_b W_b)$. Equation (3.15) yields for $c \in U^*$

$$\begin{aligned}\rho(a+c,b) &= \mathrm{Cov}(\sigma_{a+c} W_{a+c},\, \sigma_b W_b)\\ &= \mathrm{Cov}(\sigma_a W_a + \sigma_c W_c,\, \sigma_b W_b)\\ &= \rho(a,b) + \rho(c,b),\end{aligned}$$

which implies together with properties of the covariance that ρ is a bilinear form. Thus the function

$$Q: U^* \times U^* \to \mathbb{R}, \qquad Q(a,b) := s(a+b) - s(a) - s(b) = 2\rho(a,b) \tag{3.17}$$

is a bilinear form and s is thus a quadratic form. □

The cylindrical process $X = (X(t) : t \geqslant 0)$ is called *adapted to a given filtration* $\{\mathcal{F}_t\}_{t\geqslant 0}$, if $X(t)a$ is $\mathcal{F}_t$-measurable for all $t \geqslant 0$ and all $a \in U^*$. The cylindrical process X is said to have *weakly independent increments* if for all $0 \leqslant t_0 < t_1 < \dots < t_n$ and all $a_1, \dots, a_n \in U^*$ the random variables

$$(X(t_1) - X(t_0))a_1, \dots, (X(t_n) - X(t_{n-1}))a_n$$

are independent.

Definition 3.2. *An adapted cylindrical process* $(L(t) : t \geqslant 0)$ *is called a* weakly cylindrical Lévy process *if*

(a) *for all* $a_1, \dots, a_n \in U^*$ *and* $n \in \mathbb{N}$ *the stochastic process* $\big((L(t)(a_1, \dots, a_n) : t \geqslant 0\big)$ *is a Lévy process in* $\mathbb{R}^n$.

By Definition 3.2 the random variable $L(1)(a_1, \dots, a_n)$ is infinitely divisible for all $a_1, \dots, a_n \in U^*$ and the equation (3.12) implies that the cylindrical distribution of $L(1)$ is an infinitely divisible cylindrical measure.

Example 3.3. An adapted cylindrical process $(W(t) : t \geqslant 0)$ in U is called a *weakly cylindrical Wiener process*, if for all $a_1, \dots, a_n \in U^*$ and $n \in \mathbb{N}$ the $\mathbb{R}^n$-valued stochastic process

$$\big((W(t)(a_1, \dots, a_n) : t \geqslant 0\big)$$

is a Wiener process in $\mathbb{R}^n$. Here we call an adapted stochastic process $(X(t) : t \geqslant 0)$ in $\mathbb{R}^n$ a Wiener process if the increments $X(t) - X(s)$ are independent, stationary and normally distributed with expectation $E[X(t) - X(s)] = 0$ and covariance $\mathrm{Cov}[X(t) - X(s), X(t) - X(s)] = |t - s|\, C$ for a non-negative definite symmetric matrix C. If $C = \mathrm{Id}$ we call X a *standard* Wiener process. Obviously, a weakly cylindrical Wiener process is an example of a weakly cylindrical Lévy process. The characteristic function of W is given by

$$\varphi_{W(t)}(a) = \exp\left(-\tfrac{1}{2} t s(a)\right),$$

where $s : U^* \to \mathbb{R}_+$ is a quadratic form, see [11] for more details on cylindrical Wiener processes.

Example 3.4. Let ζ be an element in the algebraic dual $U^{*\prime}$, i.e. a linear function $\zeta : U^* \to \mathbb{R}$ which is not necessarily continuous. Then

$$X : U^* \to L^0(\Omega, \mathcal{F}, P), \qquad Xa := \zeta(a)$$

defines a cylindrical random variable. We call its cylindrical distribution μ a *cylindrical Dirac measure in* ζ. It follows that

$$\varphi_X(a) = \varphi_\mu(a) = e^{i\zeta(a)} \qquad \text{for all } a \in U^*.$$

We define the *cylindrical Poisson process* $(L(t) : t \geqslant 0)$ by

$$L(t)a := \zeta(a)\, n(t) \qquad \text{for all } t \geqslant 0,$$

where $(n(t) : t \geqslant 0)$ is a real valued Poisson process with intensity $\lambda > 0$. It turns out that the cylindrical Poisson process is another example of a weakly cylindrical Lévy process with characteristic function

$$\varphi_{L(t)}(a) = \exp\left(\lambda t \left(e^{i\zeta(a)} - 1\right)\right).$$

Example 3.5. Let $(Y_k : k \in \mathbb{N})$ be a sequence of cylindrical random variables each having cylindrical distribution ρ and such that $\{Y_k a : k \in \mathbb{N}\}$ is independent for all $a \in U^*$. If $(n(t) : t \geqslant 0)$ is a real valued Poisson process of intensity $\lambda > 0$ which is independent of $\{Y_k a : k \in \mathbb{N},\, a \in U^*\}$ then the *cylindrical compound Poisson process* $(L(t) : t \geqslant 0)$ is defined by

$$L(t)a := \begin{cases} 0, & \text{if } t = 0, \\ Y_1 a + \dots + Y_{n(t)} a, & \text{else,} \end{cases} \qquad \text{for all } a \in U^*.$$

The cylindrical compound Poisson process is a weakly cylindrical Lévy process with

$$\varphi_{L(t)}(a) = \exp\left(t\lambda \int_U \left(e^{i\langle u, a\rangle} - 1\right) \rho(du)\right).$$

Example 3.6. Let ρ be a Lévy measure on $\mathbb{R}$ and λ be a positive measure on a set $O \subseteq \mathbb{R}^d$. In the monograph [8] by Peszat and Zabczyk an *impulsive cylindrical process on* $L^2(O, \mathcal{B}(O), \lambda)$ is introduced in the following way: let π be the Poisson random measure on $[0,\infty)\times O\times\mathbb{R}$ with intensity measure $ds\,\lambda(d\xi)\,\rho(d\beta)$. Then for all measurable functions $f : O \to \mathbb{R}$ with compact support a random variable is defined by

$$Z(t)f := \int_0^t \int_O \int_{\mathbb{R}} f(\xi)\beta\,\tilde{\pi}(ds, d\xi, d\beta)$$

in $L^2(\Omega, \mathcal{F}, P)$ under the simplifying assumption that

$$\int_{\mathbb{R}} \beta^2\,\rho(d\beta) < \infty.$$

It turns out that the definition of $Z(t)$ can be extended to all f in $L^2(O, \mathcal{B}(O), \lambda)$ so that $Z = (Z(t) : t \geqslant 0)$ is a cylindrical process in the Hilbert space $L^2(O, \mathcal{B}(O), \lambda)$. Moreover, $(Z(t)f : t \geqslant 0)$ is a Lévy process for every $f \in L^2(O, \mathcal{B}(O), \lambda)$ and Z has the characteristic function

$$\varphi_{Z(t)}(f) = \exp\left(t \int_O \int_{\mathbb{R}\setminus\{0\}} \left(e^{if(\xi)\beta} - 1 - if(\xi)\beta \right)\,\rho(d\beta)\,\lambda(d\xi) \right), \tag{3.18}$$

see Prop. 7.4 in [8].
To consider this example in our setting we set $U = L^2(O, \mathcal{B}(O), \lambda)$ and identify U^* with U. By the results mentioned above and if we assume weakly independent increments, Lemma 3.8 tells us that the cylindrical process Z is a weakly cylindrical Lévy process in accordance with our Definition 3.2. By Corollary 2.7 it follows that there exists a cylindrical Lévy measure ν on $\mathcal{Z}(U)$ such that $\nu \circ f^{-1}$ is the Lévy measure of $(Z(t)f : t \geqslant 0)$ for all $f \in U^*$. But on the other hand, if we define a measure by

$$\nu_f : \mathcal{B}(\mathbb{R}) \to [0,\infty], \qquad \nu_f(B) := \int_O \int_{\mathbb{R}} \mathbb{1}_B(\beta f(\xi))\,\rho(d\beta)\lambda(d\xi)$$

we can rewrite (3.18) as

$$\varphi_{Z(t)}(f) = \exp\left(t \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\beta} - 1 - i\beta \right)\,v_f(d\beta) \right)$$

and by the uniqueness of the characteristics of a Levy process we see that $v_f = v \circ f^{-1}$ for all $f \in U^*$.

Example 3.7. A cylindrical process $(L(t) : t \geqslant 0)$ is induced by a stochastic process $(X(t) : t \geqslant 0)$ on U if

$$L(t)a = \langle X(t), a\rangle \qquad \text{for all } a \in U^*.$$

If X is a Lévy process on U then the induced process L is a weakly cylindrical Lévy process with the same characteristic function as X.

