A 195,747,435 vertex graph related to the Fischer group Fi23 ,part II

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A 195,747,435 VERTEX GRAPH RELATED TO THE FISCHER GROUP Fi_{23} ,II

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Introduction

In this paper and [6] we complete our investigation, begun in [5], of the structure of the 195,747,435 vertex graph $\mathcal{G}$, which is the point line collinearity graph of a certain geometry Γ associated with the Fischer group Fi_{23}. We continue the section numbering of [5] and refer the reader to Section 2 for the descriptions of $\mathcal{G}$ and the geometry Γ. Our main results are given in Section 1 and the notation we employ is to be found in Section 2. Additionally, Section 2 contains detailed information on the line orbits which is of crucial importance for many of our arguments. Diamonds, a particular configuration in $\mathcal{G}$, arise frequently - see Section 3 for their definition and properties.

Just as in [5], a denotes a fixed point of $\mathcal{G}$. Before previewing the contents of this paper, we take stock of our progress so far. As noted in [5] Theorems 2, 3 and 4 have been established. Combining together Lemmas 4.8, 4.15, 4.16, Theorem 4.13 with the definitions in (2.15) we see that Theorems 5, 7 and 8 follow. So, in $\Delta_3(a)$, we have complete knowledge of the point distribution of the lines in $\Gamma_1(x)$ for x in the G_a-orbits $\Delta_3^1(a)$, $\Delta_3^3(a)$ and $\Delta_3^4(a)$. Information about $\Delta_3^2(a)$ is almost complete, only awaiting verification of Lemma 6.6 for Theorem 6 to be proven. The remaining two G_a-orbits of $\Delta_3(a)$, $\Delta_3^5(a)$ and $\Delta_3^6(a)$, are more elusive and a complete picture of them only emerges in [6].

Here, and in [6], we take the scalpel to $\Delta_4(a)$, the fourth disc of a. Eventually we shall learn that $\Delta_4(a)$ is the union of six G_a-orbits $\Delta_4^i(a)$ $(1 \leq i \leq 6)$. The procedure we adopt in investigating each $\Delta_4^i(a)$ is first to identify G_{ax}, the stabilizer of x in G_a $(x \in \Delta_4^i(a))$. An initial step in this is the study of $\{d, x\}^\perp$ for certain $d \in \Delta_3(a)$. The bound given in Lemma 5.3(ii) is useful in reducing the number of possible cases we must consider. Having pinpointed G_{ax}, we then have access to the line orbits and their associated combinatorial descriptions as detailed in Section 2. Then we move on to determine to which G_a-orbits the points on lines in $\Gamma_1(x)$ belong. It is here that we make great use of the diamonds in $\mathcal{G}$. Unfortunately, it does not appear to be possible to deal completely with each G_a-orbit one at a time. Rather we have to content ourselves with partial information about a particular G_a-orbit and then analyse other G_a-orbits before it is possible to return to the earlier orbit and refine the information previously obtained. This neccessity of having to advance on several different fronts simultaneously is particularly marked in $\Delta_4(a)$. As a result it is difficult to keep track of what has been proved and where it has been proved. So, in Appendix B (at the end of [6]) we list the results giving the point line distribution for each G_{ax}-orbit of $\Gamma_1(x)$, $x \in \Delta_3(a) \cup \Delta_4(a)$. In passing we mention that there is an Appendix A (also at the end of [6]) in which we itemize all heptads having a certain property – this is called upon in Lemmas 8.1, 8.6, 8.8 and 12.1.

The first section of this paper contains some general observations about $\mathcal{G}$ – Lemmas 5.4–5.6 are frequently deployed, along with various numerical data, to transfer information from one G_a-orbit to another. In Section 6 we begin analysing $\Delta_4^1(a)$ - there we uncover the structure of G_{ax} ($x \in \Delta_4^1(a)$) and are able to tidy up certain matters relating to $\Delta_3^2(a)$ and $\Delta_3^5(a)$. However certain issues involving $\Delta_4^1(a)$ are not resolved before, in the following section, we next look at $\Delta_4^2(a)$. In the long Section 8 we consider $\Delta_4^4(a)$ obtaining a great deal of data about this orbit (though we cannot settle the orbits $\alpha_5(x, \mathrm{END}, +)$, $\alpha_5(x, \mathrm{END}, -)$ here, $x \in \Delta_4^4(a)$) as well as reconsidering (and finishing) $\Delta_4^2(a)$. The last three G_a-orbits of $\Delta_4(a)$, $\Delta_4^3(a)$, $\Delta_4^5(a)$ and $\Delta_4^6(a)$, plus unfinished business with the orbits $\Delta_4^1(a)$ and $\Delta_4^4(a)$ are the subject of [6].

We end this introduction with some remarks on our labelling conventions for points in $\mathcal{G}$. Usually we use x to denote the point of $\mathcal{G}$ we are currently most interested in. Additionally, whenever possible, we use the letters b (or b_i, b'), c (or c_i, c') for points of $\mathcal{G}$ in, respectively, $\Delta_1(a)$, $\Delta_2(a)$; and d, e (or d_i, d', e_i, e') for points in $\Delta_3(a)$.

5 Preliminary observations on $\Delta_4(a)$

We begin by showing that the sets $\Delta_4^i(a)$ defined in (2.15) are, in fact, all in $\Delta_4(a)$. Then, in the next result we verify that each $\Delta_4^i(a)$ is a G_a-orbit.

Lemma 5.1. *(i) Let $x \in \Gamma_0$. If $\Gamma_3(a, x) \neq \emptyset$, then $d(a, x) \leq 3$.*

(ii) If $x \in \Delta_4^i(a)$, $i \in \{1, \ldots, 6\}$, then $d(a, x) = 4$ (and so $\Gamma_3(a, x) = \emptyset$).

Proof. Part (i) follows directly from [3;Appendix 1]. Turning to part (ii), let $i = 1$ and assume that $d(a, x) \leq 3$, and argue for a contradiction. By (2.15) there exists $d \in \Delta_3^1(a) \cap \Delta_1(x)$ such that $d + x \in \alpha_0(d, D(d, a))$. If $x \in \Delta_2(a) = \Delta_2^1(a) \cup \Delta_2^2(a)$, then $x \in \Gamma_0(X)$ for each $X \in D(d, a)$ by Lemma 4.3(i). While $x \in \Delta_3^1(a) \cup \Delta_3^2(a) \cup \Delta_3^3(a) \cup \Delta_3^4(a) \cup \Delta_3^5(a) \cup \Delta_3^6(a)$ also yields, using Lemmas 4.15 and 4.16, that $x \in \Gamma_0(X)$ for each $X \in D(d, a)$. This contradicts $d + x \in \alpha_0(d, D(d, a))$. Therefore, by Lemma 4.7, $d(a, x) = 4$. Likewise we may establish (ii) for $i = 2, \ldots, 6$. □

Lemma 5.2. *For $i \in \{1, \ldots, 6\}$, $\Delta_4^i(a)$ is a G_a-orbit.*

Proof. Suppose $i = 1$. Then by (2.15) there must be a $d \in \Delta_3^1(a) \cap \Delta_1(x)$ such that $d + x \in \alpha_0(d, D(d, a))$. Now by Lemma 4.8(i) $G_{ad}^{*d} \sim L_3(4)2$ and by (2.4), $\alpha_0(d, D(d, a))$ is a G_{ad}-orbit. Combining this with Lemmas 3.2 and 4.4 shows that $\Delta_4^1(a)$ is a G_a-orbit. Similar considerations, using (2.3), (2.5) and (2.6), deal with the other cases. □

We make frequent use of our next lemma – part (ii) is particularly valuable in determining the size of $\Delta_1(x) \cap \Delta_3(a)$ for various $x \in \Delta_4(a)$.

Lemma 5.3. *Let $x \in \Delta_4(a)$ and d_1, $d_2 \in \Delta_3(a) \cap \Delta_1(x)$, with $d_1 \neq d_2$.*

(i) If $X_i \in \Gamma_3(a, d_i)$ for $i = 1, 2$, then $X_1 \neq X_2$.

(ii) $|\Delta_1(x) \cap \bigcup_{i=1}^4 \Delta_3^i(a)| \leq 23$.

Proof. Let $X_i \in \Gamma_3(a, d_i)$, $i = 1, 2$. If $X_1 = X_2$, then Lemma 3.6 implies that $\Gamma_3(a, x) \neq \emptyset$, contradicting Lemma 5.1(ii). Thus (i) holds. Since $|\Gamma_3(a)| = 23$ and $\Gamma_3(a, d) \neq \emptyset$ for $d \in \bigcup_{i=1}^4 \Delta_3^i(a)$, (ii) follows from (i). □

The following three results are all variations upon the same theme – these results aid us in identifying to which $G_{ax'}$-orbit l ($\in \Gamma_1(x')$) belongs, using information about $\Gamma_1(x)$, where $x \in \Gamma_0(l) \backslash \{x'\}$.

Lemma 5.4. *Let Λ and Λ' be distinct G_a-orbits of $\mathcal{G}$ and let $x \in \Lambda$, $x' \in \Lambda'$ be such that $d(x, x') = 1$. If $\Delta_1(x) \cap \Lambda'$ is a G_{ax}-orbit, then $\Lambda_1(x') \cap \Lambda$ is a $G_{ax'}$-orbit and*

$$|\Delta_1(x') \cap \Lambda| = [G_{ax'} : G_{ax'x}] = \frac{|\Lambda|}{|\Lambda'|} |\Delta_1(x) \cap \Lambda'|.$$

Proof. See [Lemma 3.6; 4].

□

Lemma 5.5. *Suppose that Λ and Λ' are distinct G_a-orbits of $\mathcal{G}$ and let $x \in \Lambda$, $x' \in \Lambda'$ be such that $d(x, x') = 1$. Let $\mathcal{O}_x$ be the G_{ax}-orbit of $\Gamma_1(x)$ which contains $x + x'$, and put $L = \bigcup\{\mathcal{O}_x^g \mid g \in G_a\}$. If $\Gamma_0(x + x') \not\subseteq \Lambda \cup \Lambda'$, then*

(i) $\Gamma_1(x') \cap L$ is a $G_{ax'}$-orbit of $\Gamma_1(x')$; and

(ii) $|\Gamma_1(x') \cap L| = \frac{|\mathcal{O}_x||\Lambda|}{|\Lambda'|}$.

