$X_M$-Harmonic Cohomology and Equivariant Cohomology on Riemannian Manifolds With Boundary

Al-Zamil, Qusay

2010

MIMS EPrint: 2010.96

Manchester Institute for Mathematical Sciences
School of Mathematics
The University of Manchester

Reports available from: http://eprints.maths.manchester.ac.uk/
And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097
X_M-Harmonic Cohomology and Equivariant Cohomology on Riemannian Manifolds With Boundary

Qusay S.A. Al-Zamil

October 26, 2010

Abstract

Given a Riemannian manifold M with boundary and a torus G which acts by isometries on M, we consider Witten’s coboundary operator \(d_X \mu = d + \iota_X \mu\) on invariant forms on M. In [1] we introduce the absolute \(X_M\)-cohomology \(H^*_X(M) = H^*(\Omega^*_G, d_{X_M})\) and the relative \(X_M\)-cohomology \(H^*_X(M, \partial M) = H^*(\Omega^*_G, D, d_{X_M})\) where the D is for Dirichlet boundary condition and \(\Omega^*_G\) is the invariant forms on M. Let \(\delta_{X_M}\) be the adjoint of \(d_{X_M}\) and the resulting Witten-Hodge-Laplacian is \(\Delta_{X_M} = d_{X_M} \delta_{X_M} + \delta_{X_M} d_{X_M}\) where the space \(\ker \Delta_{X_M}\) is called the \(X_M\)-harmonic forms. In this paper, we prove that the (even/odd) \(X_M\)-harmonic cohomology which is the \(X_M\)-cohomology of the subcomplex \((\ker \Delta_{X_M}, d_{X_M})\) of the complex \((\Omega^*_G, d_{X_M})\) is enough to determine the total absolute and relative \(X_M\)-cohomology. As conclusion, we infer that the free part of the absolute and relative equivariant cohomology groups are determined by the (even/odd) \(X_M\)-harmonic cohomology when the set of zeros of the corresponding vector field \(X_M\) is equal to the fixed point set \(F\) for the \(G\)-action.

Keywords: Algebraic topology, equivariant topology, manifolds with boundary, cochain complex, group actions, equivariant cohomology.

MSC 2010: 57R19, 55N91, 57R91

1 Introduction

In [3], S.Cappell, D. DeTurck et al. present the following main theorem,

**Theorem 1.1** [3]. Let \(M\) be a compact, connected, oriented smooth Riemannian manifold of dimension \(n\) with boundary. Then the cohomology of the complex \((\text{Harm}^*(M), d)\) of harmonic forms on \(M\) is given by the direct sum of the de Rham cohomology:

\[
H^k(\text{Harm}^*(M), d) \cong H^k(M, \mathbb{R}) + H^{k-1}(M, \mathbb{R})
\]

for \(k = 0, 1, \ldots, n\) and \(\text{Harm}^*(M) = \ker \Delta\) where \(\Delta\) is the Laplacian operator.

The principle idea of this paper is to adapt theorem 1.1 in terms of our operators \(d_{X_M}, \delta_{X_M}\) and \(\Delta_{X_M}\) in order to study the \(X_M\)-harmonic cohomology when the manifold in question has a boundary and then we relate the \(X_M\)-harmonic cohomology with the free part of the relative and absolute equivariant cohomology.

More precisely, in this paper, we consider a compact, connected, oriented, smooth Riemannian manifold \(M\) (with or without boundary) and we suppose \(G\) is a torus acting by isometries on \(M\) and denote by \(\Omega^*_G\) the \(k\)-forms invariant under the action of \(G\). Given \(X\) in the Lie algebra of \(G\) and...
corresponding vector field $X_M$ on $M$, in [1], we consider Witten’s coboundary operator $d_{X_M} = d + i_{X_M}$. This operator is no longer homogeneous in the degree of the invariant form: if $\omega \in \Omega^k_G$, then $d_{X_M}\omega \in \Omega^{k+1}_G \oplus \Omega^{k-1}_G$. Note then that $d_{X_M} : \Omega^*_G \to \Omega^*_G$, where $\Omega^*_G$ is the space of forms (as we call in [1]) which we denote in this paper by $\text{Harm}^+_G$. Thus, we can conclude that all of the maps in the subcomplex $\ker \delta \subset \Omega^*_G$ are $\Omega^*_G$-harmonic fields [1], i.e.