Our definition of a weakly cylindrical Lévy process is an obvious extension of the definition of a finite-dimensional Lévy processes and is exactly in the spirit of cylindrical processes. The multidimensional formulation in Definition 3.2 would already be necessary to define a finite-dimensional Lévy process by this approach and it allows us to conclude that a weakly cylindrical Lévy process has weakly independent increments. The latter property is exactly what is needed in addition to a one-dimensional formulation:

Lemma 3.8. *For an adapted cylindrical process $L = (L(t) : t \geqslant 0)$ the following are equivalent:*

(a) *L is a weakly cylindrical Lévy process;*

(b) (i) *L has weakly independent increments;*

(ii) *$(L(t)a : t \geqslant 0)$ is a Lévy process for all $a \in U^*$.*

Proof. We have only to show that (b) implies (a) for which we fix some $a_1, \dots, a_n \in U^*$. Because (3.13) implies that the characteristic functions satisfy

$$\varphi_{(L(t)-L(s))(a_1,\dots,a_n)}(\beta) = \varphi_{(L(t)-L(s))(\beta_1 a_1+\dots+\beta_n a_n)}(1)$$

for all $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ the condition (ii) implies that the increments of $((L(t)a_1, \dots, L(t)a_n) : t \geqslant 0)$ are stationary. The assumption (i) implies that

$$(L(t_1) - L(t_0))a_{k_1}, \dots, (L(t_n) - L(t_{n-1}))a_{k_n}$$

are independent for all $k_1, \dots, k_n \in \{1, \dots, n\}$ and all $0 \leqslant t_0 < \dots < t_n$. If follows that the n-dimensional random variables

$$(L(t_1) - L(t_0))(a_1, \dots, a_n), \dots, (L(t_n) - L(t_{n-1}))(a_1, \dots, a_n)$$

are independent which shows the independent increments of $(L(t)(a_1, \dots, a_n) : t \geqslant 0)$. The stochastic continuity follows by the following estimate, where we use $|\cdot|_n$ to denote the Euclidean norm in $\mathbb{R}^n$ and $c > 0$:

$$P(|(L(t)a_1, \dots, L(t)a_n)|_n > c) = P\left(|L(t)a_1|^2 + \dots + |L(t)a_n|^2 > c^2\right) \leqslant \sum_{k=1}^{n} P\left(|L(t)a_k| > \tfrac{c}{\sqrt{n}}\right),$$

which completes the proof. □

Because $(L(t)a : t \geqslant 0)$ is a one-dimensional Lévy process, we may take a càdlàg version (see e.g. Chapter 2 of [1]). Then for every $a \in U^*$ the one-dimensional Lévy-Itô decomposition implies P-a.s.

$$L(t)a = \zeta_a t + \sigma_a W_a(t) + \int_{0<|\beta|\leqslant 1} \beta\, \tilde{N}_a(t, d\beta) + \int_{|\beta|>1} \beta\, N_a(t, d\beta), \tag{3.19}$$

where $\zeta_a \in \mathbb{R}$, $\sigma_a \geqslant 0$, $(W_a(t) : t \geqslant 0)$ is a real valued standard Wiener process and N_a is the Poisson random measure defined by

$$N_a(t, B) = \sum_{0\leqslant s\leqslant t} \mathbb{1}_B(\Delta L(s)a) \qquad \text{for } B \in \mathcal{B}(\mathbb{R}\backslash\{0\}),$$

where $\Delta(f(s)) := f(s) - f(s-)$ for any càdlàg function $f : \mathbb{R} \to \mathbb{R}$. The Poisson random measure N_a gives rise to the Lévy measure ν_a by

$$\nu_a(B) := E[N_a(1, B)] \qquad \text{for } B \in \mathcal{B}(\mathbb{R}\backslash\{0\}).$$

The compensated Poisson random measure $\tilde{N}_a$ is then defined by

$$\tilde{N}_a(t, B) := N_a(t, B) - t\nu_a(B).$$

Note, that all terms in the sum on the right hand side of (3.19) are independent for each fixed $a \in U^*$. Combining with the Lévy-Khintchine formula in Theorem 2.7 yields that

$$\zeta_a = r(a), \qquad \sigma_a = s(a) \qquad \text{and} \quad \nu_a = \nu \circ a^{-1}$$

for all $a \in U^*$, where r, s and ν are the characteristics associated to the infinitely divisible cylindrical distribution of $L(1)$.
By using the Lévy-Itô decomposition (3.19) for the one-dimensional projections we define for each $t \geqslant 0$

$$\begin{aligned}
&W(t) : U^* \to L^2(\Omega, \mathcal{F}, P), \qquad W(t)a := s(a)W_a(t),\\
&M(t) : U^* \to L^2(\Omega, \mathcal{F}, P), \qquad M(t)a := \int_{0<|\beta|\leqslant 1} \beta\, \tilde{N}_a(t, d\beta),\\
&P(t) : U^* \to L^0(\Omega, \mathcal{F}, P), \qquad P(t)a := \int_{|\beta|>1} \beta\, N_a(t, d\beta).
\end{aligned}$$

The one-dimensional Lévy-Itô decomposition (3.19) is now of the form

$$L(t)a = r(a)t + W(t)a + M(t)a + P(t)a \qquad \text{for all } a \in U^*. \tag{3.20}$$

Theorem 3.9. *Let $L = (L(t) : t \geqslant 0)$ be a weakly cylindrical Lévy process in U. Then L satisfies* (3.20) *(almost surely) where*

$$\begin{aligned}
&(W(t) : t \geqslant 0) \quad \textit{is a weakly cylindrical Wiener process},\\
&(r(\cdot)t + M(t) + P(t) : t \geqslant 0) \quad \textit{is a cylindrical process.}
\end{aligned}$$

Proof. By (3.20) we know that

$$L(t)a = r(a)t + W(t)a + R(t)a \qquad \text{for all } a \in U^*,$$

where $R(t)a = M(t)a + P(t)a$. By applying this representation to every component of the n-dimensional stochastic process $(L(t)(a_1, \dots, a_n) : t \geqslant 0)$ for $a_1, \dots, a_n \in U^*$ we obtain

$$L(t)(a_1, \dots, a_n) = (r(a_1), \dots, r(a_n))t + (W(t)a_1, \dots, W(t)a_n) + (R(t)a_1, \dots, R(t)a_n).$$

But on the other hand the n-dimensional Lévy process $(L(t)(a_1, \dots, a_n) : t \geqslant 0)$ also has a Lévy-Itô decomposition where the Gaussian part is an $\mathbb{R}^n$-valued Wiener process. By uniqueness of the decomposition it follows that the Gaussian part equals $((W(t)a_1, \dots, W(t)a_n) : t \geqslant 0)$ (a.s.) which ensures that the latter is indeed a weakly cylindrical Wiener process (see the definition in Example 3.3).
Because L and W are cylindrical processes it follows that $a \mapsto r(a)t + M(t)a + P(t)a$ is also linear which completes the proof. □

One might expect that the random functions P and M are also cylindrical processes, i.e. linear mappings. But the following example shows that this is not true in general:

Example 3.10. Let $(L(t) : t \geqslant 0)$ be the cylindrical Poisson process from Example 3.4. We obtain

$$\begin{aligned}
N_a(t, B) &= \sum_{s \in [0,t]} \mathbb{1}_B(\zeta(a)\Delta n(s))\\
&= \mathbb{1}_B(\zeta(a))\, n(t)
\end{aligned}$$

for all $a \in U^*$ and $B \in \mathcal{B}(\mathbb{R}\backslash\{0\})$. The image measures $\nu \circ a^{-1}$ of the cylindrical Lévy measure ν of L are given by

$$\nu \circ a^{-1}(B) = E[N_a(1, B)] = \mathbb{1}_B(\zeta(a))\, \lambda.$$

Then we have

$$\begin{aligned} P(t)a = \int_{|\beta|>1} \beta\, N_a(t, d\beta) &= \sum_{s\in[0,t]} \Delta L(s)\, \mathbb{1}_{\{|\beta|>1\}}(\Delta L(s)) \\ &= \zeta(a) \sum_{s\in[0,t]} \Delta n(s)\, \mathbb{1}_{\{|\beta|>1\}}(\zeta(a)\Delta n(s)) \\ &= \zeta(a) n(t)\, \mathbb{1}_{\{|\beta|>1\}}(\zeta(a)). \end{aligned}$$

We obtain analogously that

$$\begin{aligned} M(t)a = \int_{|\beta|\leqslant 1} \beta\, \tilde{N}_a(t, d\beta) &= \int_{|\beta|\leqslant 1} \beta\, N_a(t, d\beta) - t \int_{|\beta|\leqslant 1} \beta\, (\nu \circ a^{-1})(d\beta) \\ &= \zeta(a)(n(t) - t\lambda)\, \mathbb{1}_{\{|\beta|\leqslant 1\}}(\zeta(a)). \end{aligned}$$

Defining the term r by

$$r(a) = \lambda\, \mathbb{1}_{\{|\beta|\leqslant 1\}}(\zeta(a))$$

gives the Lévy-Itô decomposition (3.20). But it is easy to see that none of the terms $P(t), M(t)$ and r is linear because the truncation function

$$a \mapsto \mathbb{1}_{\{|\beta|\leqslant 1\}}(\zeta(a))$$

is not linear.
For an arbitrary truncation function $h_a : \mathbb{R} \to \mathbb{R}_+$ which might even depend on $a \in U^*$ a similar calculation shows the non-linearity of the analogous terms.