Proof. See [Lemma 3.7;4] □

Lemma 5.6. *Suppose that Λ and Λ' are distinct G_a-orbits of $\mathcal{G}$ and let $x \in \Lambda$, $x' \in \Lambda'$ be such that $d(x, x') = 1$. Assume that $|\Gamma_0(x + x') \cap \Lambda| = 1$ and $|\Gamma_0(x+x') \cap \Lambda'| = 2$ and that there exists $t \in G_a$ such that t interchanges the two points in $\Gamma_0(x + x') \cap \Lambda'$. Let $\mathcal{O}_x$ and $\mathcal{O}'_{x'}$ be, respectively, the G_{ax} (respectively $G_{ax'}$)-orbit of $\Gamma_1(x)$ (respectively $\Gamma_1(x')$) containing $x + x'$ (respectively $x' + x$). Set $L = \bigcup\{\mathcal{O}_x^g \mid g \in G_a\}$ and $L' = \bigcup\{{\mathcal{O}'_{x'}}^g \mid g \in G_a\}$. Then $\Gamma_1(x') \cap L = \mathcal{O}'_{x'}$, $\Gamma_1(x) \cap L' = \mathcal{O}_x$ and therefore $L = L'$. Moreover,*

$$2|\mathcal{O}_x||\Lambda| = |\mathcal{O}'_{x'}||\Lambda'|.$$

Proof. See [Lemma 3.8; 4] □

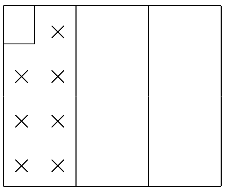
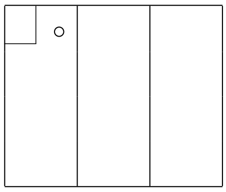
We conclude this section with a result which determines the point distribution of the G_{ax}^{*x}-orbits on $\Gamma_1(x)$ of size 40, for $x \in \Delta_3^5(a)$. Representatives of these two orbits were not given in (2.7) because they are indistinguishable when only viewed in Γ_x. So let $x \in \Delta_3^5(a)$, $\{c\} = \Delta_1(x) \cap \Delta_2^2(a)$ and $\{b\} = \Delta_1(a) \cap \Delta_2^2(x)$ (the existence of c and b follows from Lemmas 3.8(i) and 4.11(i)). Recall that G_{ax}^{*x} ($\sim 2^4 A_5$) is the stabilizer in $G_{ax}^{*x}(\cong M_{23})$ of the hexad $x+c$ and the element $X(x, b)$ of Ω_x. The orbits of G_{ax}^{*x} on $\Gamma_1(x)$ are described in (2.7).

Lemma 5.7. *Let $l \in \alpha_3(x, x+c, -)$ and $\Gamma_0(l) = \{x, y, y'\}$. Then the possibilities $\{c, y\}^\perp \cap \Delta_3^2(a) \neq \emptyset$ and $\{c, y\}^\perp \cap \Delta_3^2(a) = \emptyset$ both occur. Further, the following hold.*

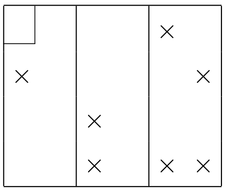
(i) If $\{c, y\}^\perp \cap \Delta_3^2(a) \neq \emptyset$, then $|\Gamma_0(l) \cap \Delta_4^3(a)| = 2$.

(ii) If $\{c, y\}^\perp \cap \Delta_3^2(a) = \emptyset$, then either $y \in \Delta_4^5(a)$, $y' \in \Delta_4^6(a)$ or $y' \in \Delta_4^5(a)$, $y \in \Delta_4^6(a)$.

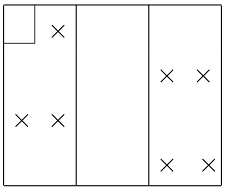
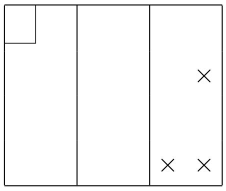
Proof. In order to work concretely in Ω_c, we take

$c + b =$  and $X(c, a) =$ 

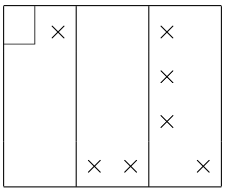
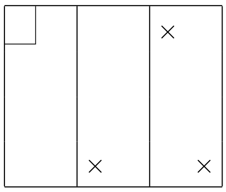
Since $c + x \in \alpha_{1,0}(c, c + b, X(c, a))$ and $\alpha_{1,0}(c, c + b, X(c, a))$ is a G_{ac}-orbit, we may assume

$c + x =$ .

Let $c + e$ be a heptad with $c + e \in \alpha_3(c, c + x)$ and let $z \in \{x, e\}^\perp \backslash \{c\}$. If we choose $c+e$ so as $(c+b)\cap(c+x)$ $(= X(x, b))$ is not in $(c+x)\cap(c+e) = T(c, z)$, then $x + z \in \alpha_3(x, x + c, -)$. Choosing

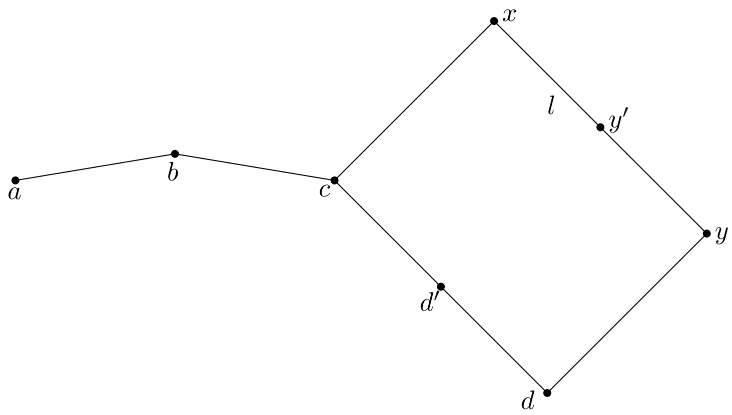
$c + e =$  (so $T(c, z) =$ )

we observe, courtesy of Theorem 4, that $e \in \Delta_3^2(a)$. Thus we have $\{c, z\}^\perp \cap \Delta_3^2(a) \neq \emptyset$ with $x + z \in \alpha_3(x, x + c, -)$. If, instead, we chose

$c + e =$  (so $T(c, z) =$ )

then, again using Theorem 4, we see that we have an instance of $\{c, z\}^\perp \cap \Delta_3^2(a) = \emptyset$ with $x + z \in \alpha_3(x, x + c, -)$. We now move on to consider (i) and (ii).

Let $X = X(x,b)$, the unique hyperplane in $\Gamma_3(c+b, c+x)$. Also let $d \in \{c,y\}^{\perp}\backslash\{x\}$ where, for part (i) we further suppose that $d \in \Delta_3^2(a)$. So we have



where $\Gamma_0(c+d) = \{c, d', d\}$. We first prove part (i). Since $d \in \Delta_3^2(a)$, by Theorem 4 $b \in \Delta_2^1(d)$. Because $l = x+y \in \alpha_3(x, x+c, -)$, $X \notin \Gamma_3(x+y)$. So, in particular, $X \notin T(c,y)$. In Ω_c we have

$$(c+b) \cap T(c,y) \subseteq (c+b) \cap (c+x) = \{X\},$$

whence we deduce that $(c+b) \cap T(c,y) = \emptyset$. Since $T(d,b) \subseteq c+b$ and $T(d,x) = T(c,y)$, (in Ω_c)

$$T(d,b) \cap T(d,x) \subseteq (c+b) \cap T(c,y) = \emptyset.$$

From Theorem 4.13(iv), in Ω_d, $|(d+c) \cap O(d,a)| = 4$ and $O(d,a) \cap T(d,b) = \emptyset$. So $d+c = ((d+c) \cap O(d,a)) \dot{\cup} T(d,b)$. Then $T(d,x) \subseteq d+c$ and $T(d,b) \cap T(d,x) = \emptyset$ force $T(d,x) \subseteq O(d,a)$. Consequently $|(d+y) \cap O(d,a)| = 4$. From $X(d,a) \in T(d,b)$, $T(d,b) \subseteq d+c$ and $T(d,b) \cap T(d,x) = \emptyset$ (in Ω_d) we also note that $y \notin \Gamma_0(X(d,a))$. Therefore $y \in \Delta_4^3(a)$ by definition (see (2.15)(xi)). Since $d' \in \Delta_3^2(a)$ by Theorem 4 and $y' \in \Delta_1(d')$ by Lemma 3.10, similarly we have $y' \in \Delta_4^3(a)$, and this proves part (i).

For part (ii), as $c+x \notin \Gamma_1(X(c,a))$, by Lemma 3.11(ii) we may suppose d is chosen so that $\Gamma_3(a,d) \neq \emptyset$. By assumption $d \notin \Delta_3^2(a)$ and hence, by Theorem 4 we may assume, without loss of generality, that $d \in \Delta_3^3(a)$ and $d' \in \Delta_3^4(a)$.

From $\Gamma_3(a,x) = \emptyset$ and Lemma 3.6 we see that $y, y' \notin \Gamma_0(X(d,a))$ because $X(d,a) = X(c,a)$. Using Lemma 3.10, $y \in \Delta_4^6(a)$ and $y' \in \Delta_4^5(a)$ by definition (see (2.15)). This completes the proof of the lemma. □

By Lemma 5.7 we can distinguish between the two G_{ax}^{*x}-orbits of size 40 on $\Gamma_1(x)$ by observing the configuration at c. The set of lines in $\alpha_3(x, x+c, -)$ satisfying part (i) of Lemma 5.7 will be labelled $\alpha_3^{(1)}(x, x+c, -)$ and the remaining lines in $\alpha_3(x, x+c, -)$, (satisfying part (ii)) will be labelled $\alpha_3^{(2)}(x, x+c, -)$.

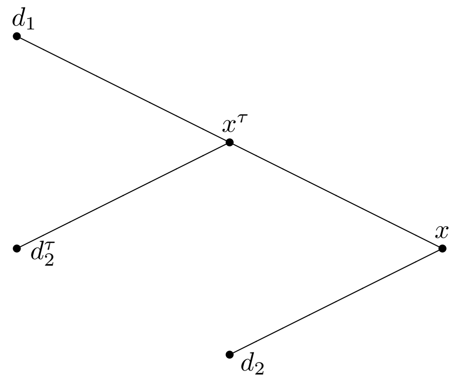
6 A first look at $\Delta_4^1(a)$

In this section we begin our investigation of the point distribution of lines in $\Gamma_1(x)$ for $x \in \Delta_4(a)$. Some of this comes as easy corollaries of results about lines incident with points in $\Delta_3(a)$.

For the whole of this section we assume $x \in \Delta_4^1(a)$. By definition (see (2.15)) $\Delta_1(x) \cap \Delta_3^1(a) \neq \emptyset$. We first show that this set has a unique point.

Lemma 6.1. $|\Delta_1(x) \cap \Delta_3^1(a)| = 1$.

Proof. For a contradiction assume that d_1, $d_2 \in \Delta_1(x) \cap \Delta_3^1(a)$ with $d_1 \neq d_2$. Hence $D(a, d_1) \cap D(a, d_2) = \emptyset$ by Lemma 5.3(i). Let $X \in \Gamma_3(a, d_1)$ and $Y \in \Gamma_3(a, d_2)$. So, in Ω_a, X is an element of $D(a, d_1)$ and Y is an element of $D(a, d_2)$. If $\tau := \tau(X)$ then $x^\tau \neq x$ by Lemma 3.2. Since $\tau \in Q(a)$, $d_2^\tau \in \Gamma_0(Y)$ and we have



where $d_2 \neq d_2^\tau$ by Lemma 3.4. Furthermore $d_2^\tau \in \Delta_1(d_2)$ by Lemma 4.1 because

$x, x^\tau \notin \Gamma_0(Y)$. If $\Gamma_0(d_2 + d_2^\tau) = \{d_2, c, d_2^\tau\}$ then $c \in \Delta_2^1(a)$ by Lemma 4.16 and [3;Appendix 1]. However Lemma 3.10 implies that $c \in \Delta_1(d_1)$ and so, in Ω_a, $D(a, d_1) \cup D(a, d_2) \subseteq T(a, c)$ by Theorem 4.13(iii). But then $D(a, d_1) \cap D(a, d_2) \neq \emptyset$, a contradiction.

□

Lemma 6.2. *(i)* $|\Delta_4^1(a)| = 2^{13}.3.5.11.23$.

(ii) $G_{ax}^{*a} \sim L_3(2).2$ *and* $Q(a)_x \cong 2$.