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge (*\beta)$$

which is defined on $\Omega^*_G(M)$, where $\ast : \Omega^*_G \to \Omega^{n-*}_G$ is the Hodge star operator and then it leads us to the formal adjoint $\delta_{X_M} = -(\mp 1)^n * d_{X_M} * : \Omega^*_G \to \Omega^{n-*}_G$ of $d_{X_M}$. The resulting \textit{Witten-Hodge-Laplacian}

$$\Delta_{X_M} = (d_{X_M} + \delta_{X_M})^2 = d_{X_M} \delta_{X_M} + \delta_{X_M} d_{X_M} : \Omega^*_G \to \Omega^*_G.$$

When the manifold in question $M$ is closed we define the $X_M$-cohomology which is the cohomology of the complex $(\Omega^*_G, d_{X_M})$ where $d_{X_M}^2 = 0$ because the forms are invariant (see [1] for details) and we denote it by $H^+_M(M)$. In this setting, Witten [4] introduces the definition of the $X_M$-\textit{harmonic forms} (as we call in [1]) which we denote in this paper by $\text{Harm}^+_M(M) = \text{Harm}^+_M(M) + \text{Harm}^-_M(M)$; then it is the kernel of the Witten-Hodge-Laplacian operator $\Delta_{X_M}$ (following [1]), i.e.

$$\text{Harm}^+_M(M) = \ker \Delta_{X_M} \cap \Omega^*_G = \{ \omega \in \Omega^*_G | \Delta_{X_M} \omega = 0 \}.$$

Clearly, $\text{Harm}^+_M(M) \subset \Omega^*_G$, but $\Delta_{X_M}$ and $d_{X_M}$ commute which means that the coboundary operator $d_{X_M}$ preserves the $X_M$-harmonicity of invariant forms. i.e.

$$\text{Harm}^+_M(M) \xrightarrow{d_{X_M}} \text{Harm}^+_M(M).$$

Hence, $(\text{Harm}^+_M(M), d_{X_M})$ is a subcomplex of the $\mathbb{Z}_2$-graded complex $(\Omega^*_G, d_{X_M})$. Therefore, we can compute the $X_M$-cohomology of this complex which we call the $X_M$-harmonic cohomology and denote by $H^+(\text{Harm}^+_M(M), d_{X_M})$.

In the boundaryless case, we have proved that the space of $X_M$-\textit{harmonic fields} $\mathcal{H}^+_M = \ker d_{X_M} \cap \ker \delta_{X_M}$ equal to the space of $X_M$-harmonic forms [1], i.e.

$$\text{Harm}^+_M(M) = \mathcal{H}^+_M.$$

Thus, we can conclude that all of the maps in the subcomplex $(\text{Harm}^+_M(M), d_{X_M})$ are zero which means that

$$H^+(\text{Harm}^+_M(M), d_{X_M}) = \text{Harm}^+_M(M) = \mathcal{H}^+_M.$$

But, proposition 2.6 of [1] asserts that $H^+_{\mathcal{H}^+_M}(M) \cong \mathcal{H}^+_M$, hence,

$$H^+(\text{Harm}^+_M(M), d_{X_M}) \cong H^+_{\mathcal{H}^+_M}(M). \quad (1.1)$$

From another hand, eq.(1.1) is no longer true when the manifold in question has a boundary because the space of $X_M$-harmonic forms $\text{Harm}^+_M(M)$ no longer coincides with the space of $X_M$-harmonic fields $\mathcal{H}^+_M[1]$. Therefore, the main purpose of this paper is to study the $X_M$-harmonic cohomology when the manifold in question has a boundary and the result is theorem 2.3.