Example 3.11. Let $(L(t) : t \geqslant 0)$ be the cylindrical compound Poisson process introduced in Example 3.5. If we define for $a \in U^*$ a sequence of stopping times recursively by $T_0^a := 0$ and $T_n^a := \inf\{t > T_{n-1}^a : |\Delta L(t)a| > 1\}$ then it follows that

$$\int_{|\beta|>1} \beta\, N_a(t, d\beta) = J_1(a) + \cdots + J_{N_a(t, B_1^c)}(a),$$

where $B_1^c = \{\beta \in \mathbb{R} : |\beta| > 1\}$ and

$$J_n(a) := \int_{|\beta|>1} \beta\, N_a(T_n^a, d\beta) - \int_{|\beta|>1} \beta\, N_a(T_{n-1}^a, d\beta).$$

We say that a cylindrical Lévy process $(L(t), t \geq 0)$ is *weak order* 2 if $E\,|L(t)a|^2 < \infty$ for all $a \in U^*$ and $t \geqslant 0$. In this case, we can decompose L according to

$$L(t)a = r_2(a)t + W(t)a + M_2(t)a \qquad \text{for all } a \in U^*, \tag{3.21}$$

where $r_2(a) = r(a) + \int_{|\beta|>1} \beta\, \nu_a(d\beta)$ and

$$M_2(t) : U^* \to L^2(\Omega, \mathcal{F}, P), \qquad M_2(t)a := \int_{\mathbb{R}\backslash\{0\}} \beta\, \tilde{N}_a(t, d\beta).$$

In this representation it turns out that all terms are linear:

Corollary 3.12. *Let $L = (L(t) : t \geqslant 0)$ be a weakly cylindrical Lévy process of weak order 2 on U. Then L satisfies* (3.21) *with*

$$\begin{aligned} &r_2 : U^* \to \mathbb{R} \quad \textit{linear}, \\ &(W(t) : t \geqslant 0) \quad \textit{is a weakly cylindrical Wiener process}, \\ &(M_2(t)) : t \geqslant 0) \quad \textit{is a cylindrical process.} \end{aligned}$$

Proof. Let $a, b \in U^*$ and $\gamma \in \mathbb{R}$. Taking expectation in (3.21) yields

$$r_2(\gamma a + b)t = E[L(t)(\gamma a + b)] = \gamma E[L(t)a] + E[L(t)b] = \gamma r_2(a)t + r_2(b)t.$$

Thus, r_2 is linear and since also W and L in (3.21) are linear it follows that M_2 is a cylindrical process. □

But our next example shows that the assumption of finite second moments is not necessary for a "cylindrical" version of the Lévy-Itô decomposition:

Example 3.13. Let $(L(t) : t \geqslant 0)$ be a weakly cylindrical Lévy process which is induced by a Lévy process $(X(t) : t \geqslant 0)$ on U, i.e.

$$L(t)a = \langle X(t), a\rangle \qquad \text{for all } a \in U^*, t \geqslant 0.$$

The Lévy process X can be decomposed according to

$$X(t) = rt + W(t) + \int_{0<\|u\|\leqslant 1} u\,\tilde{Y}(t, du) + \int_{\|u\|>1} uY(t, du),$$

where $r \in U$, $(W(t) : t \geqslant 0)$ is an U-valued Wiener process and

$$Y(t, C) = \sum_{s\in[0,t]} \mathbb{1}_C(\Delta X(s)) \qquad \text{for } C \in \mathcal{B}(U),$$

see [12]. Obviously, the cylindrical Lévy process L is decomposed according to

$$L(t)a = \langle r, a\rangle t + \langle W(t), a\rangle + \langle \int_{0<\|u\|\leqslant 1} u\,\tilde{Y}(t, du), a\rangle + \langle \int_{\|u\|>1} uY(t, du), a\rangle,$$

for all $a \in U^*$. All terms appearing in this decomposition are linear even for a Lévy process X without existing weak second moments, i.e. with $E\langle X(1), a\rangle^2 = \infty$.
More specificially and for comparison with Example 3.10 let $(X(t) : t \geqslant 0)$ be a Poisson process on U, i.e. $X(t) = u_0 n(t)$ where $u_0 \in U$ and $(n(t) : t \geqslant 0)$ is a real valued Poisson process with intensity $\lambda > 0$. Then we obtain

$$\int_{0<\|u\|\leqslant 1} u\,\tilde{Y}(t, du) = \begin{cases} 0, & \|u_0\| > 1, \\ (n(t) - \lambda t)u_0, & \|u_0\| \leqslant 1. \end{cases}$$

4 Integration

For the rest of this paper we will always assume that our cylindrical Lévy process $(L(t), t \geq 0)$ is *weakly càdlàg*, i.e. the one-dimensional Lévy processes $(L(t)a, t \geq 0)$ are càdlàg for all $a \in U^*$.

4.1 Covariance operator

Let L be a weakly cylindrical Lévy process of weak order 2 with decomposition (3.21). Then the prescription

$$M_2(t) : U^* \to L^2(\Omega, \mathcal{F}, P), \qquad M_2(t)a = \int_{\mathbb{R}\setminus\{0\}} \beta \, \tilde{N}_a(t, d\beta) \tag{4.22}$$

defines a cylindrical process $(M_2(t) : t \geqslant 0)$ which has weak second moments. Thus, we can define the covariance operators:

$$\begin{aligned} Q_2(t) : U^* \to U^{*\prime}, \qquad (Q_2(t)a)(b) &= E\left[(M_2(t)a)(M_2(t)b)\right] \\ &= E\left[\left(\int_{\mathbb{R}\setminus\{0\}} \beta \, \tilde{N}_a(t, d\beta)\right)\left(\int_{\mathbb{R}\setminus\{0\}} \beta \, \tilde{N}_b(t, d\beta)\right)\right], \end{aligned}$$

where $U^{*\prime}$ denotes the algebraic dual of U^*. In general one can not assume that the image $Q_2(t)a$ is in the bidual space U^{**} or even U as one might expect for ordinary U-valued stochastic processes with weak second moments. We give a counterexample for that fact after we know that there is no need to consider all times t:

Lemma 4.1. *We have $Q_2(t) = tQ_2(1)$ for all $t \geqslant 0$.*

Proof. The characteristic function of the 2-dimensional random variable $(M_2(t)a, M_2(t)b)$ satisfies for all $\beta_1, \beta_2 \in \mathbb{R}$:

$$\begin{aligned} \varphi_{M_2(t)a, M_2(t)b}(\beta_1, \beta_2) &= E\left[\exp\left(i(\beta_1 M_2(t)a + \beta_2 M_2(t)b)\right)\right] \\ &= E\left[\exp(iM_2(t)(\beta_1 a + \beta_2 b))\right] \\ &= \left(E\left[\exp(iM_2(1)(\beta_1 a + \beta_2 b))\right]\right)^t \\ &= \left(\varphi_{M_2(1)a, M_2(1)b}(\beta_1, \beta_2)\right)^t. \end{aligned}$$

This relation enables us to calculate

$$\begin{aligned} &\frac{\partial}{\partial \beta_2}\frac{\partial}{\partial \beta_1}\varphi_{M_2(t)a, M_2(t)b}(\beta_1, \beta_2) \\ &\quad = \frac{\partial}{\partial \beta_2}\frac{\partial}{\partial \beta_1}\left(\varphi_{M_2(1)a, M_2(1)b}(\beta_1, \beta_2)\right)^t \\ &\quad = t(t-1)\left(\varphi_{M_2(1)a, M_2(1)b}(\beta_1, \beta_2)\right)^{t-2}\frac{\partial}{\partial \beta_2}\varphi_{M_2(1)a, M_2(1)b}(\beta_1, \beta_2)\frac{\partial}{\partial \beta_1}\varphi_{M_2(1)a, M_2(1)b}(\beta_1, \beta_2) \\ &\qquad\quad + t\left(\varphi_{M_2(1)a, M_2(b)}(\beta_1, \beta_2)\right)^{t-1}\frac{\partial}{\partial \beta_2}\frac{\partial}{\partial \beta_1}\varphi_{M_2(1)a, M_2(1)b}(\beta_1, \beta_2). \end{aligned}$$

By recalling that

$$\frac{\partial}{\partial \beta_1}\varphi_{M_2(1)a, M_2(1)b}(\beta_1, \beta_2)|_{\beta_1=0, \beta_2=0} = i\, E[M_2(1)a] = 0,$$

the representation above of the derivative can be used to obtain

$$\begin{aligned} -E[(M_2(t)a)(M_2(t)b)] &= \frac{\partial}{\partial \beta_2}\frac{\partial}{\partial \beta_1}\varphi_{M_2(t)a, M_2(t)b}(\beta_1, \beta_2)|_{\beta_1=0, \beta_2=0} \\ &= t\frac{\partial}{\partial \beta_2}\frac{\partial}{\partial \beta_1}\varphi_{M_2(1)a, M_2(1)b}(\beta_1, \beta_2)|_{\beta_1=0, \beta_2=0} \\ &= -tE[(M_2(1)a)(M_2(1)b)], \end{aligned}$$

which completes our proof. ☐

Because of Lemma 4.1 we can simplify our notation and write Q_2 for $Q_2(1)$.