Proof. By Lemma 6.1 we have $\{d\} = \Delta_3^1(a) \cap \Delta_1(x)$ for some $d \in \Gamma_0$. Then $|\Delta_1(d) \cap \Delta_4^1(a)| = 240$ by (2.4) and (2.15). Taking this together with Lemmas 4.8(i) and 6.1 we get

$$|\Delta_4^1(a)| = 240 \times 2^9.11.23 = 2^{13}.3.5.11.23.$$

For part (ii), Lemma 6.1 implies that $G_{ax} \leq G_{ad}$. By Lemma 4.8(i) $Q(a)_d \cong 2^2$. In fact, $Q(a)_d = Q(d)_a = \langle \tau(Y_1), \tau(Y_2) \rangle$ where $\{Y_1, Y_2\} = \Gamma_3(a, d)$. Also $x^{\tau(Y_i)} \neq x$ by Lemma 3.2 because $x \notin \Gamma_0(Y_i)$ $(i = 1, 2)$. However $\tau(Y_1)\tau(Y_2) \in G_{ax}$, whence $Q(a)_x = \langle \tau(Y_1)\tau(Y_2) \rangle \cong 2$. Since $[G_{ad} : G_{ax}] = 240$ we then have $[G_{ad}^{*a} : G_{ax}^{*a}] = 120$. Examining the maximal subgroups of G_{ad}^{*a} $(\sim L_3(4)2)$ using [1] we see that $G_{ax}^{*a} \sim L_3(2).2$ is the only possibility.

□

The next result shows that x is collinear with points in other $\Delta_3(a)$ G_a-orbits.

Lemma 6.3. *(i)* $|\Delta_1(x) \cap \Delta_3^2(a)| = 7$.

(ii) Let $e \in \Delta_1(x) \cap \Delta_3^2(a)$. *Then* $e + x \in \alpha_{2,0}(e, O(e, a), X(e, a))$.

Proof. Let $\{d\} = \Delta_1(x) \cap \Delta_3^1(a)$. By considering the set of 21 heptads $\{d + c \mid c \in \Delta_2^1(a) \cap \Delta_1(d)\}$ in Γ_d we see there exists $c \in \Delta_2^1(a) \cap \Delta_1(d)$ with $c \in \Delta_2^1(x)$. Since $\Gamma_3(a, x) = \emptyset$, $T(c, a) \cap T(c, x) = \emptyset$ in Ω_c. Hence there exists $e \in \{c, x\}^\perp$ such that $|(c + e) \cap T(c, a)| = 1$ or 3 in Ω_c. Since $x \in \Delta_4(a)$ we must have $|(c + e) \cap T(c, a)| = 1$ by Theorem 3, whence, again by Theorem 3, $e \in \Delta_3^2(a)$. Thus $\Delta_1(x) \cap \Delta_3^2(a) \neq \emptyset$.

Let e be any point in $\Delta_1(x)\cap\Delta_3^2(a)$. Then we have $e{+}x \in \alpha_{i,0}(e, O(e,a), X(e,a))$ for some $i = 0, 2$ or 4 by (2.5) because $\Gamma_3(a,x) = \emptyset$. Moreover, by Lemma 3.2 $\tau(X(e,a))$ interchanges the two points in $\Gamma_0(e{+}x)\backslash\{e\}$. If $e{+}x \in \alpha_{0,0}(e, O(e,a), X(e,a))$, then we may use (2.5) and Lemma 5.6 to show that $x+e$ lies in a G_{ax}-orbit on $\Gamma_1(x)$ of size

$$\frac{2.8.|\Delta_3^2(a)|}{|\Delta_4^1(a)|} = \frac{16.2^8.3.5.11.23}{2^{13}.3.5.11.23} = \frac{1}{2}$$

which is clearly impossible. Similarly if $e + x \in \alpha_{4,0}(e, O(e,a), X(e,a))$, then $x+e$ lies in a G_{ax}-orbit of size

$$\frac{2.56.|\Delta_3^2(a)|}{|\Delta_4^1(a)|} = \frac{112.2^8.3.5.11.23}{2^{13}.3.5.11.23} = \frac{7}{2}$$

which is again untenable. Therefore $e{+}x \in \alpha_{2,0}(e, O(e,a), X(e,a))$ which proves part (ii). Part (i) now follows from part (ii) using Lemmas 4.8(ii) and 6.2(i).

$\square$

Lemma 6.4. $a \in \Delta_4^1(x)$.

Proof. By Lemma 6.3 we can choose $e \in \Delta_3^2(a) \cap \Delta_1(x)$ and we have $e + x \in \alpha_{2,0}(e, O(e,a), X(e,a))$. Let $e{+}x = \begin{array}{|cc|cc|cc|}\hline \square & & \times & \times & \times & \times \\ & \times & & & & \\ & \times & & & & \\ & \times & & & & \\ \hline \end{array}$ with $O(e,a)$ and $X(e,a)$ as in (2.5) (with e playing the part of x there). Examining the MOG in [2] and using Theorem 4.13(iv) we see that there exists a heptad $e + c$ with $c \in \Delta_2^2(a) \cap \Delta_1(e) \cap \Delta_2^1(x)$ and $|T(c,x) \cap O(e,a)| = 1$ in Ω_e. (In fact, there are exactly two such heptads, namely

$$\begin{array}{|cc|cc|cc|}\hline \square & \times & \times & \times & \times & \times \\ & & & & & \times \\ & & & & & \times \\ & & & & & \times \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|cc|cc|cc|}\hline \square & \times & \times & \times & & \times \\ & & & & \times & \\ & & & & \times & \\ & & & & \times & \\ \hline \end{array}.)$$

Put $\{b\} = \{a,c\}^{\perp}$. Then $b \in \Delta_2^1(e)$ by Theorem 4 and $T(e,b) \cap O(e,a) = \emptyset$ in Ω_e by Theorem 4.13(iv). Hence $|T(e,b) \cap T(c,x)| = 2$ in Ω_e, which implies that $b \in \Delta_3^1(x)$ by Lemma 4.14(i). Since $d(a,x) = 4$, the result now follows from Theorem 5. $\square$

The following result gives an explicit geometric description of the group G_{ax}^{*x}.

Lemma 6.5. *Let $b \in \Delta_1(a) \cap \Delta_3^1(x)$ and $d \in \Delta_3^1(a) \cap \Delta_1(x)$. Then $D(x,b) \cap (x+d) = \emptyset$ in Ω_x and G_{ax}^{*a} is the stabilizer in G_x^{*x} ($\cong M_{23}$) of the heptad $x+d$ and the duad $D(x,b)$.*

Proof. First we note that b and d are the unique points in $\Delta_1(a) \cap \Delta_3^1(x)$ and $\Delta_3^1(a) \cap \Delta_1(x)$ respectively by Lemmas 6.1 and 6.4. Therefore G_{ax}^{*x} is contained in the stabilizer of $x+d$ and $D(x,b)$ in G_x^{*x}. Let $\Gamma_0(x+d) = \{x, x', d\}$. Assume that $D(x,b) \cap (x+d) \neq \emptyset$ and let $D(x,b) = \{X_1, X_2\}$. Suppose that $X_1 \in \Gamma_3(x+d)$ and $X_2 \notin \Gamma_3(x+d)$. Then $\tau := \tau(X_1)\tau(X_2)$ fixes a and interchanges d and x' by Lemma 3.2. However $x' \in \Delta_4^1(a)$ by definition (see (2.15)), which gives a contradiction. Hence $X_1,\ X_2 \in \Gamma_3(x+d)$. So Theorem 5 implies that either $d \in \Delta_2^1(b)$ or $x' \in \Delta_2^1(b)$. Since $x' \in \Delta_4(a)$ we must have $d \in \Delta_2^1(b)$. By Theorem 3 $|\Gamma_3(a,b,d)| = |\Gamma_3(x,b,d)| = 2$ because $a \in \Delta_3^1(d)$ and $x \in \Delta_3^1(b)$. Therefore $\Gamma_3(a,x) \neq \emptyset$ because $|\Gamma_3(b,d)| = 3$. We now have a contradiction to Lemma 5.1(ii) and thus we conclude that $D(x,b) \cap (x+d) = \emptyset$. In M_{23} the stabilizer of a heptad and a duad disjoint from the heptad is isomorphic to $L_3(2).2$. Therefore the result now follows by Lemma 6.2(ii). $\square$

We are now in a position to describe the G_{ax}-orbits of $\Gamma_1(x)$ by their intersection with $D(x,b)$ (denoted by DUAD) and the heptad $x+d$ (where $\{d\} = \Delta_3^1(a) \cap \Delta_1(x)$ and $\{b\} = \Delta_1(a) \cap \Delta_3^1(x)$). These orbits are listed in (2.9) (with b there playing the role of d).

Lemma 6.6. *Let $\{d\} = \Delta_3^1(a) \cap \Delta_1(x)$ and $\{b\} = \Delta_1(a) \cap \Delta_3^1(x)$.*

(i) If $l \in \alpha_{3,2}(x, x+d, \mathrm{DUAD})$, then $|\Gamma_0(l) \cap \Delta_3^2(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_4^1(a)| = 2$.

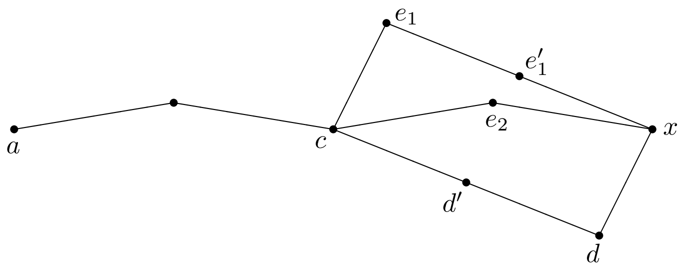
(ii) Let $e \in \Delta_3^2(a) \cap \Delta_1(x)$. Then $e+x \in \alpha_{2,0}(e, O(e,a), X(e,a))$ if and only if $x+e \in \alpha_{3,2}(x, x+d, \mathrm{DUAD})$.

(iii) If $l \in \alpha_{3,0}^{\mathfrak{L}}(x, x+d, \mathrm{DUAD})$, then $|\Gamma_0(l) \cap \Delta_3^6(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_4^1(a)| = 2$.

(iv) If $l \in \alpha_{1,2}(x, x+d, \mathrm{DUAD})$, then $|\Gamma_0(l) \cap \Delta_3^6(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_4^1(a)| = 2$.

Proof. Let $e \in \Delta_3^2(a) \cap \Delta_1(x)$. For $y \in \Delta_4^1(a) \cap \Delta_1(e)$ we have $e + y \in \alpha_{2,0}(e, O(e,a), X(e,a))$ by Lemma 6.3(ii). So, by Lemma 3.2, $|\Gamma_0(e+y) \cap \Delta_3^2(a)| = 1$ and $|\Gamma_0(e+y) \cap \Delta_4^1(a)| = 2$ and therefore, by (2.5), $\Delta_1(e) \cap \Delta_4^1(a)$ is a G_{ae}-orbit. Consequently, using Lemma 5.4, $\Delta_1(x) \cap \Delta_3^2(a)$ is a G_{ax}-orbit whence, by Lemma 6.3(i) and (2.9), $\{x + e' \mid e' \in \Delta_3^2(a)\} = \alpha_{3,2}(x, x+d, \mathrm{DUAD})$. This proves parts (i) and (ii).