In the remainder of this introduction we recall necessary results from [1] and [2] when $\partial M \neq \emptyset$. In [1], we define two types of $X_M$-cohomology, the absolute $X_M$-cohomology $H^+_{X_M}(M)$ and the relative $X_M$-cohomology $H^+_{X_M}(M, \partial M)$. The first is the cohomology of the complex $(\Omega^*_G, d_{X_M})$, while the second is the cohomology of the subcomplex $(\Omega^*_G, d_{X_M}, \omega)$, where $\omega \in \Omega^*_G, \partial$ if it satisfies $i^* \omega = 0$
(the D is for Dirichlet boundary condition). One also defines \( \Omega^{\pm}_{G,N}(M) = \{ \alpha \in \Omega^+_{G}(M) \mid \iota^*(\ast \alpha) = 0 \} \) (Neumann boundary condition). Clearly, the Hodge star provides an isomorphism

\[
\ast : \Omega^\pm_{G,D} \xrightarrow{\sim} \Omega^{n-\pm}_{G,N}
\]

where we write \( n \pm \) for the parity (modulo 2) resulting from subtracting an even/odd number from \( n \). Furthermore, because \( d_{ex} \) and \( \iota^* \) commute, it follows that \( d_{ex} \) preserves Dirichlet boundary conditions while \( \delta_{ex} \) preserves Neumann boundary conditions. In fact, the space \( \mathcal{H}^\pm_{X_M}(M) \) is infinite dimensional and so is much too big to represent the \( X_M \)-cohomology, hence, we restrict \( \mathcal{H}^\pm_{X_M}(M) \) into each of two finite dimensional subspaces, namely \( \mathcal{H}^\pm_{X_M,D}(M) \) and \( \mathcal{H}^\pm_{X_M,N}(M) \) with the obvious meanings (Dirichlet and Neumann \( X_M \)-harmonic fields, respectively). There are therefore two different candidates for \( X_M \)-harmonic representatives when the boundary is present. This construction firstly leads us to present the \( X_M \)-Hodge-Morrey decomposition theorem which states that

\[
\Omega^\pm_G(M) = \mathcal{E}^\pm_{X_M}(M) \oplus C^\pm_{X_M}(M) \oplus \mathcal{H}^\pm_{X_M}(M) \tag{1.2}
\]

where \( \mathcal{E}^\pm_{X_M}(M) = \{ d_{ex} \alpha \mid \alpha \in \Omega^\pm_{G,D}(M) \} \) and \( C^\pm_{X_M}(M) = \{ \delta_{ex} \beta \mid \beta \in \Omega^\pm_{G,N}(M) \} \). This decomposition is orthogonal with respect to the \( L^2 \)-inner product given above.

In addition, we present the \( X_M \)-Friedrichs Decomposition Theorem which states that

\[\mathcal{H}^\pm_{X_M}(M) = \mathcal{H}^\pm_{X_M,D}(M) \oplus \mathcal{H}^\pm_{X_M,co}(M) \tag{1.3}\]
\[\mathcal{H}^\pm_{X_M}(M) = \mathcal{H}^\pm_{X_M,N}(M) \oplus \mathcal{H}^\pm_{X_M,ex}(M) \tag{1.4}\]

where \( \mathcal{H}^\pm_{X_M,ex}(M) = \{ \xi \in \mathcal{H}^\pm_{X_M}(M) \mid \xi = d_{ex} \sigma \} \) and \( \mathcal{H}^\pm_{X_M,co}(M) = \{ \eta \in \mathcal{H}^\pm_{X_M}(M) \mid \eta = \delta_{ex} \alpha \} \). These give the orthogonal \( X_M \)-Hodge-Morrey-Friedrichs decomposition [1],

\[\Omega^\pm_G(M) = \mathcal{E}^\pm_{X_M}(M) \oplus C^\pm_{X_M}(M) \oplus \mathcal{H}^\pm_{X_M,ex}(M) \oplus \mathcal{H}^\pm_{X_M,co}(M) \tag{1.5}\]
\[= \mathcal{E}^\pm_{X_M}(M) \oplus C^\pm_{X_M}(M) \oplus \mathcal{H}^\pm_{X_M,D}(M) \oplus \mathcal{H}^\pm_{X_M,co}(M) \tag{1.6}\]

The two decompositions are related by the Hodge star operator. The consequence for \( X_M \)-cohomology is that each class in \( \mathcal{H}^\pm_{X_M}(M) \) is represented by a unique \( X_M \)-harmonic field in \( \mathcal{H}^\pm_{X_M,co}(M) \), and each relative class in \( \mathcal{H}^\pm_{X_M}(M, \partial M) \) is represented by a unique \( X_M \)-harmonic field in \( \mathcal{H}^\pm_{X_M,D}(M) \). The Hodge star operator \( \ast \) induces an isomorphism

\[\mathcal{H}^\pm_{X_M}(M) \cong \mathcal{H}^{n-\pm}_{X_M}(M, \partial M). \tag{1.7}\]

We call eq.(1.7) the \( X_M \)-Poincaré-Lefschetz duality.