Example 4.2. For the cylindrical Poisson process in Example 3.10 we have

$$M_2(t) = \int_{\mathbb{R}\setminus\{0\}} \beta \, \tilde{N}_a(t, d\beta) = \zeta(a)(n(t) - \lambda t) \qquad \text{for all } a \in U^*.$$

It follows that

$$\begin{aligned}(Q_2 a)(b) &= E\left[(M_2(1)a)(M_2(1)b)\right] \\ &= \zeta(a)\zeta(b) E\left[\left|n(1) - \lambda\right|^2\right] \\ &= \zeta(a)\zeta(b)\lambda.\end{aligned}$$

If we choose ζ discontinuous then $Q_2(a)$ is discontinuous and thus $Q_2(a) \notin U^{**}$.

Definition 4.3. *The cylindrical process M_2 is called* strong *if the covariance operator*

$$Q_2 : U^* \to U^{*\prime}, \qquad Q_2 a(b) = E\left[\left(\int_{\mathbb{R}\setminus\{0\}} \beta \, \tilde{N}_a(1, d\beta)\right)\left(\int_{\mathbb{R}\setminus\{0\}} \beta \, \tilde{N}_b(1, d\beta)\right)\right],$$

maps to U.

Lemma 4.4. *If the cylindrical Lévy measure ν of the cylindrical Lévy process M_2 extends to a Radon measure then M_2 is strong.*

Proof. It is easily seen that the operator

$$G : U^* \to L^2(U, \mathcal{B}(U), \nu), \qquad Ga = \langle \cdot, a \rangle \, \mathbb{1}_U(\cdot)$$

is a closed operator and therefore G is continuous. Thus, we have that

$$\begin{aligned}\Big((Q_2 a)(b)\Big)^2 &\leqslant E\left|M_2(1)a\right|^2 E\left|M_2(1)b\right|^2 \\ &= E\left|M_2(1)a\right|^2 \int_{\mathbb{R}\setminus\{0\}} \beta^2 \, (\nu \circ b^{-1})(d\beta) \\ &= E\left|M_2(1)a\right|^2 \int_U \left|\langle u, b \rangle\right|^2 \, \nu(du) \\ &\leqslant E\left|M_2(1)a\right|^2 \left\|G\right\|^2 \left\|b\right\|^2,\end{aligned}$$

which completes the proof. □

If M_2 is strong then the covariance operator Q_2 is a symmetric positive linear operator which maps U^* to U. A factorisation lemma (see e.g. Proposition III.1.6 (p.152) in [16]) implies that there exists a Hilbert subspace $(H_{Q_2}, [\cdot,\cdot]_{H_{Q_2}})$ of U such that

(a) $Q_2(U^*)$ is dense in H_{Q_2};

(b) for all $a, b \in U^*$ we have: $[Q_2 a, Q_2 b]_{H_{Q_2}} = \langle Q_2 a, b \rangle$.

Moreover, if i_{Q_2} denotes the natural embedding of H_{Q_2} into U we have

(c) $Q_2 = i_{Q_2} i_{Q_2}^*$.

The Hilbert space H_{Q_2} is called the *reproducing kernel Hilbert space associated with Q_2*.

Example 4.5. We have the following useful formulae:

$$\operatorname{Cov}(M_2(1)a,\, M_2(1)b) = \langle Q_2 a, b\rangle = [i^*_{Q_2} a, i^*_{Q_2} b]_{H_{Q_2}}.$$

In particular, we have

$$E\,|M_2(1)a|^2 = \left\| i^*_{Q_2} a \right\|^2_{H_{Q_2}}. \tag{4.23}$$

Remark 4.6. Assume that $(L(t) : t \geqslant 0)$ is a weakly cylindrical Lévy process of weak order 2 in U with $E[L(t)a] = 0$ for all $a \in U^*$. Then its decomposition according to Corollary 3.12 is given by

$$L(t)a = W(t)a + M_2(t)a \qquad \text{for all } a \in U^*,$$

where $W = (W(t) : t \geqslant 0)$ is a weakly cylindrical Wiener process and M_2 is of the form (4.22) with covariance operator Q_2. The covariance operator Q_1 of W,

$$Q_1 : U^* \to U^{*\prime}, \qquad (Q_1(a))(b) = E[(W(1)a)(W(1)b)]$$

may exhibit similar behaviour to Q_2 in that it might be discontinuous, see [11] for an example. Consequently, we call L a *strongly cylindrical Lévy process of weak order 2* if both Q_1 and Q_2 map to U. By independence of W and M_2 it follows that

$$Q : U^* \to U \qquad (Qa)(b) := (Q_1 a)(b) + (Q_2 a)(b)$$

is the covariance operator of L. As before the operator Q can be factorised through a Hilbert space H_Q.

4.2 Representation as a Series

Theorem 4.7. *If the cylindrical process M_2 of the form* (4.22) *is strong then there exist a Hilbert space H with an orthonormal basis $(e_k)_{k\in\mathbb{N}}$, $F \in L(H,U)$ and uncorrelated real valued càdlàg Lévy processes $(m_k)_{k\in\mathbb{N}}$ such that*

$$M_2(t)a = \sum_{k=1}^{\infty} \langle F e_k, a\rangle m_k(t) \qquad \text{in } L^2(\Omega,\mathcal{F},P) \text{ for all } a \in U^*. \tag{4.24}$$

Proof. Let $Q_2 : U^* \to U$ be the covariance operator of $M_2(1)$ and $H = H_{Q_2}$ its reproducing kernel Hilbert space with the inclusion mapping $i_{Q_2} : H \to U$ (see the comments after Lemma 4.4). Because the range of $i^*_{Q_2}$ is dense in H and H is separable there exists an orthonormal basis $(e_k)_{k\in\mathbb{N}} \subseteq \operatorname{range}(i^*_{Q_2})$ of H. We choose $a_k \in U^*$ such that $i^*_{Q_2} a_k = e_k$ for all $k \in \mathbb{N}$ and define $m_k(t) := M_2(t)a_k$. Then by using the equation (4.23) we obtain that

$$\begin{aligned}
E\left|\sum_{k=1}^{n} \langle i_{Q_2} e_k, a\rangle m_k(t) - M_2(t)a\right|^2 &= E\left| M_2(t)\left(\sum_{k=1}^{n} \langle i_{Q_2} e_k, a\rangle a_k - a\right)\right|^2 \\
&= t\left\| i^*_{Q_2}\left(\sum_{k=1}^{n} \langle i_{Q_2} e_k, a\rangle a_k - a\right)\right\|^2_H \\
&= t\left\| \sum_{k=1}^{n} [e_k, i^*_{Q_2} a]_H e_k - i^*_{Q_2} a\right\|^2_H \\
&\to 0 \qquad \text{for } n \to \infty.
\end{aligned}$$

Thus, M_2 has the required representation and it remains to establish that the Lévy processes $m_k := (m_k(t) : t \geqslant 0)$ are uncorrelated. For any $s \leqslant t$ and $k, l \in \mathbb{N}$ we have:

$$\begin{aligned} E[m_k(s)m_l(t)] &= E[M_2(s)a_k M_2(t)a_l] \\ &= E[M_2(s)a_k(M_2(t)a_l - M_2(s)a_l)] + E[M_2(s)a_k M_2(s)a_l]. \end{aligned}$$

The first term is zero by Lemma 3.8 and for the second term we obtain

$$E[M_2(s)a_k M_2(s)a_l] = s\langle Q_2 a_k, a_l\rangle = s[i^*_{Q_2} a_k, i^*_{Q_2} a_l]_H = s[e_k, e_l]_H = s\delta_{k,l}.$$

Hence, $m_k(s)$ and $m_l(t)$ are uncorrelated. □

Remark 4.8. The proof of Theorem 4.7 shows that the real valued Lévy processes m_k can be chosen as

$$m_k(t) = \int_{\mathbb{R}\setminus\{0\}} \beta\, \tilde{N}_{a_k}(t, d\beta) \qquad \text{for all } t \geqslant 0,$$

where $\tilde{N}_{a_k}$ is the compensated Poisson random measure. Because of the choice of a_k the relation (4.23) yields that

$$E\,|m_k(t)|^2 = tE\,|M_2(1)a_k|^2 = t\left\|i^*_{Q_2} a_k\right\|^2_{H_{Q_2}} = t\,\|e_k\|^2_{H_{Q_2}} = t \tag{4.25}$$

for all $k \in \mathbb{N}$ implying that

$$\int_{\mathbb{R}\setminus\{0\}} \beta^2(\nu \circ a_k^{-1})(d\beta) = 1. \tag{4.26}$$

An interesting question is the reverse implication of Theorem 4.7. Under which condition on a family $(m_k)_{k\in\mathbb{N}}$ of real valued Lévy processes can we construct a cylindrical Lévy process via the sum (4.24)?