For (iii), let $c \in \Delta_2^1(a) \cap \Delta_1(d)$. Since $\Gamma_3(a,x) = \emptyset$, there exist e_1, $e_2 \in \{c,x\}^{\perp} \backslash \{d\}$ with $e_1 \in \Delta_3^6(a)$ and $e_2 \in \Delta_3^2(a)$. We have



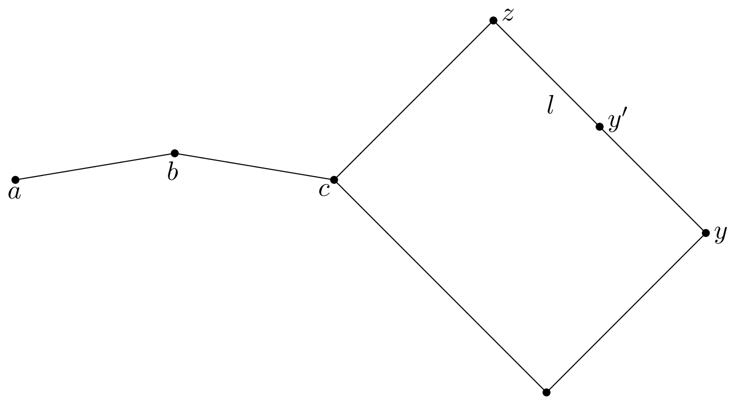
where $\Gamma_0(c+d) = \{c, d', d\}$ and $\Gamma_0(x+e_1) = \{x, e_1', e_1\}$. Since $(x+e_1) \cap (x+d) = T(x,c) = (x+e_2) \cap (x+d)$, $(x+d) \cap \mathrm{DUAD} = \emptyset$ and $|(x+e_2) \cap \mathrm{DUAD}| = 2$ in Γ_x, $x + e_1 \in \alpha_{3,0}^{\mathcal{L}}(x, x+d, \mathrm{DUAD})$ by definition (see (2.9)). We have $e_1' \in \Delta_1(d')$ by Lemma 3.10. Therefore Theorem 4.13 and (2.15) imply that $e_1' \in \Delta_2^1(a) \cup \Delta_3^1(a) \cup \Delta_3^3(a) \cup \Delta_4^1(a)$ because $d' \in \Delta_3^1(a)$. However $e_1 \in \Delta_3^6(a)$ and $x \in \Delta_4^1(a)$ which means that $e_1' \notin \Delta_2^1(a) \cup \Delta_3^1(a) \cup \Delta_3^3(a)$ by Lemmas 4.15 and 5.1(ii). This proves part (iii).

Finally we turn to part (iv). Let $c \in \Delta_2^1(x) \cap \Delta_1(b)$ (c exists by Theorem 4.13(iii)). For each $y \in \{c,x\}^{\perp}$, $y \in \Delta_2^1(b)$ and the heptad $x + y$ contains $D(x,b)$ ($=$ DUAD) in Ω_x. Therefore $a \in \Delta_3^i(y)$ for $i = 1$,2 or 6 and using symmetry (see Lemma 4.12) $y \in \Delta_3^i(a)$ for $i = 1$,2 or 6. Since $\Gamma_3(a,x) = \emptyset$, $\Gamma_3(T(c,x)) \cap \Gamma_3(a) = \emptyset$. Thus there exists y $\in \{c,x\}^{\perp}$ with y $\in \Delta_3^6(a)$. By (2.9) and part (i), $x + y \in \alpha_{1,2}(x, x+d, \mathrm{DUAD})$ because $x + y$ contains DUAD in Ω_x. Let $\Gamma_0(x+y) = \{x, y', y\}$. Then $b \in \Delta_3^1(y')$. Note that $\Gamma_3(a, y') = \emptyset$, for otherwise we have $X \in \Gamma_3(a, y')$ with $X \notin \Gamma_3(y' + x)$. But then $\tau(X)$ interchanges y and x by Lemma 3.2, which is impossible as $y \in \Delta_3(a)$ and $x \in \Delta_4(a)$. Hence $a \in \Delta_4^1(y')$ by the definition of $\Delta_4^1(y')$. Appealing to Lemma 6.4 completes the proof of part (iv) and hence of the lemma. $\square$

We return, temporarily, to the orbit $\Delta_3^5(a)$ – the information given in our next result is required in Lemma 6.8.

Lemma 6.7. *Let* $z \in \Delta_3^5(a)$ *and* $l \in \alpha_3(z, z+c, +)$ *(where* $\{c\} = \Delta_1(z) \cap \Delta_2^2(a)$*). Then* $|\Gamma_0(l) \cap \Delta_4^1(a)| = |\Gamma_0(l) \cap \Delta_4^3(a)| = 1$.

Proof. We have



where $\Gamma_0(l) = \{z, y', y\}$ and $\{b\} = \{a, c\}^{\perp}$. Since $\Gamma_3(a, z) = \emptyset$ we must have $X(c, a) \notin \Gamma_3(T(c, y))$. Let $X = X(z, b)$. By definition $y, y' \in \Gamma_0(X)$ and so $X \in \Gamma_3(T(c, y))$. Hence, of the 5 heptads in $\Gamma_1(c, T(c, y))$, one lies in $\Gamma_1(X(c, a)) \cap \alpha_3(c, c+b)$, two lie in $\alpha_3(c, c+b) \backslash \Gamma_1(X(c, a))$ and two lie in $\alpha_1(c, c+b) \backslash \Gamma_1(X(c, a))$. Therefore $\{c, y\}^{\perp}$ consists of one point d in $\Delta_3^2(a)$, two points in $\Delta_3^6(a)$ and two points in $\Delta_3^5(a)$ (including z).

In Ω_d, $|(d+c) \cap O(d, a)| = 4$ and $T(d, b) \cap O(d, a) = \emptyset$ by Theorem 4.13(iv). Since $z + y \in \alpha_3(z, z+c, +)$, $X \in \Gamma_3(z+y)$. Hence $X \in \Gamma_3(b, c, d, z, y)$ by Lemmas 3.6 and 4.3. Therefore $T(d, z) \cap T(d, b) \neq \emptyset$ in Ω_d, whence $T(d, z) \not\subseteq O(d, a)$. Also $X(d, a)$ $(= X(c, a)) \notin \Gamma_3(T(d, z))$ because $T(d, z) = T(c, y)$. Thus there exists z_1, $z_2 \in \{d, z\}^{\perp} \backslash \{c\}$ with $d + z_1 \in \alpha_{4,0}(d, O(d, a), X(d, a))$ and $d + z_2 \in \alpha_{2,0}(d, O(d, a), X(d, a))$. So, by (2.15) and Lemma 6.3(ii), we have $z_1 \in \Delta_4^3(a)$ and $z_2 \in \Delta_4^1(a)$. However $X \in \Gamma_3(T(z, d))$ implies that $z + z_i \in \alpha_3(z, z+c, +)$ for $i = 1, 2$. We have $\Delta_4^1(a) \neq \Delta_4^3(a)$ by Lemma 6.3(ii) and (2.15)(xi) because $\alpha_{2,0}(d, O(d, a), X(d, a))$ and $\alpha_{4,0}(d, O(d, a), X(d, a))$ are distinct G_{ad}-orbits. Since $\alpha_3(z, z+c, +)$ is a G_{az}^{*z}-orbit on $\Gamma_1(z)$ by (2.7) it follows that $|\Gamma_0(l) \cap \Delta_4^1(a)| = |\Gamma_0(l) \cap \Delta_4^3(a)| = 1$, as required. $\square$

Before considering $\Delta_4^1(a)$ again we summarize current progress on Theorem 9. Lemma 4.11(i) gives the size of $\Delta_3^5(a)$ and the structure of G_{az} $(z \in \Delta_3^5(a))$. In Lemmas 5.7 and 6.7 the point distribution for $\alpha_3^{(1)}(z, z+c, -)$, $\alpha_3^{(2)}(z, z+c, -)$ and $\alpha_3(z, z+c, +)$ is elucidated (where $\{c\} = \Delta_1(z) \cap \Delta_2^2(a)$). It remains to examine $\alpha_1(z, z+c, +)$ and $\alpha_1(z, z+c, -)$ and this will be done in, respectively, Lemmas 9.6(i),(ii) and 13.5(i), when we have learned more about $\Delta_4^5(a)$ and $\Delta_4^6(a)$.

Lemma 6.8. *Let* $\{d\} = \Delta_3^1(a) \cap \Delta_1(x)$ *and* $\{b\} = \Delta_1(a) \cap \Delta_3^1(x)$. *For* $l \in \alpha_{1,1}(x, x+d, \mathrm{DUAD})$ *we have that*

(i) $|\Gamma_0(l) \cap \Delta_3^5(a)| = |\Gamma_0(l) \cap \Delta_4^3(a)| = 1$*; and*

(ii) if $e \in \Gamma_0(l) \cap \Delta_3^5(a)$*, then* $l \in \alpha_3(e, e+c, +)$ *(where* $\{c\} = \Delta_2^2(a) \cap \Delta_1(e)$*).*

Proof. Let $y \in \Delta_3^5(a)$ and $k \in \alpha_3(y, y+c', +)$ where $\{c'\} = \Delta_2^2(a) \cap \Delta_1(y)$. Then, by Lemma 6.7, $|\Gamma_0(k) \cap \Delta_4^1(a)| = |\Gamma_0(k) \cap \Delta_4^3(a)| = 1$. If $z \in \Gamma_0(k) \cap \Delta_4^1(a)$, then by Lemmas 4.11(i), 5.5 and 6.2(i) we have that k lies in a G_{az}-orbit on $\Gamma_1(z)$ of size 42. Hence $k \in \alpha_{1,1}(z, z+d', \mathrm{DUAD})$ (where $\{d'\} = \Delta_1(z) \cap \Delta_3^1(a)$) by (2.9). Since $\Delta_4^1(a)$ is a G_a-orbit and $\alpha_{1,1}(x, x+d, \mathrm{DUAD})$ is a G_{ax}-orbit we have part (i). Appealing to Lemma 5.5 again yields part (ii).

$\square$

The point distribution of the remaining G_{ax}-orbits will be dealt with in Sections 8 and 12.

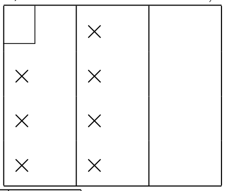
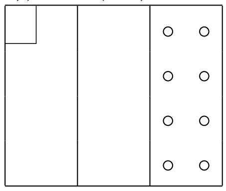
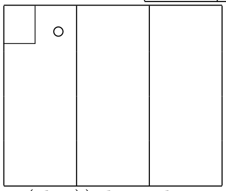
7 A first look at $\Delta_4^2(a)$

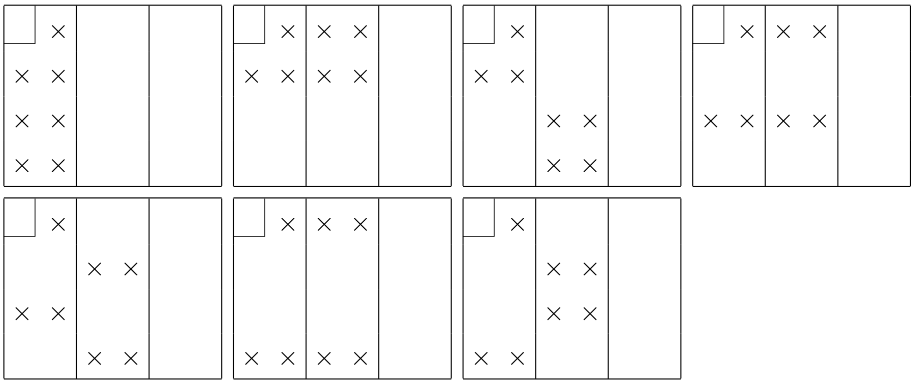
In this short section we study some of the points distance one from a point in the G_a-orbit $\Delta_4^2(a)$. Let $x \in \Delta_4^2(a)$ be fixed for the whole of this section. More specifically, in this section, after pinning down $|\Delta_4^2(a)|$ and G_{ax}, we will determine the point distribution for lines in $\alpha_0(x, 0(x, a))$ and $\alpha_4(x, 0(x, a))$.