In order to prove the results for next section we will need the following theorem which is proved in [2].

**Theorem 1.2** [2]. Let \( M \) be a compact, oriented smooth Riemannian manifold of dimension \( n \) with boundary and with an action of a torus \( G \) which acts by isometries on \( M \). If an \( X_M \)-harmonic field \( \lambda \in \mathcal{H}^\pm_{X_M}(M) \) vanishes on the boundary \( \partial M \), then \( \lambda \equiv 0 \), i.e.

\[\mathcal{H}^\pm_{X_M,N}(M) \cap \mathcal{H}^\pm_{X_M,D}(M) = \{ 0 \} \tag{1.8}\]

As a consequence of Theorem 1.2, we obtain the following result.

**Corollary 1.3** [2]

\[\mathcal{H}^\pm_{X_M}(M) = \mathcal{H}^\pm_{X_M,ex}(M) + \mathcal{H}^\pm_{X_M,co}(M) \tag{1.9}\]

where “+” is not a direct sum.
Acknowledgment The author wishes to express deep gratitude towards his supervisor Dr. James Montaldi for his reading the first draft of this paper and for his helpful suggestions.

2 Main results

In this section, we use the symbol + between the spaces to indicate a direct sum whereas we reserve the symbol ⊕ for an orthogonal (with respect to $L^2$-inner product) direct sum unless otherwise indicated.

We begin with the following remark.

**Remark 2.1** We need to define the following subspaces:

$$E_{X_M}^\pm (M) = \{ d_{X_M} \alpha \mid \alpha \in \Omega^\pm_G(M) \}$$

and

$$cE_{X_M}^\pm (M) = \{ \delta_{X_M} \alpha \mid \alpha \in \Omega^\pm_G(M) \}.$$

But, the $X_M$-Hodge-Morrey decomposition (1.2) implies the following decompositions:

$$E_{X_M}^\pm = E_{X_M}^\pm (M) = E_{X_M}^\pm (M) \oplus \mathcal{H}^\pm_{X_M,ex}(M)$$

and

$$cE_{X_M}^\pm = cE_{X_M}^\pm (M) = C_{X_M}^\pm (M) \oplus \mathcal{H}^\pm_{X_M,co}(M).$$

2.1 The image of the Witten-Hodge-Laplacian operator

The image of the Witten-Hodge-Laplacian operator $\Delta_{X_M}$ will be most important to obtain our main theorem 2.3. We therefore need first to prove the following lemma 2.2.

**Lemma 2.2** Let $M$ be a compact, connected, oriented smooth Riemannian manifold of dimension $n$ with boundary and with an action of a torus $G$ which acts by isometries on $M$. Then the Witten-Hodge-Laplacian operator $\Delta_{X_M} = d_{X_M} \delta_{X_M} + \delta_{X_M} d_{X_M} : \Omega^\pm_G(M) \to \Omega^\pm_G(M)$ is surjective.

**Proof:** We need to prove that $\Delta_{X_M}(\Omega^\pm_G(M)) = \Omega^\pm_G(M)$. Clearly, $\Delta_{X_M}(\Omega^\pm_G(M)) \subset \Omega^\pm_G(M)$, so we only need to prove the converse. To do so, we will first compute the image of $\Delta_{X_M}$ on each summand of the $X_M$-Hodge-Morrey decomposition (1.2).