Remark 4.9. Let $(L(t) : t \geqslant 0)$ be a strongly cylindrical Lévy process with decomposition $L(t) = W(t) + M_2(t)$. By Remark 4.6 the covariance operator Q of L can be factorised through a Hilbert space H_Q and so Theorem 4.7 can be generalised as follows. There exist an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H_Q, $F \in L(H_Q, U)$ and uncorrelated real valued Lévy processes $(m_k)_{k\in\mathbb{N}}$ such that

$$L(t)a = \sum_{k=1}^{\infty} \langle Fe_k, a\rangle m_k(t) \qquad \text{in } L^2(\Omega, \mathcal{F}, P) \text{ for all } a \in U^*.$$

As the stochastic processes m_k can be choosen as $m_k(t) = L(t)a_k$ for some $a_k \in U^*$ it follows that for all $t \geq 0, k \in \mathbb{N}$

$$m_k(t) = W(t)a_k + \int_{\mathbb{R}\setminus\{0\}} \beta\, \tilde{N}_{a_k}(t, d\beta).$$

4.3 Integration

In this section we introduce a cylindrical integral with respect to the cylindrical process $M_2 = (M_2(t) : t \geqslant 0)$ in U. Because M_2 has weakly independent increments and is of weak order 2 we can closely follow the analysis for a cylindrical Wiener process as was considered in [11]. The integrand is a stochastic process with values in $L(U, V)$, the set of bounded linear operators from U to V, where V denotes a separable Banach space. For that purpose we assume for M_2 the representation according to Theorem 4.7:

$$M_2(t)a = \sum_{k=1}^{\infty} \langle i_{Q_2} e_k, a\rangle m_k(t) \qquad \text{in } L^2(\Omega, \mathcal{F}, P) \text{ for all } a \in U^*,$$

where H_{Q_2} is the reproducing kernel Hilbert space of the covariance operator Q_2 with the inclusion mapping $i_{Q_2} : H_{Q_2} \to U$ and an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H_{Q_2}. The real valued Lévy processes $(m_k(t) : t \geqslant 0)$ are defined by $m_k(t) = M_2(t)a_k$ for some $a_k \in U^*$ with $i_Q^* a_k = e_k$, see Remark 4.8.

Definition 4.10. *The set $C(U, V)$ contains all random variables $\Phi : [0, T]\times\Omega \to L(U, V)$ such that:*

(a) *$(t, \omega) \mapsto \Phi^*(t, \omega)f$ is $\mathcal{B}[0, T] \otimes \mathcal{F}$ measurable for all $f \in V^*$;*

(b) *$(t, \omega) \mapsto \langle\Phi(t, \omega)u, f\rangle$ is predictable for all $u \in U$ and $f \in V^*$.*

(c) *$\int_0^T E \|\Phi^*(s, \cdot)f\|^2_{U^*}\, ds < \infty$ for all $f \in V^*$.*

As usual we neglect the dependence of $\Phi \in C(U, V)$ on ω and write $\Phi(s)$ for $\Phi(s, \cdot)$ as well as for the dual process $\Phi^*(s) := \Phi^*(s, \cdot)$ where $\Phi^*(s, \omega) \in L(V, U)$ denotes the dual (or adjoint) operator of $\Phi(s, \omega) \in L(U, V)$.
We define the candidate for a stochastic integral:

Definition 4.11. *For $\Phi \in C(U, V)$ we define*

$$I_t(\Phi)f := \sum_{k=1}^{\infty} \int_0^t \langle\Phi(s)i_{Q_2} e_k, f\rangle\, m_k(ds) \qquad \text{in } L^2(\Omega, \mathcal{F}, P)$$

for all $f \in V^*$ and $t \in [0, T]$.

For a predictable mapping $h : [0, t]\times\mathbb{R}\times\Omega \to \mathbb{R}$ the stochastic integral $\int_{[0,t]\times\mathbb{R}\setminus\{0\}} h(s, \beta)\, \tilde{N}_a(ds, d\beta)$ exists if

$$\int_{[0,t]\times\mathbb{R}\setminus\{0\}} E\left[(h(s, \beta))^2\right]\, \nu_a(d\beta)\, ds < \infty,$$

see for example Chapter 4 in [1]. Thus, the stochastic integral

$$\int_0^t \langle\Phi(s)i_{Q_2} e_k, f\rangle\, m_k(ds) = \int_{[0,t]\times\mathbb{R}\setminus\{0\}} \langle\Phi(s)i_{Q_2} e_k, f\rangle\, \beta\, \tilde{N}_{a_k}(ds, d\beta)$$

exists because property (c) in Definition 4.10 together with (4.26) implies

$$\begin{aligned} &\int_{[0,t]\times\mathbb{R}\setminus\{0\}} E\left[\left(\langle\Phi(s)i_{Q_2} e_k, f\rangle\, \beta\right)^2\right]\, (\nu \circ a_k^{-1})(d\beta)\, ds \\ &\qquad = \int_{[0,t]} E\left[\left(\langle i_{Q_2} e_k, \Phi^*(s)f\rangle\right)^2\right]\, ds \int_{\mathbb{R}\setminus\{0\}} \beta^2(\nu \circ a_k^{-1})\,(d\beta) \\ &\qquad \leqslant \|i_{Q_2} e_k\|^2 \int_0^t E\,\|\Phi^*(s)f\|^2\, ds < \infty. \end{aligned}$$

Before we establish that the sum of these integrals in Definition 4.11 converges we derive a simple generalisation of Itô's isometry for stochastic integrals with respect to compensated Poisson random measures.

Lemma 4.12. *Let $(h_i(t) : t \in [0,T])$ for $i = 1, 2$ be two predictable real valued processes with*

$$\int_0^T E\,|h_i(s)|^2\,ds < \infty$$

and let $m_1 := (M_2(t)a : t \in [0,T])$ and $m_2 := (M_2(t)b : t \in [0,T])$ for $a, b \in U^*$. Then we have

$$E\left[\left(\int_0^T h_1(s)\,m_1(ds)\right)\left(\int_0^T h_2(s)\,m_2(ds)\right)\right] = \mathrm{Cov}(m_1(1), m_2(1))\,E\left[\int_0^T h_1(s)h_2(s)\,ds\right].$$

Proof. Let g_i, $i = 1, 2$, be simple processes of the form

$$g_i(s) = \xi_{i,0}\,\mathbb{1}_{\{0\}}(s) + \sum_{k=1}^{n-1} \xi_{i,k}\,\mathbb{1}_{(t_k,t_{k+1}]}(s) \tag{4.27}$$

for $0 = t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n = T$ and a sequence of random variables $\{\xi_{i,k}\}_{k=0,\ldots,n-1}$ such that $\xi_{i,k}$ is $\mathcal{F}_{t_k}$-measurable and $\sup_{k=0,\ldots,n-1} |\xi_{i,k}| < C$ P-a.s. We obtain

$$\begin{aligned}E\left[\left(\int_0^T g_1(s)\,m_1(ds)\right)\left(\int_0^T g_2(s)\,m_2(ds)\right)\right] &= Cov(m_1(1), m_2(1)) \sum_{k=1}^{n-1} E[\xi_{1,k}\xi_{2,k}](t_{k+1} - t_k)\\ &= Cov(m_1(1), m_2(1)) E\left[\int_0^T g_1(s)g_2(s)\,ds\right].\end{aligned}$$

For the processes h_i there exist simple processes $(g_i^{(n)})$ of the form (4.27) such that

$$E\left[\int_0^T (g_i^{(n)}(s) - h_i(s))^2\,ds\right] \to 0 \qquad \text{for } n \to \infty. \tag{4.28}$$

Itô's isometry implies that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\int_0^T g_i^{(n_k)}(s)\,m_i(ds) \to \int_0^T h_i(s)\,m_i(ds) \qquad P\text{-a.s. for } k \to \infty$$

for $i = 1, 2$. By applying Lebesgue's dominated convergence theorem we obtain

$$E\left[\left(\int_0^T g_1^{(n_k)}(s)\,m_1(ds)\right)\left(\int_0^T g_2^{(n_k)}(s)\,m_2(ds)\right)\right] \to E\left[\left(\int_0^T h_1(s)\,m_1(ds)\right)\left(\int_0^T h_2(s)\,m_2(ds)\right)\right].$$

On the other hand, (4.28) implies that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$E[g_i^{(n_k)}(s) - h_i(s)] \to 0 \qquad \text{Lebesgue almost everywhere for } k \to \infty.$$

Lebesgue's dominated convergence theorem again implies that

$$\int_0^T E\left[g_1^{(n_k)}(s)g_2^{(n_k)}(s)\right]\,ds \to \int_0^T E\left[h_1(s)h_2(s)\right]\,ds \qquad \text{for } k \to \infty,$$

which completes the proof.