The remaining line orbit $\alpha_2(x, 0(x, a))$ must await the verification, in Lemma 8.4, that $\Delta_4^4(a) \neq \Delta_4^6(a)$, and so is dealt with in Lemma 8.12.

Our first two results are used in the identification of G_{ax}.

Lemma 7.1. $|\Delta_1(x) \cap \Delta_3^2(a)| \geq 15$

Proof. Let $d \in \Delta_1(x) \cap \Delta_3^2(a)$ with $d + x \in \alpha_{0,0}(d, O(d, a), X(d, a))$. By (2.5) we may assume that $d+x =$  where $O(d, a) =$  and $X(d, a) =$  For all $c \in \Delta_2^1(a) \cap \Delta_1(d)$ we have $d + c \in \alpha_{0,1}(d, O(d, a), X(d, a))$ by Theorem 4.13(iv) and $\alpha_{0,1}(d, O(d, a), X(d, a))$ consists of the heptads



So we have $d+c \in \alpha_3(d, d+x)$ for all $c \in \Delta_2^1(a) \cap \Delta_1(d)$. Fix $c \in \Delta_2^1(a) \cap \Delta_1(d)$. Then $c \in \Delta_2^1(x)$ and $T(c, a) \cap T(c, x) = \emptyset$ in Ω_c because $\Gamma_3(a, x) = \emptyset$ by Lemma 5.1(ii). Using Lemma 3.11(ii) either

(1) there exists $y \in \{c, x\}^\perp$ with $|(c + y) \cap T(c, a)| = 2$ in Ω_c; or

(2) there exists y_1, $y_2 \in \{c, x\}^\perp \backslash \{d\}$, $y_1 \neq y_2$ with $|(c + y_i) \cap T(c, a)| = 1$ for $i = 1,2$ in Ω_c.

If (1) holds, then $y \in \Delta_3^1(a)$ by (2.15)(iii). However Theorem 5 and $x \in \Delta_4(a)$ imply that $x \in \Delta_4^1(a)$ which contradicts Lemma 6.3(ii) because $d + x \in \alpha_{0,0}(d, O(d,a), X(d,a))$. So (2) holds and $y_2 \in \Delta_3^2(a)$ by Lemma 4.14(ii). Since $|\Delta_2^1(a) \cap \Delta_1(d)| = 7$, to prove the lemma it is enough to show that if $c' \in \Delta_2^1(a) \cap \Delta_1(d)$ and $c' \neq c$, then $c' \notin \Delta_1(y_1) \cup \Delta_1(y_2)$. Assume $c' \in \Delta_1(y_i)$ for some $i = 1, 2$. Then $c, c' \in \{y_i, d\}^{\perp}$ and $c, c' \in \Gamma_0(X(d,a))$ by Theorem 4.13(iv). Therefore Lemma 3.6 yields $y_i \in \Gamma_0(X(d,a))$, whence $x \in \Gamma_0(X(d,a))$ by Lemma 3.6 again. We now have a contradiction to Lemma 5.1(ii) and so the lemma is proved. $\square$

Lemma 7.2. *If $G_{ax} \sim 2^4.L_3(2)$, then for any $d \in \Delta_1(x) \cap \Delta_3^2(a)$ we have $|\Delta_1(d) \cap \Delta_4^2(a)| = 16$.*

Proof. We assume the result is false and argue for a contradiction. By (2.5), (2.15) and Lemmas 3.2 and 6.3(ii) we must have $|\Delta_1(d) \cap \Delta_4^2(a)| = 16 + 112 = 128$ and $\Delta_4^2(a) = \Delta_4^3(a)$. Also, $G_{ax} \sim 2^4.L_3(2)$ together with Lemma 5.2 imply that $|\Delta_4^2(a)| = |\Delta_4^3(a)| = 2^{11}.3.5.11.23$. Using (2.7), Lemmas 5.7(i) and 6.7 there exists $e \in \Delta_3^5(a) \cap \Delta_1(x)$ with $|\Delta_1(e) \cap \Delta_4^3(a)| \geq 140$. Furthermore, for each line $l \in \alpha_3^{(1)}(e, e+c, -) \cup \alpha_3(e, e+c, +)$ (where $\{c\} = \Delta_2^2(a) \cap \Delta_1(e)$), we have $|\Gamma_0(l) \cap \Delta_3^5(a)| = 1$. Therefore, since $\Delta_4^3(a)$ and $\Delta_3^5(a)$ are G_a-orbits we have at least n lines in $\Gamma_1(x)$, incident with a point in $\Delta_3^5(a)$ where

$$n \geq \frac{140.|\Delta_3^5(a)|}{|\Delta_4^3(a)|} = \frac{140.2^{12}.3.7.11.23}{2^{11}.3.5.11.23} = 392.$$

This contradicts the fact that $\Gamma_1(x) = 253$, so proving the lemma. $\square$

Lemma 7.3. *(i) $G_{ax} \cong A_8$ with $Q(a)_x = 1$ and $Q(x)_a = 1$. Further G_x^{*x} is the stabilizer in M_{23} of an octad of Ω_x.*

(ii) $|\Delta_4^2(a)| = 2^{12}.11.23$.

(iii) $|\Delta_1(x) \cap \Delta_3^2(a)| = 15$ and, for $e \in \Delta_1(x) \cap \Delta_3^2(a)$, $|\Delta_1(e) \cap \Delta_4^2(a)| = 16$.

Proof. By the definition of $\Delta_4^2(a)$ (see (2.15)) there exists $d \in \Delta_1(x) \cap \Delta_3^2(a)$ such that

$$d + x \in \alpha_{0,0}(d, O(d,a), X(d,a)).$$

Recall that $G_{ad} \sim [2^7]L_3(2)$, $G_{ad}^{*d} \sim 2^3L_3(2)$ and $Q(a) \cap Q(d) = \langle \tau(X(a,d)) \rangle$. From (2.5) the stabilizer of $d+x$ in G_{ad}^{*d} is isomorphic to $L_3(2)$. Since $\tau(X(d,a)) \notin G_x$ by Lemma 3.2, we conclude that

(7.3.1) $G_{adx} \sim 2^3L_3(2)$ with $G_{adx} \cap Q(d) \cong 2^3$ and $G_{adx} \cap Q(a) \cap Q(d) = 1$.

Lemmas 4.13(iv) and 6.3(ii) and (2.5) imply that for $y \in \Delta_1(d) \cap \Delta_4^2(a)$ we have

$$d + x \in \alpha_{0,0}(d, O(d,a), X(d,a)) \cup \alpha_{4,0}(d, O(d,a), X(d,a)).$$

Hence, as $\Delta_4^2(a)$ is a G_a-orbit, Lemma 3.2 and (2.5) give

(7.3.2) $|\Delta_1(d) \cap \Delta_4^2(a)| = 16$ or 128.

We now show that $Q(a)_x = 1$. For $g \in Q(a)_x$ we have that g fixes $X(a,d)$, whence $d^g = d$ by Lemma 5.3(i). Thus $Q(a)_x \leq G_{adx}$ and therefore, as $Q(a)_x \trianglelefteq G_{adx}$, we get $Q(a)_x \leq Q(d)$. So $Q(a)_x \leq G_{adx} \cap Q(a) \cap Q(d) = 1$ by (7.3.1).

Using [1] to examine the subgroups of $G_a^{*a} \cong M_{23}$ which can contain a subgroup of the shape $2^3L_3(2)$ we deduce that

(7.3.3) the shape of G_{ax} is one of $2^3L_3(2)$, $2^4L_3(2)$, 2^4A_7, A_8, M_{22} and M_{23}.

Set $n = |\Delta_1(x) \cap \Delta_3^2(a)|$.

Suppose G_{ax} has shape 2^4A_7, M_{22} or M_{23}. Then, by Lemma 5.2, we have, respectively, $|\Delta_4^2(a)| = 2^{11}.11.23$, $2^{11}.23$ and 2^{11}. Thus $|\Delta_4^2(a)| \leq 2^{11}.11.23$. Combining (7.3.2) and Lemma 4.8(ii) yields

$$2^8.3.5.11.23.16 \leq |\Delta_4^2(a)|.n \leq 2^{11}.11.23.n.$$

Thus $n \geq 30$ which is impossible by Lemma 5.3(ii). So G_{ax} is not of shape 2^4A_7, M_{22} or M_{23}.

Now assume that $G_{ax} \sim 2^3L_3(2)$. Then $|\Delta_4^2(a)| = 2^{12}.3.5.11.23$. Using (7.3.2) and Lemma 4.8(ii) gives

$$2^8.3.5.11.23.128 \geq |\Delta_4^2(a)|.n = 2^{12}.3.5.11.23.n,$$

whence $n \leq 8$. This situation is ruled out by Lemma 7.1.

Next we consider the case $G_{ax} \sim 2^4L_3(2)$. So $|\Delta_4^2(a)| = 2^{11}.3.5.11.23$ and, by Lemma 7.2, $|\Delta_1(d) \cap \Delta_4^2(a)| = 16$ which then yields

$$2^8.3.5.11.23.16 = |\Delta_4^2(a)|.n = 2^{11}.3.5.11.23.n.$$

But then $n = 2$, contrary to Lemma 7.1. Therefore, in view of (7.3.3), $G_{ax} \cong A_8$ is the only possibility. Clearly we then get that $Q(x)_a = 1$. Consulting [1] we further conclude that G^{*x}_{ax} is the stabilizer in M_{23} of an octad in Ω_x. Also

$$|\Delta_4^2(a)| = \frac{2^{18}.3^2.5.7.11.23}{2^6.3^2.5.7} = 2^{12}.11.23$$

If $|\Delta_1(d) \cap \Delta_4^2(a)| = 128$, then we see that $n = 120$, against Lemma 5.3(ii). Thus, by (7.3.2), $|\Delta_1(d) \cap \Delta_4^2(a)| = 16$ and then we obtain part (iii), so proving the lemma.

□

We will use $O(x, a)$ to denote the octad of Ω_x stabilized by G^{*x}_{ax} ($\cong G_{ax} \cong A_8$). Referring to (2.10) we see that G_{ax} has 3 orbits on $\Gamma_1(x)$.

Corollary 7.4. *Let* $d \in \Delta_1(x) \cap \Delta_3^2(a)$. *Then*

(i) $x + d \in \alpha_0(x, O(x, a))$;

(ii) $d + x \in \alpha_{0,0}(d, O(d, a), X(d, a))$; *and*

(iii) $\Gamma_0(d + x)\backslash\{d\} \subseteq \Delta_4^2(a)$.

Proof. Parts (i) and (ii) follow from (2.5), (2.10) and Lemmas 5.4 and 7.3(iii). Now part (ii) and (2.15) give part (iii). □

Lemma 7.5. *Let* $l \in \alpha_4(x, O(x, a))$. *Then* $|\Gamma_0(l) \cap \Delta_3^6(a)| = |\Gamma_0(l) \cap \Delta_4^3(a)| = 1$.