It is clear that

$$\Delta_{X_M}(\mathcal{E}^\pm_{X_M}(M)) = d_{X_M} \delta_{X_M}(\mathcal{E}^\pm_{X_M}(M)) \subset E^\pm_{X_M}.$$

Now, let $\beta \in E^\pm_{X_M}$, then $\beta = d_{X_M} \alpha$ and by applying the $X_M$-Hodge-Morrey decomposition (1.2) on $\alpha$ we get $\alpha = d_{X_M} \sigma + \delta_{X_M} \rho + \kappa$, so

$$\beta = d_{X_M} \alpha = d_{X_M} \delta_{X_M} \rho$$

but also by (1.2), $\rho$ can be written as $\rho = d_{X_M} \varepsilon + \delta_{X_M} \pi + \kappa$ which implies that

$$\beta = d_{X_M} \alpha = d_{X_M} \delta_{X_M} \rho = d_{X_M} \delta_{X_M} d_{X_M} \varepsilon \in \Delta_{X_M}(\mathcal{E}^\pm_{X_M}(M)).$$

Hence, $\Delta_{X_M}(\mathcal{E}^\pm_{X_M}(M)) = E^\pm_{X_M}$. Likewise, $\Delta_{X_M}(\mathcal{C}^\pm_{X_M}(M)) = cE^\pm_{X_M}$. Clearly, $\Delta_{X_M}(\mathcal{H}^\pm_{X_M}(M)) = 0$. Using the above equations together with remark 2.1, we obtain

$$\Delta_{X_M}(\Omega^\pm_G(M)) = E^\pm_{X_M} + cE^\pm_{X_M}$$

$$= (\mathcal{E}^\pm_{X_M}(M) \oplus \mathcal{H}^\pm_{X_M,ex}(M)) + (\mathcal{C}^\pm_{X_M}(M) \oplus \mathcal{H}^\pm_{X_M,co}(M)).$$

(2.1)
Let $M$ be a compact, connected, oriented smooth Riemannian manifold of dimension $\delta$. We define the map $\omega \in \Omega^\pm_G(M)$ then the $X_M$-Hodge-Morrey decomposition (1.2) together with corollary 1.3 assert that $\omega$ can be decomposed as

$$\omega = d_{X_M} \alpha + \delta_{X_M} \beta + (d_{X_M} \rho + \delta_{X_M} \sigma) \in E^\pm_{X_M}(M) \oplus C^\pm_{X_M}(M) \oplus (H^\pm_{X_M, \text{ex}}(M) + H^\pm_{X_M, \text{co}}(M))$$ \hspace{1cm} (2.2)$$

Rearranging eq.(2.2), we get that $\omega \in \Delta_{X_M}(\Omega^\pm_G(M))$ as desired. Thus, $\Delta_{X_M}$ is surjective.

Now, it is time to present the following fundamental theorem which is analogous to theorem 1.1.

**Theorem 2.3** Let $M$ be a compact, connected, oriented smooth Riemannian manifold of dimension $n$ with boundary and with an action of a torus $G$ which acts by isometries on $M$. Then the (even or odd) $X_M$-harmonic cohomology of the subcomplex $(\text{Harm}^\pm_{X_M}(M), d_{X_M})$ completely determines the total $X_M$-cohomology of the complex $(\Omega^*_G, d_{X_M})$ and it is given by the direct sum:

$$H^\pm(\text{Harm}^\pm_{X_M}(M), d_{X_M}) \cong H^\pm_{X_M}(M) + H^\pm_{X_M}(M) = H^\pm_{X_M}(M)$$ \hspace{1cm} (2.3)$$

**Proof:** Applying the definition of the $X_M$-cohomology of the subcomplex $(\text{Harm}^\pm_{X_M}(M), d_{X_M})$, we obtain that

$$H^\pm(\text{Harm}^\pm_{X_M}(M), d_{X_M}) = \ker d_{X_M} |_{\text{Harm}^\pm_{X_M}(M)}/\ker d_{X_M} (\text{Harm}^\pm_{X_M}(M))$$

where $\ker d_{X_M} |_{\text{Harm}^\pm_{X_M}(M)} = \ker d_{X_M} \cap \text{Harm}^\pm_{X_M}(M)$. But, the $X_M$-Hodge-Morrey-Friedrichs decomposition (1.5) implies the following decomposition

$$\ker d_{X_M} |_{\text{Harm}^\pm_{X_M}(M)} = \mathcal{E}^\pm_{X_M}(M) \oplus \mathcal{H}^\pm_{X_M, \text{N}}(M) \oplus \mathcal{H}^\pm_{X_M, \text{ex}}(M) = \mathcal{H}^\pm_{X_M, \text{N}}(M) \oplus E_{X_M} \text{Harm}^\pm_{X_M}(M)$$