□

Lemma 4.13. *$I_t(\Phi): V^* \to L^2(\Omega, \mathcal{F}, P)$ is a well-defined cylindrical random variable in V which is independent of the representation of L, i.e. of $(e_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$.*

Proof. We begin to establish the convergence in $L^2(\Omega, \mathcal{F}, P)$. For that, let $m, n \in \mathbb{N}$ and we define for simplicity $h(s) := i_{Q_2}^* \Phi^*(s) f$. Doob's maximal inequality and Lemma 4.12 imply

$$\begin{aligned}
E\left|\sup_{0\leqslant t\leqslant T} \sum_{k=m+1}^{n} \int_0^t \langle \Phi(s) i_{Q_2} e_k, f\rangle\, m_k(ds)\right|^2 \\
&\leqslant 4 \sum_{k=m+1}^{n} \left(\int_{\mathbb{R}\setminus\{0\}} \beta^2\, (\nu \circ a_k^{-1})(d\beta)\right) \int_0^T E\, [e_k, h(s)]_{H_{Q_2}}^2\, ds \\
&\leqslant 4 \sum_{k=m+1}^{\infty} \int_0^T E\left[[e_k, h(s)]_{H_{Q_2}} e_k, h(s)\right]_{H_{Q_2}} ds \\
&= 4 \sum_{k=m+1}^{\infty} \sum_{l=m+1}^{\infty} \int_0^T E\left[[e_k, h(s)]_{H_{Q_2}} e_k, [e_l, h(s)]_{H_{Q_2}} e_l\right]_{H_{Q_2}} ds \\
&= 4 \int_0^T E\, \|(\mathrm{Id} - p_m) h(s)\|_{H_{Q_2}}^2\, ds,
\end{aligned}$$

where $p_m : H_{Q_2} \to H_{Q_2}$ denotes the projection onto the span of $\{e_1, \dots, e_m\}$. Because $\|(\mathrm{Id} - p_m) h(s)\|_{H_{Q_2}}^2 \to 0$ P-a.s. for $m \to \infty$ and

$$\int_0^T E\, \|(\mathrm{Id} - p_m) h(s)\|_{H_{Q_2}}^2\, ds \leqslant \left\|i_{Q_2}^*\right\|_{U^* \to H_{Q_2}}^2 \int_0^T E\, \|\Phi^*(s, \cdot) f\|_{U^*}^2\, ds < \infty$$

we obtain by Lebesgue's dominated convergence theorem the convergence in $L^2(\Omega, \mathcal{F}, P)$. Because the processes $\{m_k\}_{k\in\mathbb{N}}$ are uncorrelated Lemma 4.12 enables us to derive an analogue of Itô's isometry:

$$\begin{aligned}
E\left|\sum_{k=1}^{\infty} \int_0^t \langle \Phi(s) i_{Q_2} e_k, f\rangle\, m_k(ds)\right|^2 &= \sum_{k=1}^{\infty} E\left|\int_0^t \langle \Phi(s) i_{Q_2} e_k, f\rangle\, m_k(ds)\right|^2 \\
&= \sum_{k=1}^{\infty} E\, |m_k(1)|^2 \int_0^t E\, |\langle \Phi(s) i_{Q_2} e_k, f\rangle|^2\, ds \\
&= \sum_{k=1}^{\infty} \int_0^t E\left[\left[e_k, i_{Q_2}^* \Phi^*(s) f\right]_{H_{Q_2}}^2\right] ds \\
&= \int_0^t \left\|i_{Q_2}^* \Phi^*(s) f\right\|_{H_{Q_2}}^2\, ds, \qquad (4.29)
\end{aligned}$$

where we used (4.23) to obtain

$$E\, |m_k(1)|^2 = \left\|i_{Q_2}^* a_k\right\|^2 = \|e_k\|^2 = 1.$$

To prove the independence of the given representation of M_2 let $(d_l)_{l\in\mathbb{N}}$ be an other orthonormal basis of H_{Q_2} and $w_l \in U^*$ such that $i_{Q_2}^* w_l = d_l$ and $(n_l(t) : t \geqslant 0)$ Lévy processes defined by $n_l(t) = M_2(t) w_l$. As before we define in $L^2(\Omega, \mathcal{F}, P)$:

$$\tilde{I}_t(\Phi) f := \sum_{l=1}^{\infty} \int_0^t \langle \Phi(s) i_{Q_2} d_l, f\rangle\, n_l(ds) \qquad \text{for all } f \in V^*.$$

Lemma 4.12 enables us to compute the covariance:

$$\begin{aligned}
&E\Big[\big(I_t(\Phi)f\big)\big(\tilde{I}_t(\Phi)f\big)\Big]\\
&\quad=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}E\left[\left(\int_0^t \langle\Phi(s)i_{Q_2}e_k,f\rangle\, m_k(ds)\right)\left(\int_0^t \langle\Phi(s)i_{Q_2}d_l,f\rangle\, n_l(ds)\right)\right]\\
&\quad=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\mathrm{Cov}(m_k(1),n_l(1))E\left[\int_0^t \langle\Phi(s)i_{Q_2}e_k,f\rangle\langle\Phi(s)i_{Q_2}d_l,f\rangle\, ds\right]\\
&\quad=\int_0^t E\left[\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[e_k,d_l]_{H_{Q_2}}\left[e_k,i_{Q_2}^*\Phi^*(s)f\right]_{H_{Q_2}}\left[d_l,i_{Q_2}^*\Phi^*(s)f\right]_{H_{Q_2}}\, ds\right]\\
&\quad=\int_0^t E\left\|i_{Q_2}^*\Phi^*(s)f\right\|_{H_{Q_2}}^2\, ds.
\end{aligned}$$

By using Itô's isometry (4.29) we obtain

$$\begin{aligned}
&E\left[\left|I_t(\Phi)f-\tilde{I}_t(\Phi)f\right|^2\right]\\
&\quad=E\Big[\left|I_t(\Phi)f\right|^2\Big]+E\Big[\left|\tilde{I}_t(\Phi)f\right|^2\Big]-2E\Big[\big(I_t(\Phi)f\big)\big(\tilde{I}_t(\Phi)f\big)\Big]\\
&\quad=0,
\end{aligned}$$

which proves the independence of $I_t(\Phi)$ on $(e_k)_{k\in\mathbb{N}}$ and $(a_k)_{k\in\mathbb{N}}$. The linearity of $I_t(\Phi)$ is obvious and hence the proof is complete. □

Our next definition is not very surprising:

Definition 4.14. *For $\Phi\in C(U,V)$ we call the cylindrical random variable*

$$\int_0^t \Phi(s)\, dM_2(s) := I_t(\Phi)$$

a cylindrical stochastic integral with respect to M_2.

In the proof of Lemma 4.13 we already derived Itô's isometry:

$$E\left|\left(\int_0^t \Phi(s)\, dM_2(s)\right)f\right|^2=\int_0^t E\left\|i_{Q_2}^*\Phi^*(s)f\right\|_{H_{Q_2}}^2\, ds$$

for all $f\in V^*$.

Remark 4.15. If a strongly cylindrical Lévy process L is of the form $L(t)=W(t)+M_2(t)$ one can utilise the series representation in Remark 4.9 to define a stochastic integral with respect to L by the same approach as in this subsection. But on the other hand we can follow [2] and define

$$\int \Phi(s)\, dL(s) := \int \Phi(s)\, dW(s)+\int \Phi(s)\, dM_2(s),$$

where the stochastic integral with respect to the cylindrical Wiener process W is defined analogously, see [11] for details. This approach allows even more flexibility because one can choose different integrands Φ_1 and Φ_2 for the two different integrals on the right hand side.

5 Cylindrical Ornstein-Uhlenbeck process

Let V be a separable Banach space and let $(M_2(t) : t \geqslant 0)$ be a strongly cylindrical Lévy process of the form (4.22) on a separable Banach space U with covariance operator Q_2 and cylindrical Lévy measure ν. We consider the Cauchy problem

$$\begin{aligned} dY(t) &= AY(t)\,dt + C\,dM_2(t) \qquad \text{for all } t \geqslant 0, \\ Y(0) &= Y_0, \end{aligned} \tag{5.30}$$

where $A : \text{dom}(A) \subseteq V \to V$ is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geqslant 0}$ on V and $C : U \to V$ is a linear, bounded operator. The initial condition is given by a cylindrical random variable $Y_0 : V^* \to L^0(\Omega, \mathcal{F}, P)$. In addition, we assume that Y_0 is continuous when $L^0(\Omega, \mathcal{F}, P)$ is equipped with the topology of convergence in probability.

Remark 5.1. In this section we focus on the random noise M_2 for simplicity. But because of Remark 4.15 our results in this section on the Cauchy problem (5.30) can easily be generalised to the Cauchy problem of the form

$$dY(t) = AY(t)\,dt + C_1\,dW(t) + C_2\,dM_2(t),$$

where $(W(t) : t \geqslant 0)$ is a strongly cylindrical Wiener process.