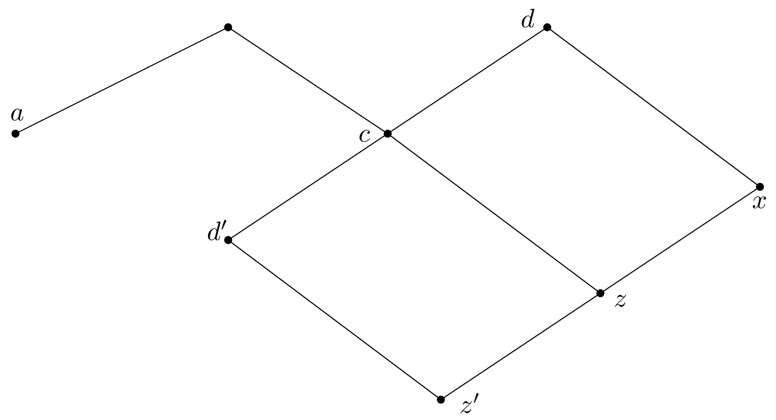
Proof. Let $d \in \Delta_1(x) \cap \Delta_3^2(a)$; so $d + x \in \alpha_{0,0}(d, O(d, a), X(d, a))$ by Corollary 7.4(ii). We may choose $c \in \Delta_1(d)$ with $c \in \Delta_2^1(a) \cap \Delta_2^1(x)$. If $\Gamma_0(c + d) = \{c, d', d\}$, then $d' \in \Delta_3^2(a)$ by Lemma 4.6(i). Since $X(d, a) \notin \Gamma_3(T(d', x))$, $y \notin \Gamma_0(X(d, a))$ for each $y \in \{d', x\}^{\perp}\backslash\{d\}$ by Lemma 3.6. Moreover, by Theorem 4.13(iv), the heptad $d' + d$ has empty intersection with the octad $O(d', a)$ in $\Omega_{d'}$, which means that $T(d', x) \cap O(d', a) = \emptyset$ in $\Omega_{d'}$. Therefore either,

(1) $|\{d', x\}^{\perp} \cap \Delta_4^1(a)| = 4$; or

(2) $|\{d', x\}^{\perp} \cap \Delta_4^2(a)| = 2$ and $|\{d', x\}^{\perp} \cap \Delta_4^3(a)| = 2$.

Assume (1) holds. By Lemma 3.10 c is collinear with the point in $\Gamma_0(x+y)\backslash\{x,y\}$ for each $y \in \{d',x\}^\perp$. Using Lemma 3.11(ii) we can choose $y' \in \{c,x\}^\perp$ with $y' \neq d$ and $\Gamma_3(a,y') \neq \emptyset$, and, by Lemma 3.10, $y' \in \Gamma_0(x+y)\backslash\{x,y\}$ for some y $\in \{d',x\}^\perp$. Let $X \in \Gamma_3(a,y')$. Then $x,y \notin \Gamma_0(X)$, whence $x^{\tau(X)} = y$ by Lemma 3.2. Since $\tau(X) \in G_a$ we must have $\Delta_4^2(a) = \Delta_4^1(a)$. This clearly contradicts Lemmas 6.2(i) and 7.3(ii). Thus (1) cannot hold and we are in case (2).

We now examine $\{c,x\}^\perp$. By Theorem 5, if $e \in \Delta_3^1(a)$, then $\Delta_1(e)\cap\Delta_4(a) \subseteq \Delta_4^1(a)$. Hence $\{c,x\}^\perp \cap \Delta_3^1(a) = \emptyset$. Furthermore $T(c,x) \cap T(c,a) = \emptyset$ in Ω_c because $\Gamma_3(a,x) = \emptyset$. Therefore we must have $|\{c,x\}^\perp \cap \Delta_3^2(a)| = 3$ and $|\{c,x\}^\perp \cap \Delta_3^6(a)| = 2$. Let $y \in \{c,x\}^\perp$ and $\Gamma_0(x+y) = \{x,y',y\}$. If $y \in \Delta_3^2(a)$, then $y' \in \Delta_4^2(a)$ by Corollary 7.4(iii). So by (2), if $y \in \Delta_3^6(a)$ we must have $y' \in \Delta_4^3(a)$. Assume $z \in \Delta_3^6(a) \cap \{c,x\}^\perp$ and $\Gamma_0(x+z) = \{x,z',z\}$. So $z' \in \Delta_4^3(a)$.



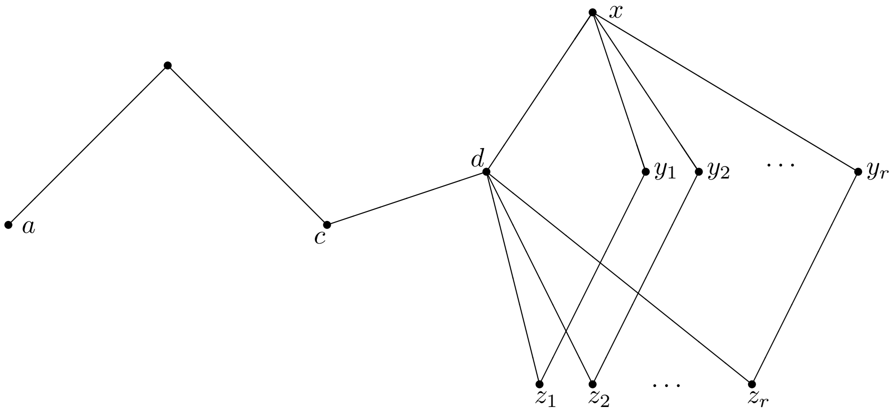
In $\Omega_x, (x+y) \cap O(x,a) = \emptyset$ for all $y \in \Delta_3^2(a) \cap \{c,x\}^\perp$ because $x+y \in \alpha_0(x,O(x,a))$ by Corollary 7.4(i). Hence $T(x,c) \cap O(x,a) = \emptyset$ and $|(x+z) \cap O(x,a)| = 4$ in Ω_x. Therefore $x+z \in \alpha_4(x,O(x,a))$ and since $\alpha_4(x,O(x,a))$ is a G_{ax}-orbit, the lemma is proved. $\square$

8 A first look at $\Delta^4_4(a)$

We delay the exploration of $\Delta^3_4(a)$ until Section 11 and turn our attention instead to $\Delta^4_4(a)$. For the whole of Section 8, we let x be a fixed point in $\Delta^4_4(a)$.

Recall from (2.15)(xii) that $\Delta_1(x) \cap \Delta^3_3(a) \neq \emptyset$. We show that, in fact, $|\Delta_1(x) \cap \Delta^3_3(a)| = 11$. This is done in several stages. The set Y_d, which we now define, plays an important role in these arguments. For $d \in \Delta_1(x) \cap \Delta^3_3(a)$ with $d + x \in \alpha_{1,0}(d, d + c, X(d, a))$ (where $\{c\} = \Delta_1(d) \cap \Delta^2_2(a)$), we put

$$Y_d := \{y \in \Delta^3_3(a) \cap \Delta_1(x) \cap \Delta^1_2(d) |\ \text{there exists } z \in \{y, d\}^{\perp} \cap \Delta^1_3(a)\}.$$



($z_i \in \Delta^1_3(a)$, $Y_d = \{y_1, y_2, ..., y_r\}$)

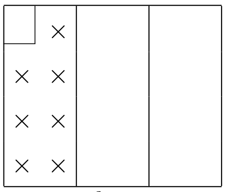
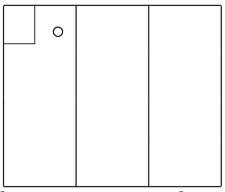
Lemma 8.1. *Suppose $d \in \Delta_1(x) \cap \Delta^3_3(a)$ with $d + x \in \alpha_{1,0}(d, d + c, X(d, a))$ (where $\{c\} = \Delta_1(d) \cap \Delta^2_2(a)$). Then*

(i) Y_d is a G_{adx}-orbit and $|Y_d| = 10$;

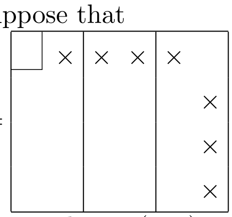
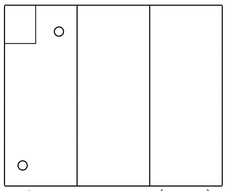
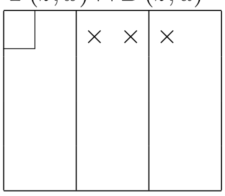
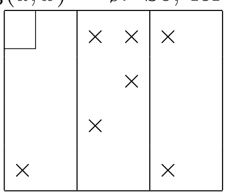
(ii) $|\Delta_1(x) \cap \Delta^3_3(a)| \geq 11$; and

(iii) there exists a G_{ax}-orbit in $\Delta_1(x) \cap \Delta^3_3(a)$ of size at least 10.

Proof. Parts (ii) and (iii) follow from part (i). For part (i), by (2.3), without loss of generality we may assume $d + x = \begin{array}{|cc|cc|cc|} \hline \square & & \times & & \times & \times \\ \times & & \times & & & \\ & & & & & \times \\ & & & & \times & \\ \hline \end{array}$, where $d + c =$

 and $X(d,a) =$  in Ω_d. Using Appendix A we see there are exactly 10 heptads intersecting $d+c$ in just $X(d,a)$ and $d+x$ in three elements of Ω_d (namely $h_1, h_2, h_3, h_4, h_5, h_6, h_9, h_{11}, h_{14}, h_{15}$). Therefore there are 10 lines $l \in \Gamma_1(d)$ with $l \in \alpha_{1,1}(d, d+c, X(d,a)) \cap \alpha_3(d, d+x)$. Furthermore $|\Gamma_0(l) \cap \Delta_3^1(a)| = 1$ for each such l by Lemma 4.15 and so the set $S := \Delta_1(d) \cap \Delta_3^1(a) \cap \Delta_2^1(x)$ contains exactly 10 points. By Lemma 4.8(iii), $G_{ad} \sim 2^5 A_6$ and G_{ad} is transitive on $\{y \in \Delta_1(d) | d+y \in \alpha_{1,0}(d, d+c, X(d,a))\}$ by (2.3) and Lemma 3.2. Hence $G_{adx} \cong A_5$ and S is a G_{adx}-orbit. Moreover $S \subseteq \Gamma_0(X(d,a))$.

Fix $z \in S$. Since $\Gamma_3(a,x) = \emptyset$ by Lemma 5.1(ii) we must have $\Gamma_3(a,y) = \emptyset$ for at least three points $y \in \{z,x\}^\perp$. By considering the possible heptads in Γ_z together with (2.15) we conclude that $\{z,x\}^\perp \backslash \{d\}$ consists of one point $e \in \Delta_3^3(a)$ and three points in $\Delta_4^1(a)$. For example, without loss of generality we may suppose that

$z+d =$  where $D(z,a) =$  by (2.4) and Lemma 4.15. Then $T(z,x) \cap D(z,a) = \emptyset$ in Ω_z because $\Gamma_3(a,x) = \emptyset$. So, for example, $T(z,x) =$  and we get $z+e =$  in this case.