where $E_{X_M} \text{Harm}^\pm_{X_M}(M) = E^\pm_{X_M}(M) \cap \text{Harm}^\pm_{X_M}(M)$. But $d_{X_M}(\text{Harm}^\pm_{X_M}(M)) \subset \ker d_{X_M} |_{\text{Harm}^\pm_{X_M}(M)}$, then we obtain a direct sum decomposition

$$H^\pm(\text{Harm}^\pm_{X_M}(M), d_{X_M}) = \ker d_{X_M} |_{\text{Harm}^\pm_{X_M}(M)}/d_{X_M} (\text{Harm}^\pm_{X_M}(M)) = \mathcal{H}^\pm_{X_M, \text{N}}(M) + E_{X_M} \text{Harm}^\pm_{X_M}(M)$$

However, the $X_M$-Hodge isomorphism theorem [1] asserts that $H^\pm_{X_M}(M) \cong H^\pm_{X_M, \text{N}}(M)$. Hence, we only need to prove that

$$E_{X_M} \text{Harm}^\pm_{X_M}(M) \cong \ker d_{X_M} |_{\text{Harm}^\pm_{X_M}(M)}/d_{X_M} (\text{Harm}^\pm_{X_M}(M)) \cong H^\pm_{X_M}(M).$$

We define the map $\delta_{X_M}$ as follows:

$$\delta_{X_M}([\phi]) = [\delta_{X_M} \phi] \in H^\pm_{X_M}(M), \quad \forall [\phi] \in E_{X_M} \text{Harm}^\pm_{X_M}(M)/d_{X_M} (\text{Harm}^\pm_{X_M}(M))$$

To prove $\delta_{X_M}$ is a well-defined:

Let $\theta_1 + \theta_2 = d_{X_M} \beta$, for some $\beta \in \text{Harm}^\pm_{X_M}(M)$. i.e. $\Delta_{X_M} \beta = (d_{X_M} \delta_{X_M} + \delta_{X_M} d_{X_M}) \beta = 0.$
\[ \delta_X \theta_1 - \delta_X \theta_2 = \delta_X d_X \beta = -d_X \delta_X \beta = d_X \sigma \in d_X \Omega^\pm \] (2.4)

Moreover, \( \delta_X \beta \) is \( X_M \)-harmonic as \( \Delta_X (\delta_X \beta) = \delta_X d_X \delta_X \beta = \delta_X^2 (\theta_1 - \theta_2) = 0 \). It means that \( \delta_X (\theta_1 - \theta_2) \in d_X \text{Harm}^\pm_{X_M} \). Thus, \( \delta_X \) is a well-defined.

Next, we prove \( \delta_X \) is one-to-one. To this end, let \( \phi \in E_X \text{Harm}^\pm_{X_M} (M) \) and \( \delta_X \phi \in d_X \Omega^\pm_G \). We only need to prove \( \phi \in d_X (\text{Harm}^\mp_{X_M} (M)) \). So, \( \phi = d_X \beta \), and therefore

\[ \Delta_X \beta = (d_X \delta_X + \delta_X d_X) \beta = d_X \delta_X + \delta_X \phi \in d_X \Omega^\pm_G \]

Thus, \( \Delta_X \beta = d_X \eta \) for some \( \eta \in \Omega^\pm_G \), but \( \Delta_X \beta \) is onto by lemma (2.2) then we can write \( \eta = \Delta_X \sigma \). Hence, \( \Delta_X \beta = d_X \eta = d_X \sigma = \Delta_X d_X \sigma = \Delta_X (\delta_X - d_X) \sigma \) which implies that \( \beta - d_X \sigma \in \text{Harm}^\mp_{X_M} (M) \). Hence, we can rewrite \( \phi = d_X \beta \) as follows, \( \phi = d_X (\beta - d_X \sigma) \in d_X (\text{Harm}^\mp_{X_M} (M)) \).