To find an appropriate meaning of a solution of (5.30) let $T : \text{dom}(T) \subseteq U \to V$ be a closed densely defined linear operator acting with dual operator $T^* : \text{dom}(T^*) \subseteq V^* \to U^*$. If X is a cylindrical random variable in U then we obtain a linear map TX with domain $\text{dom}(T^*)$ by the prescription

$$TX : \text{dom}(T^*) \subseteq V^* \to L^0(\Omega, \mathcal{F}, P), \qquad (TX)a := X(T^*a).$$

If $\text{dom}(T^*) = V^*$ then TX defines a new cylindrical random variable in V. If μ_X denotes the cylindrical distribution of X then the cylindrical distribution μ_{TX} of TX is given by

$$\mu_{TX}(Z(a_1, \dots, a_n; B)) = \mu_X(Z(T^*a_1, \dots, T^*a_n; B)),$$

for all $a_1, \dots, a_n \in V^*$, $B \in \mathcal{B}(\mathbb{R}^n)$ and $n \in \mathbb{N}$. By applying this definition the operator C appearing in the Cauchy problem (5.30) defines a new cylindrical process $CM_2 := (CM_2(t) : t \geqslant 0)$ in V by

$$CM_2(t)a = M_2(t)(C^*a) \qquad \text{for all } a \in V^*.$$

The cylindrical process CM_2 is a cylindrical Lévy process in V with covariance operator CQ_2C^* and cylindrical Lévy measure ν_{CM_2} given by

$$\nu_{CM_2}(Z(a_1, \dots, a_n; B)) = \nu_{M_2}(Z(C^*a_1, \dots, C^*a_n; B)).$$

Definition 5.2. *An adapted, cylindrical process* $(Y(t) : t \geqslant 0)$ *in* V *is called a* weak cylindrical solution *of* (5.30) *if*

$$Y(t)a = Y_0a + \int_0^t AY(s)a\,ds + (CM_2(t))a \qquad \textit{for all } a \in \textit{dom}(A^*).$$

Definition 5.2 extends the concept of a solution of stochastic Cauchy problems on a Hilbert space or a Banach space driven by a Lévy process to the cylindrical situation, see [9] for the case of a Hilbert space and [12] for the case of a Banach space. The following example illustrates this generalisation.

Example 5.3. Let $\tilde{N}$ be a compensated Poisson random measure in U. Then a weak solution of

$$\begin{aligned} dZ(t) &= AZ(t)\,dt + \int_{0<\|u\|} C\,d\tilde{N}(dt,du) \qquad \text{for all } t \geqslant 0, \\ Z(0) &= Z_0 \end{aligned} \tag{5.31}$$

is a stochastic process $Z = (Z(t) : t \geqslant 0)$ in V such that P-a.s.

$$\langle Z(t), a\rangle = \langle Z(0), a\rangle + \int_0^t \langle Z(s), A^*a\rangle\,ds + \int_{[0,t]\times U} \langle C(u), a\rangle\,\tilde{N}(ds,du) \tag{5.32}$$

for all $a \in \operatorname{dom}(A^*)$ and $t \geqslant 0$. These kinds of equations in Hilbert spaces are considered in [2] and [8] and in Banach spaces in [12].
If we define a cylindrical Lévy process $(M_2(t) : t \geqslant 0)$ by

$$M_2(t)a := \int_U \langle u, a\rangle\,\tilde{N}(t,du),$$

then it follows that the induced cylindrical process $(Y(t) : t \geqslant 0)$ with $Y(t)a = \langle Z(t), a\rangle$ where Z is a weak solution of (5.31) is a weak cylindrical solution of

$$\begin{aligned} dY(t) &= AY(t)\,dt + C\,dM_2(t), \\ Y(0) &= Y_0 \end{aligned}$$

in the sense of Definition 5.2 with $Y_0a := \langle Z_0, a\rangle$.

A Cauchy problem of the form (5.31) might not have a solution in the traditional sense. But a cylindrical solution always exists:

Theorem 5.4. *For every Cauchy problem of the form* (5.30) *there exists a unique weak cylindrical solution* $(Y(t) : t \geqslant 0)$ *which is given by*

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)C\,dM_2(s) \qquad \textit{for all } t \geqslant 0.$$

Proof. We define the stochastic convolution by the cylindrical random variable

$$X(t) := \int_0^t S(t-v)C\,dM_2(v) \qquad \text{for all } t \geqslant 0.$$

To ensure that the cylindrical stochastic integral exists we need only to check that the integrand satisfies the condition (c) in Definition 4.10 which follows from

$$\int_0^t \|S^*(t-v)a\|_{V^*}^2\,dv = \int_0^t \|S(v)a\|_V^2\,dv < \infty,$$

because of the exponential estimate of the growth of semigroups, i.e.

$$\|S(t)a\|_V \leqslant Ce^{\gamma t} \qquad \text{for all } t \geqslant 0, \tag{5.33}$$

where $C \in (0,\infty)$ and $\gamma \in \mathbb{R}$ are some constants. By using standard properties of strongly continuous semigroups we calculate for $a \in V^*$ that

$$\begin{aligned}
\int_0^t AX(r)a\,dr &= \int_0^t X(r)(A^*a)\,dr \\
&= \int_0^t \left(\int_0^r S(r-v)C\,dM_2(v) \right) (A^*a)\,dr \\
&= \sum_{k=1}^{\infty} \int_0^t \int_0^r \langle S(r-v)Ci_{Q_2}e_k, A^*a\rangle\, m_k(dv)\,dr \\
&= \sum_{k=1}^{\infty} \int_0^t \int_r^t \langle S(r-v)Ci_{Q_2}e_k, A^*a\rangle\, dr\, m_k(dv) \\
&= \sum_{k=1}^{\infty} \int_0^t \langle Ci_{Q_2}e_k, S^*(t-v)a - a\rangle\, m_k(dv) \\
&= X(t)a - M_2(t)(C^*a), \qquad (5.34)
\end{aligned}$$

where we have used the stochastic Fubini theorem for Poisson stochastic integrals (see Theorem 5 in [2]), the application of which is justified by the estimate (5.33). For convenience we define

$$Z(t) := S(t)Y_0 \qquad \text{for all } t \geqslant 0.$$

Proposition 1.2.2 in [17] guarantees that the adjoint semigroup satisfies

$$\int_0^t S^*(r)A^*a\,dr = S^*(t)a - a \qquad \text{for all } a \in \operatorname{dom}(A^*),$$

in the sense of Bochner integrals. Thus, we have

$$\int_0^t AZ(r)a\,dr = \int_0^t Y_0(S^*(r)A^*a)\,dr = Y_0 \int_0^t S^*(r)A^*a\,dr = Z(t)a - Y_0a.$$

The assumption on the continuity of the initial condition Y_0 enables the change of the integration and the application of the initial condition Y_0. Together with (5.34) this completes our proof. □

The cylindrical process $(Y(t) : t \geqslant 0)$ given in Theorem 5.4 is called a *cylindrical Ornstein-Uhlenbeck process.*

For all $t \geqslant 0$, let $C_t(\Omega, V)$ be the linear space of all adapted cylindrical random variables in V which are $\mathcal{F}_t$-measurable. A family $\{Z_{s,t} : 0 \leqslant s \leqslant t\}$ of mappings

$$Z_{s,t} : C_s(\Omega, V) \to C_t(\Omega, V)$$

is called a *cylindrical flow* if $Z_{t,t} = \mathrm{Id}$ and for each $0 \leqslant r \leqslant s \leqslant t$

$$Z_{r,t} = Z_{s,t} \circ Z_{r,s} \quad P\text{-a.s.}$$

In relation to the cylindrical Ornstein-Uhlenbeck process in Theorem 5.4 we define

$$Z_{s,t}X := S(t-s)X + \int_s^t S(t-r)C\,dM_2(r) \qquad \text{for } X \in C_s(\Omega, V) \qquad (5.35)$$

and for all $0 \leqslant s \leqslant t$.

Proposition 5.5.

(a) *The family* $\{Z_{s,t} : 0 \leqslant s \leqslant t\}$ *as given by* (5.35) *is a cylindrical flow.*

(b) *For all* $a_1, \dots, a_n \in U^*$ *the stochastic process* $(Y(t)(a_1, \dots, a_n) : t \geqslant 0)$ *in* $\mathbb{R}^n$ *is a time-homogeneous Markov process.*

Proof. (a) This is established by essentially the same argument as that given in the proof of Proposition 4.1 of [2].
(b) For each $0 \le s \le t$, $\mathrm{a}_{(\mathrm{n})} = (a_1, \dots, a_n) \in V^{*n}$, $f \in B_b(\mathbb{R}^n), n \in \mathbb{N}$, we have

$$\begin{aligned} E\Big[f(Y(t)(a_1, \dots, a_n))|\mathcal{F}_s\Big] \\ &= E\Big[f(Z_{0,t}Y(0)a_1, \dots, Z_{0,t}Y(0)a_n)|\mathcal{F}_s\Big] \\ &= E\Big[f\big((Z_{s,t} \circ Z_{0,s})Y(0)a_1, \dots, (Z_{s,t} \circ Z_{0,s})Y(0)a_n\big)|\mathcal{F}_s\Big] \\ &= E\Big[f(S(t-s)Z_{0,s}Y(0)(a_1, \dots, a_n) + \left(\int_s^t S(t-u)C\, dM_2(u)\right)(a_1, \dots, a_n))|\mathcal{F}_s\Big]. \end{aligned}$$