To prove $|Y_d| = 10$ it is enough to show that e is not collinear with any point in $S \backslash \{z\}$. Assume $z' \in S \cap \Delta_1(e)$ with $z' \neq z$, and argue for a contradiction. Then $z, z' \in \Gamma_0(X(d,a))$, whence Lemma 3.6 implies that $e \in \Gamma_0(X(d,a))$. However this gives $d, e \in \Gamma_0(X(d,a)) \cap \Delta_1(x)$ with $d \neq e$, contrary to Lemma 5.3(i). So $|Y_d| = 10$. The fact that Y_d is a G_{adx}-orbit follows because S is a G_{adx}-orbit and the proof of part (i) is complete. $\square$

Lemma 8.2. *Let $\mathcal{B}_x$ be a G_{ax}-orbit on $\Gamma_1(x)$ and suppose there exists $l \in \mathcal{B}_x$ and $d \in \Gamma_0(l) \cap \Delta_3^3(a)$ with $l \in \alpha_{1,0}(d, d+c, X(d,a))$ (where $\{c\} = \Delta_1(d) \cap \Delta_2^2(a)$). Let $\mathcal{D}_x = \{y \in \Delta_1(x) \cap \Delta_3^3(a) | x+y \in \mathcal{B}_x\}$.*

(i) If $y \in \Delta_1(x) \cap \Delta_3^3(a)$ with $y+x \in \alpha_{1,0}(y, y+c', X(y,a))$ (for $\{c'\} = \Delta_1(y) \cap \Delta_2^2(a)$), then $y \in \mathcal{D}_x$.

(ii) $Y_d \subseteq \mathcal{D}_x$.

(iii) For each $y \in \mathcal{D}_x$, $G_{ayx} \cong A_5$.

Proof. We first show part (i) holds. Let y be as given in part (i), and set $L = \{(\alpha_{1,0}(d, d+c, X(d,a)))^g | \, g \in G_a\}$. By (2.3) $\alpha_{1,0}(d, d+c, X(d,a))$ is a G_{ad}-orbit which contains $l = d+x$. Also, by Lemma 3.2, $\tau(X(d,a))$ interchanges the two points in $\Gamma_0(d+x) \backslash \{d\}$. Employing Lemma 5.6 with $\Delta_3^3(a)$, $\Delta_4^4(a)$, d, x and $\mathcal{B}_x$ playing, respectively, the roles of Λ , Λ', x, x' and $\vartheta'_{x'}$ yields $\Gamma_1(x) \cap L = \mathcal{B}_x$. Now $\Delta_3^3(a)$ being a G_a-orbit and $\alpha_{1,0}(y, y+c', X(y,a))$ being a G_{ay}-orbit implies that $y+x \in L$. Therefore $x+y \in \Gamma_1(x) \cap L = \mathcal{B}_x$, whence $y \in \mathcal{D}_x$.

Turning to (ii), since Y_d is a G_{adx}-orbit, either $Y_d \subseteq \mathcal{D}_x$ or $Y_d \cap \mathcal{D}_x = \emptyset$. We assume the latter and argue for a contradiction. Then for all $d_1 \in Y_d$, $d_1 + x \in \alpha_{3,0}(d_1, d_1+c_1, X(d_1,a))$ (where $\{c_1\} = \Delta_1(d_1) \cap \Delta_2^2(a)$) by part(i) and Lemma 5.1(ii). Fix $d_1 \in Y_d$ and let $e \in \Delta_3^1(a) \cap \{d, d_1\}^\perp$. Since $\Gamma_3(a,x) = \emptyset$, $d+e \in \alpha_{1,1}(d, d+c, X(d,a))$ and $d_1 + e \in \alpha_{1,1}(d_1, d_1+c_1, X(d_1,a))$, we have $T(d,d_1) \cap (d+c) = \emptyset$ in Ω_d and $T(d_1,d) \cap (d_1+c_1) = \emptyset$ in Ω_{d_1}. Therefore Lemma 3.11(ii) implies that $\{d+x' \mid x' \in \{d,d_1\}^\perp \backslash \{e\}\}$ consists of three lines in $\alpha_{1,0}(d, d+c, X(d,a))$ and one line in $\alpha_{3,0}(d, d+c, X(d,a))$. Similarly $\{d_1 + x' \mid x' \in \{d, d_1\}^\perp \backslash \{e\}\}$ consists of three lines in $\alpha_{1,0}(d_1, d_1+c_1, X(d_1,a))$ and one line in $\alpha_{3,0}(d_1, d_1+c_1, X(d_1,a))$. Thus there exists $x_1 \in \{d,d_1\}^\perp \backslash \{e, x\}$ with $d + x_1 \in \alpha_{1,0}(d, d+c, X(d,a))$ and $d_1 + x_1 \in \alpha_{1,0}(d_1, d_1+c_1, X(d_1,a))$. Since G_{ad} is transitive on $\alpha_{1,0}(d, d+c, X(d,a))$ and $\tau(X(d,a)) \notin G_x$ by Lemma 3.2, there exists $h \in G_{ad}$ with $x_1^h = x$. Moreover $e^h \in \Delta_3^1(a) \cap \{d, d_1^h\}^\perp$ and thus $d_1^h \in Y_d$, whence $d_1^h + x \in \alpha_{3,0}(d_1^h, d_1^h + c_1^h, X(d_1^h, a))$. However $d_1^h + x = (d_1 + x_1)^h \in \alpha_{1,0}(d_1^h, d_1^h + c_1^h, X(d_1^h, a))$ by the flag-transitivity of G. From this contradiction we conclude that (ii) holds.

From Lemma 4.8(iii) $G_{ad} \sim 2^5 A_6$ with $G_{ad} \cap Q(d) = \langle \tau(X(d,a)) \rangle$. Combining (2.3) and Lemma 3.2 yields $H \cong A_5$. Since $\mathcal{D}_x$ is a G_{ax}-orbit and, by part (i), $d \in \mathcal{D}_x$, we obtain (iii).

$\square$

Lemma 8.3. *Let $\mathcal{D}_x$ be the G_{ax}-orbit described in Lemma 8.2. Then $|\mathcal{D}_x| \neq 12$.*

Proof. Let $d \in \Delta_1(x) \cap \Delta_3^3(a)$ with $d+x \in \alpha_{1,0}(d, d+c, X(d,a))$ ($\{c\} = \Delta_1(d) \cap \Delta_2^2(a)$). Supposing $|\mathcal{D}_x| = 12$, we seek a contradiction. Then, by Lemmas 8.1(i) and 8.2, $\mathcal{D}_x = \{d\} \cup Y_d \cup \{e\}$. Put $H = G_{adx}$. By Lemma 8.2(iii) $H \cong A_5$.Again by Lemma 8.1(i), Y_d is H-invariant and therefore $H \leq G_{aex}$. Now $\mathcal{D}_x$ is a G_{ax}-orbit and so $d^g = e$ for some $g \in G_{ax}$. Hence $G_{aex} = H = H^g$ for some $g \in G_{ax}$. Suppose $e^g \neq d$. Then $e^g \in Y_d$ and H fixes e^g. This contradicts Lemma 8.1(i). Therefore we must have $e^g = d$ and hence $g^2 \in H$. Thus $H \trianglelefteq \langle H, g\rangle$ with $|\langle H, g\rangle| = 2^3.3.5$. Since $|C_{M_{23}}(\vartheta)| = 15$ for any 5-element ϑ of M_{23} (see [1]) we conclude that $\langle H, g\rangle \cong S_5$. Now $H \trianglelefteq G_{ax}$ yields the untenable $H \leq G_{ad'x}$ for all $d' \in \mathcal{D}_x$ and so as $[G_{ax} : \langle H, g\rangle] = 6$ we deduce that $G_{ax} \cong S_6$.

Consulting [1] we see that $G_{ax} \cong G_x^{*a}$ is contained in a subgroup of G_a^{*a} isomorphic to either a group of shape $L_3(4)2$ or A_8. These subgroups of G_a^{*a} are, respectively, the stabilizer of a duad and an octad of Ω_a. Consequently $G'_{ax} \cong A_6$ fixes at least two hyperplanes in $\Gamma_3(a)$. Now $A_5 \cong H = G_{adx}$ ($\leq G'_{ax}$) fixes at most three hyperplanes, two of which are $X(a,d)$ and $X(a,e)$. So without loss of generality we have that G'_{ax} fixes $X(a,d)$. Since $X(a,d') \neq X(a,d)$ for any $d' \in (\Delta_1(x) \cap \Delta_3^3(a))\backslash\{d\}$ by Lemma 3.6, we conclude that $G'_{ax} \leq G_{adx}$. This contradicts the fact that $[G_{ax} : G_{adx}] = |\mathcal{D}_x| = 12$ and so the proof of the lemma is complete. $\square$

Lemma 8.4. *Let $d \in \Delta_3^3(a)$ with $\{c\} = \Delta_1(d) \cap \Delta_2^2(a)$.*

(i) If $d \in \Delta_1(x)$, then $d + x \in \alpha_{1,0}(d, d+c, X(d,a))$.

(ii) If $l \in \alpha_{1,0}(d, d+c, X(d,a))$, then $|\Gamma_0(l) \cap \Delta_4^4(a)| = 2$.

(iii) If $l \in \alpha_{3,0}(d, d+c, X(d,a))$, then $|\Gamma_0(l) \cap \Delta_4^6(a)| = 2$.

(iv) $\Delta_4^4(a) \neq \Delta_4^6(a)$.

(v) $\Delta_1(d) \cap \Delta_4^4(a)$ and $\Delta_1(d) \cap \Delta_4^6(a)$ are G_{ad}-orbits of length 192 and 160 respectively.

Proof. Let $d \in \Delta_3^3(a)$. For any $l \in \alpha_{i,0}(d, d+c, X(d,a))$ $(i = 1,3)$, $l \notin \Gamma_1(X(d,a))$. Therefore $\tau(X(d,a))$ interchanges the two points in $\Gamma_0(l)\backslash\{d\}$ by Lemma 3.2. Hence the two points in $\Gamma_0(l)\backslash\{d\}$ lie in the same G_a-orbit. By (2.5) and (2.15), to prove the lemma it is enough to show that for all $l \in \alpha_{3,0}(d, d+c, X(d,a))$, $\Gamma_0(l) \cap \Delta_4^4(a) = \emptyset$.

Suppose, for a contradiction, that there exists $l \in \alpha_{3,0}(d, d+c, X(d,a))$ and $y \in \Gamma_0(l) \cap \Delta_4^4(a)$. Since $\tau(X(d,a))$ interchanges y and the point in $\Gamma_0(l)\backslash\{d, y\}$ we may appeal to Lemma 5.6 to show that d lies in a G_{ay}-orbit $\mathcal{D}_1$ on $\Delta_1(y)$ of size

$$n_1 = \frac{2.|\alpha_{3,0}(d, d+c, X(d+a))|.|\Delta_3^3(a)|}{|\Delta_4^4(a)|}.$$

Using (2.3) and Lemma 4.8(iii) we get

$$n_1 = \frac{2.80.2^{10}.7.11.23}{|\Delta_4^4(a)|} = \frac{2^{15}.5.7.11.23}{|\Delta_4^4(a)|}.$$

However, by definition there exists $e \in \Delta_1(y) \cap \Delta_3^3(a)$ with $e + y \in \alpha_{1,0}(e, e+c', X(e,a))$ $(\{c'\} = \Delta_1(e) \cap \Delta_2^2(a))$. By a similar argument to the above we can show that e lies in a G_{ay}-orbit $\mathcal{D}_2$ on $\Delta_1(y)$ of size

$$\begin{aligned} n_2 &= \frac{2.|\alpha_{1,0}(d, d+c, X(d,a))|.|\Delta_3^3(a)|}{\Delta_4^4(a)} \\ &= \frac{2^{16}.3.7.11.23}{|\Delta_4^4(a)|}. \end{aligned}$$

Hence $\mathcal{D}_1 \neq \mathcal{D}_2$ and $n_1/n_2 = 5/6$. Since, for all $z \in \Delta_1(d) \cap \Delta_4^4(a)$, $d + z \in \alpha_{i,0}(d, d+c, X(d,a))$ for $i = 1,3$, we must have $|\Delta_1(y) \cap \Delta_3^3(a)| = n_1 + n_2$. Then Lemma 5.3(ii) implies that $n_1 + n_2 = 11$ or 22. However $n_1 + n_2 \neq 11$ otherwise $n_1 = 5$ and $n_2 = 6$, contrary to Lemma 8.1(iii). Therefore $n_1 = 10$ and $n_2 = 12$. However we now have a G_{ay}-orbit $\mathcal{D}_2$ on $\Delta_1(y)$ of size 12 with $e \in \mathcal{D}_2$ where $e \in \Delta_3^3(a)$ and $e + y \in \alpha_{1,0}(e, e+c', X(e,a))$. This contradicts Lemma 8.3 and so the proof is complete. □

Set n:= $|\Delta_1(x) \cap \Delta_3^3(a)|$.