Finally, to prove \( \delta_X \) is onto. Given \( \alpha \in \ker d_X \), we should find \( \phi \in E_X \text{Harm}^\pm_{X_M} (M) \) such that \( \delta_X \phi - \alpha \in \ker d_X \). Applying lemma (2.2) on \( \alpha \), then we can write \( \alpha = \Delta_X \beta \) and then we take \( \phi = d_X \beta \). one should notice that \( \Delta_X \phi = \Delta_X \delta_X \beta = d_X \Delta_X \beta = d_X \alpha = 0 \), so \( \alpha \in \ker d_X \). Thus, \( \phi \in E_X \text{Harm}^\pm_{X_M} (M) \). Now,

\[ \delta_X \phi - \alpha \in \ker d_X \Omega^\pm_G \]

So, \( \delta_X \phi - \alpha \in \ker d_X \Omega^\pm_G \), as desired. Hence \( \delta_X \phi - \alpha \in \ker d_X \Omega^\pm_G \), as desired. Hence \( \delta_X \) is bijection map. So, eq.(2.3) holds.

In addition, \( \Delta_X \) and \( \delta_X \) commute. Hence, the coboundary operator \( \delta_X \) preserves the \( X_M \)-harmonicity of invariant forms. i.e.

\[ \text{Harm}^\pm_{X_M} (M) \xrightarrow{\delta_X} \text{Harm}^\mp_{X_M} (M) \]

Thus, \( (\text{Harm}^\pm_{X_M} (M), \delta_X) \) is a subcomplex of the \( \Omega^\pm_G \)-graded complex \( (\Omega^\pm_G, \delta_X) \). Therefore, we can compute the \( X_M \)-cohomology of this complex which we denote by \( H^\pm (\text{Harm}^\pm_{X_M} (M), \delta_X) \). So, applying the Hodge star to the isomorphism given by theorem 2.3 and replace \( n \) by \( \pm \) and then using \( X_M \)-Poincaré-Lefschetz duality (1.7) to obtain the following corollary.

**Corollary 2.4**

\[ H^\pm (\text{Harm}^\pm_{X_M} (M), \delta_X) \cong H^\pm_{X_M} (M, \partial M) + H^\mp_{X_M} (M, \partial M) = H^\pm_{X_M} (M, \partial M) \]

### 3 Conclusions

In [1], we elucidate the connection between the \( X_M \)-cohomology groups and the relative and absolute equivariant cohomology groups (i.e. \( H^G_{\mp} (M) \) and \( H^G_{\pm} (M, \partial M) \)) which are modules over \( \mathbb{R}[u_1, \ldots, u_r] \) and the result is the following theorem.
Theorem 3.1 [1]. Let \( \{X_1, \ldots, X_\ell \} \) be a basis of the Lie algebra \( g \) and \( \{u_1, \ldots, u_\ell \} \) the corresponding coordinates and let \( X = \sum_j s_j X_j \in g \). If the set of zeros \( N(X_M) \) of the corresponding vector field \( X_M \) is equal to the fixed point set \( F \) for the \( G \)-action then

\[
H_{X_M}^\pm(M, \partial M) \cong H_G^\pm(M, \partial M) / m_X H_G^\pm(M, \partial M) \cong H^\pm(F, \partial F),
\]

(3.1)

and

\[
H_{X_M}^\pm(M) \cong H_G^\pm(M) / m_X H_G^\pm(M) \cong H^\pm(F)
\]

(3.2)

where \( m_X = \langle u_1 - s_1, \ldots, u_\ell - s_\ell \rangle \) is the ideal of polynomials vanishing at \( X \).

We conclude that theorem 3.1, theorem 2.3 and corollary 2.4 prove the following theorem:

Theorem 3.2 With the hypotheses of the theorem 3.1. Then the (even or odd) \( X_M \)-harmonic cohomology of the subcomplexes \( (\text{Harm}_M^X(M), d_{X_M}) \) and \( (\text{Harm}_M^X(M), \delta_{X_M}) \) completely determine the free part of the absolute and relative equivariant cohomology groups, i.e.

\[
H^\pm(\text{Harm}_M^X(M), d_{X_M}) \cong H_G^\pm(M, \partial M) / m_X H_G^\pm(M, \partial M) \cong H^\pm(F)
\]

and

\[
H^\pm(\text{Harm}_M^X(M), \delta_{X_M}) \cong H_G^\pm(M, \partial M) / m_X H_G^\pm(M, \partial M) \cong H^\pm(F, \partial F).
\]

References


School of Mathematics,
University of Manchester,
Oxford Road,
Manchester M13 9PL,
UK.

Qusay.Abdul-Aziz@postgrad.manchester.ac.uk