Now since the random vector $\left(\int_s^t S(t-u)C\, dM_2(u)\right) \mathrm{a}_{(\mathrm{n})}$ is measurable with respect to $\sigma\left(\{M_2(v)a - M_2(u)a; s \le u \le v \le t,\, a \in V^*\}\right)$ we can use standard arguments for proving the Markov property for SDEs driven by $\mathbb{R}^n$-valued Lévy processes (see e.g. section 6.4.2 in [1]) to deduce that

$$E\Big[f(Y(t)(a_1, \dots, a_n))|\mathcal{F}_s\Big] = E\Big[f(Y(t)(a_1, \dots, a_n))|Y(s)(a_1, \dots, a_n)\Big],$$

which completes the proof. □

Although the Markov process $(Y(t)\,\mathrm{a}_{(\mathrm{n})} : t \geqslant 0)$ is a projection of a cylindrical Ornstein-Uhlenbeck process it is not in general an Ornstein-Uhlenbeck process in $\mathbb{R}^n$ in its own right. Indeed, if this were to be the case we would expect to be able to find for every $\mathrm{a}_{(\mathrm{n})} \in V^{*n}$ a matrix $Q_{\mathrm{a}_{(\mathrm{n})}} \in \mathbb{R}^{n \times n}$ and a Lévy process $(l_{\mathrm{a}_{(\mathrm{n})}}(t) : t \geqslant 0)$ in $\mathbb{R}^n$ such that

$$Y(t)\,\mathrm{a}_{(\mathrm{n})} = e^{tQ_{\mathrm{a}_{(\mathrm{n})}}} Y(0)\,\mathrm{a}_{(\mathrm{n})} + \left(\int_0^t e^{(t-s)Q_{\mathrm{a}_{(\mathrm{n})}}} C\, dl_{\mathrm{a}_{(\mathrm{n})}}(s)\right).$$

That this does not hold in general is shown by the following example:

Example 5.6. On the Banach space $V = L^p(\mathbb{R})$, $p > 1$ we define the translation semigroup $(S(t))_{t \geqslant 0}$ by $(S(t)f)x = f(x+t)$ for $f \in V$. For an arbitrary real valued random variable $\xi \in L^0(\Omega, \mathcal{F}, P)$ we define the initial condition by $Y_0 g := g(\xi)$ for all $g \in L^q(\mathbb{R})$ where $q^{-1} + p^{-1} = 1$. Then we obtain

$$(S(t)Y_0)g = Y_0 S^*(t) g = g(\xi - t) \qquad \text{for every } g \in L^q(\mathbb{R}).$$

If $(Y(t)g : t \geqslant 0)$ were an Ornstein-Uhlenbeck process it follows that there exists $\lambda_g \in \mathbb{R}$ and a random variable ζ_g such that

$$g(\xi - t) = e^{\lambda_g t} \zeta_g \qquad P\text{-a.s.} \tag{5.36}$$

To see that the last line cannot be satisfied take $g = \mathbb{1}_{(0,1)}$ and take ξ to be a Bernoulli random variable. Then we have

$$g(\xi - t) = \mathbb{1}_{(0,1)}(\xi - t) = \xi\, \mathbb{1}_{(0,1)}(t),$$

which cannot be of the form (5.36).

It follows from the Markov property that for each $\mathrm{a}_{(\mathrm{n})} \in V^{*n}$ there exists a semigroup of linear operators $(T_{\mathrm{a}_{(\mathrm{n})}}(t) : t \geqslant 0)$ defined for each $f \in B_b(\mathbb{R}^n)$ by

$$T_{\mathrm{a}_{(\mathrm{n})}}(t)f(\beta) = E[f(Y(t)\,\mathrm{a}_{(\mathrm{n})})|Y(0)\,\mathrm{a}_{(\mathrm{n})} = \beta].$$

The semigroup is of *cylindrical Mehler type* in that for all $b \in V$,

$$T_{\mathrm{a}_{(\mathrm{n})}}(t)f(\pi_{\mathrm{a}_{(\mathrm{n})}} b) = \int_V f(\pi_{S^*(t)\,\mathrm{a}_{(\mathrm{n})}} b + \pi_{\mathrm{a}_{(\mathrm{n})}} y)\,\rho_t(dy), \tag{5.37}$$

where ρ_t is the cylindrical law of $\int_0^t S(t-s)C\,dM_2(s)$.
We say that the cylindrical Ornstein-Uhlenbeck process Y has an *invariant cylindrical measure* μ if for all $\mathrm{a}_{(\mathrm{n})} = (a_1, \ldots, a_n) \in V^{*n}$ and $f \in B_b(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} T_{\mathrm{a}_{(\mathrm{n})}}(t)f(\beta)\,(\mu \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}})(d\beta) = \int_{\mathbb{R}^n} f(\beta)\,(\mu \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}})(d\beta) \qquad \text{for all } t \geqslant 0, \tag{5.38}$$

or equivalently

$$\int_V T_{\mathrm{a}_{(\mathrm{n})}}(t)f(\pi_{\mathrm{a}_{(\mathrm{n})}} b)\,\mu(db) = \int_V f(\pi_{\mathrm{a}_{(\mathrm{n})}} b)\,\mu(db) \qquad \text{for all } t \geqslant 0.$$

By combining (5.38) with (5.37) we deduce that a cylindrical measure μ is an invariant measure for $(Y(t) : t \geqslant 0)$ if and only if it is *self-decomposable* in the sense that

$$\mu \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}} = \mu \circ \pi^{-1}_{S^*(t)\,\mathrm{a}_{(\mathrm{n})}} * \rho_t \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}$$

for all $t \geqslant 0$, $\mathrm{a}_{(\mathrm{n})} \in V^{*n}$.

Proposition 5.7.

(a) *For each $a \in V^*$ the following are equivalent:*

  (i) $\rho_t \circ a^{-1}$ *converges weakly as $t \to \infty$;*

  (ii) $\left(\int_0^t S(r)C\,dM_2(r)\right) a$ *converges in distribution as $t \to \infty$.*

(b) *If $\rho_t \circ a^{-1}$ converges weakly for every $a \in V^*$ then the prescription*

$$\rho_\infty : \mathcal{Z}(V) \to [0,1], \qquad \rho_\infty(Z(a_1, \ldots, a_n; B)) := wk\text{-}\lim_{t\to\infty} \rho_t \circ \pi^{-1}_{a_1,\ldots,a_n}(B)$$

defines an invariant cylindrical measure ρ_∞ for Y. Moreover, if μ is another such cylindrical measure then

$$\mu \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}} = \left(\rho_\infty \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}\right) * \left(\gamma \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}}\right),$$

where γ is a cylindrical measure such that $\gamma \circ \pi^{-1}_{\mathrm{a}_{(\mathrm{n})}} = \gamma \circ \pi^{-1}_{S^*(t)\,\mathrm{a}_{(\mathrm{n})}}$ for all $t \geqslant 0$.

(c) *If an invariant measure exists then it is unique if $(S(t) : t \geqslant 0)$ is stable, i.e. $\lim_{t\to\infty} S(t)x = 0$ for all $x \in V$.*

Proof. The arguments of Lemma 3.1, Proposition 3.2 and Corollary 6.2 in [4] can be easily adapted to our situation. □

In order to derive a simple sufficient condition implying the existence of a unique invariant cylindrical measure we assume that the semigroup $(S(t) : t \geqslant 0)$ is exponentially stable, i.e. there exists $R > 1$, $\lambda > 0$ such that $\|S(t)\| \leqslant Re^{-\lambda t}$ for all $t \geqslant 0$.

Corollary 5.8.
If $(S(t) : t \geqslant 0)$ is exponentially stable then there exists a unique invariant cylindrical measure.

Proof. For every $t_1 > t_2 > 0$ and $a \in V^*$ the Itô's isometry (4.29) implies that

$$\begin{aligned} E\left|\left(\int_0^{t_1} S(r)C\,dM_2(r)\right)a - \left(\int_0^{t_2} S(r)C\,dM_2(r)\right)a\right|^2 \\ &= \int_{t_2}^{t_1} \left\|i_{Q_2}^* C^* S^*(r)a\right\|_{H_{Q_2}}^2\,dr \\ &\leqslant \|i_{Q_2}\|^2 \|C\|^2 \|a\|^2 \int_{t_2}^{t_1} \|S(r)\|^2\,dr \\ &\to 0 \qquad \text{as } t_1, t_2 \to \infty, \end{aligned}$$

because of the exponential stability. Consequently, the integral $\left(\int_0^t S(r)C\,dM_2(r)\right)a$ converges in mean square and Proposition 5.7 completes the proof. □

An obvious and important question is whether a cylindrical Ornstein-Uhlenbeck process is induced by a stochastic process in V. This will be the objective of forthcoming work but here we give a straightforward result in this direction, within the Hilbert space setting:

Lemma 5.9. *Let V be a separable Hilbert space and assume that*

$$\sum_{k=1}^{\infty} \int_0^t \|S(r)Ci_k e_k\|^2\,dr < \infty \qquad \text{for all } t \geqslant 0.$$

If the initial condition Y_0 is induced by a random variable in V then the cylindrical weak solution Y of (5.30) *is induced by a stochastic process in V.*

Proof. For all $m < n$

$$E\left\|\sum_{k=m+1}^{n} \int_0^t S(t-r)Ci_{Q_2}e_k\,m_k(dr)\right\|^2 = \sum_{k=m+1}^{n} \int_0^t \left\|S(r)Ci_{Q_2}e_k\right\|^2\,dr \to 0 \text{ as } m, n \to \infty,$$

and it follows by completeness that there exists a V-valued random variable Z in $L^2(\Omega, \mathcal{F}, P; V)$ such that

$$Z = \sum_{k=1}^{\infty} \int_0^t S(t-r)Ci_{Q_2}e_k\,m_k(dr) \qquad \text{in } L^2(\Omega, \mathcal{F}, P; V),$$

which completes the proof by Theorem 5.4. □

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