Lemma 8.5. *Let $d \in \Delta_1(x) \cap \Delta_3^3(a)$. Then,*

(i) $[G_{ax} : G_{adx}] = n$;

(ii) G_{ax} acts transitively on $\Delta_1(x) \cap \Delta_3^3(a)$; and

(iii) $n = 11, 14, 16, 21, 22$ *or* 23.

Proof. By Lemma 5.2 $\Delta_4^4(a)$ is a G_a-orbit, and so

$$|\Delta_4^4(a)| = \frac{2^{18}3^2.5.7.11.23}{|G_{ax}|}.$$

Counting edges, and using Lemma 4.8(iii), also gives

$$|\Delta_4^4(a)| = \frac{|\Delta_3^3(a)|.96.2}{n} = \frac{2^{16}.3.7.11.23}{n}.$$

Therefore $|G_{ax}| = 2^2.3.5.n$ and hence, by Lemma 8.2(iii), $[G_{ax} : G_{adx}] = n$, which proves (i).

Part (ii) follows from part (i).

Since $\Delta_4^4(a)$ is an integer, n must divide $2^{16}.3.7.11.23$. By (ii) and Lemma 8.3, $n \neq 12$. Furthermore $11 \leq n \leq 23$ by Lemmas 5.1(ii) and 8.1(ii). This yields the list of possible values for n in part (iii). $\square$

Lemma 8.6. $n \neq 16$

Proof. We show that $n = 16$ leads to a contradiction. Let $d \in \Delta_1(x) \cap \Delta_3^3(a)$. Appealing to Lemmas 8.2(iii) and 8.5 we have $[G_{ax} : G_{adx}] = 16$, $G_x^{*x} \cong G_{ax}$ and $G_{adx} \cong A_5$. By examining possible minimal normal subgroups of G_{ax} and using the fact that $|C_{M_{23}}(\vartheta)| = 15$ for any 5-element ϑ in M_{23}, we deduce that $G_{ax} \sim 2^4 A_5$ with $N = O_2(G_{ax})$ an elementary abelian subgroup of order 16. Since M_{23} has only two conjugacy classes of elementary abelian subgroups of order 16, we infer that G_x^{*x} is a subgroup of either $2^4 A_7$ (the stabilizer of a heptad) or $2^4(3 \times A_5) : 2$ (the stabilizer of a triad).

We consider the former possibility first. Then $G_{ax}^{*x} \leq G_{xl}^{*x}$ for some $l \in \Gamma_1(x)$. Since N acts transitively upon $\{x + d' \mid d' \in \Delta_1(x) \cap \Delta_3^3(a)\}$, all the heptads $x+d'$ in this set must intersect in a common element of the heptad l in Ω_x. That is all the lines $x + d'$ ($d' \in \Delta_1(x) \cap \Delta_3^3(a)$) are in $\Gamma_1(X)$ for some hyperplane X. Since $|Y_d| = 10$ by Lemma 8.1(i), there are 10 points in the set

$$S := \{z \in \Delta_3^1(a) \cap \Delta_2^1(x) \cap \Delta_1(d) \mid \text{ there exists } d' \in (\{z, x\}^\perp \backslash \{d\}) \cap \Delta_3^3(a)\}.$$

Therefore Lemma 3.6 implies that $z \in \Gamma_0(X)$ for each $z \in S$. However, using the Appendix A, we see that

$$(d + x) \cap \bigcap_{z \in S} (d + z) = \emptyset.$$

This contradiction rules out the case when G_{ax}^{*x} is a subgroup of 2^4A_7.

In the other case we have that G_{ax}^{*x} ($\sim 2^4A_5$) is contained in the pointwise stabilizer of two elements of Ω_x which is isomorphic to $L_3(4)$. So G_{ax}^{*x} is isomorphic to a parabolic subgroup of $L_3(4)$, whence G_{ax}^{*x} acts transitively (by conjugation) upon $(N^{*x})^{\#}$. Since N acts regularly upon $\{x + d' \mid d' \in \Delta_1(x) \cap \Delta_3^3(a)\}$, the conjugation action of G_{adx}^{*x} ($\cong A_5$) on $(N^{*x})^{\#}$ is permutation isomorphic to G_{adx} acting on $(\Delta_1(x) \cap \Delta_3^3(a)) \backslash \{d\}$. However Y_d is a G_{adx}-orbit and $|Y_d| = 10$ by Lemma 8.1(i). With this contradiction we have established Lemma 8.6. $\square$

We are now in a position to show that n = 11.

Lemma 8.7. *(i)* $|\Delta_1(x) \cap \Delta_3^3(a)| = 11$.

(ii) $|\Delta_4^4(a)| = 2^{16}.3.7.23$.

(iii) $G_{ax} \cong L_2(11)$ *and* $G_{ax} \cap Q(x) = 1$.

Proof. (i) Suppose $n \neq 11$. Then from Lemmas 8.5(iii) and 8.6, $n \in \{14, 21, 22, 23\}$. Considering possible minimal normal subgroups of G_{ax} yields the existence of elements of orders 5×7, 5×11 or 5×23 in M_{23}, which is impossible. Thus $n = 11$.

(ii)Combining part (i) with (2.3), (2.15)(xii) and Lemma 4.8(iii),

$$\begin{aligned} |\Delta_4^4(a)| &= \frac{|\Delta_3^3(a)|.192}{n} \\ &= 2^{16}.3.7.23 \end{aligned}$$

(iii) From part (i) and Lemmas 8.2(iii) and 8.5(ii), $G_{ax} \cap Q(x) = 1$ with G_{ax}^{*x} containing a subgroup of index 11 isomorphic to A_5. A perusal of [1] now yields (iii).

$\square$

Using [1] we see that G_{ax}^{*x} is the stabilizer of an 11-element set called an endecad (the symmetric difference of a heptad and an octad, intersecting in two elements of Ω_x) and an element X of Ω_x disjoint from the endecad. The orbits of G_{ax}^{*x} ($\cong L_2(11)$) are described in (2.12). The next few results give the point distribution for lines in some of these G_{ax}^{*x}-orbits.

Lemma 8.8. *Let $d \in \Delta_1(x) \cap \Delta_3^3(a)$. Then $x+d \in \alpha_1(x, \mathrm{END}, -)$ with $|\Gamma_0(x+d) \cap \Delta_3^3(a)| = 1$ and $|\Gamma_0(x+d) \cap \Delta_4^4(a)| = 2$.*

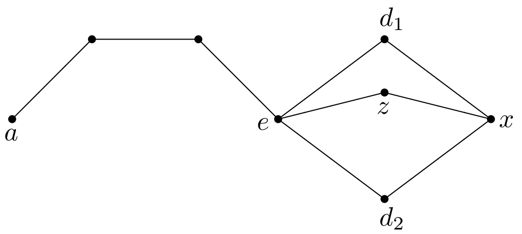
Proof. By Lemma 8.7(i) $|\Delta_1(x) \cap \Delta_3^3(a)| = 11$. Therefore (2.12) implies that $\{x+y \mid y \in \Delta_1(x) \cap \Delta_3^3(a)\}$ is equal to one of $\alpha_1(x, \mathrm{END}, -)$, $\alpha_1(x, \mathrm{END}, +)$ or $\alpha_5(x, \mathrm{END}, +)$. Suppose $\{x+y \mid y \in \Delta_1(x) \cap \Delta_3^3(a)\}$ equals one of $\alpha_1(x, \mathrm{END}, +)$ and $\alpha_5(x, \mathrm{END}, +)$. Then the 11 lines in $\{x+y \mid y \in \Delta_1(x) \cap \Delta_3^3(a)\}$ are incident with the same hyperplane $X \in \Gamma_3(x)$. By Lemma 8.1(i) for each of the 10 points $y \in (\Delta_1(x) \cap \Delta_3^3(a)) \backslash \{d\}$ there exists $z \in \Delta_3^1(a) \cap \{d, y\}^\perp$, whence, by Lemma 3.6, $d+y \in \Gamma_1(X)$ for each $y \in Y_d$. However, consulting Appendix A reveals that the 10 heptads $h_1, h_2, h_3, h_4, h_5, h_6, h_9, h_{11}, h_{14}, h_{15}$ (see Lemma 8.1) have empty intersection. Thus we deduce that $\{x+y \mid y \in \Delta_1(x) \cap \Delta_3^3(a)\} = \alpha_1(x, \mathrm{END}, -)$. By (2.15) we have $\Gamma_0(d+x) \backslash \{d\} \subseteq \Delta_4^4(a)$, and we have the lemma. □

Lemma 8.9. *(i) Let $l \in \alpha_3(x, \mathrm{END}, -)$. Then $|\Gamma_0(l) \cap \Delta_4^1(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_4^4(a)| = 2$.*

(ii) Let $l \in \alpha_3(x, \mathrm{END}, +)$. Then $|\Gamma_0(l) \cap \Delta_4^1(a)| = |\Gamma_0(l) \cap \Delta_4^4(a)| = |\Gamma_0(l) \cap \Delta_4^6(a)| = 1$.

(iii) If $l \in \alpha_3(x, \mathrm{END}, -) \cup \alpha_3(x, \mathrm{END}, +)$ with $\{y\} = \Gamma_0(l) \cap \Delta_4^1(a)$ and $\{e\} = \Delta_1(y) \cap \Delta_3^1(a)$, then $e \in \Delta_2^1(x)$.

Proof. Let $d_1, d_2 \in \Delta_1(x) \cap \Delta_3^3(a)$ with $d_1 \neq d_2$. Then Lemmas 8.1(i), 8.4(i) and 8.7(i) imply that there exists $e \in \Delta_3^1(a) \cap \{d_1, d_2\}^\perp$. We have



where $z \in \{e, x\}^\perp \backslash \{d_1, d_2\}$. Then $z \notin \Gamma_0(X(d_i, a))$ $(i = 1, 2)$ by Lemma 3.6 because $\Gamma_3(a, x) = \emptyset$. Therefore $\Gamma_3(a, z) = \emptyset$, otherwise for any $X \in \Gamma_3(a, z)$ we would have $X \in \Gamma_3(e)$ by Lemma 4.15(i), which is impossible because $\Gamma_3(a, e) = \{X(a, d_1), X(a, d_2)\}$ by Lemma 4.3(ii). Thus $z \in \Delta_4^1(a)$

by definition (see (2.15)(ix)). However $x+d_1, x+d_2 \in \alpha_1(x, \mathrm{END}, -)$ by Lemma 8.8 and, as heptads in Ω_x, $x+d_1$ and $x+d_2$ intersect END in different elements. Thus $T(x,e)$ and END have empty intersection in Ω_x. Hence $x+z \in \alpha_1(x, \mathrm{END}, +) \cup \alpha_3(x, \mathrm{END}, +) \cup \alpha_3(x, \mathrm{END}, -)$ by (2.12) because $x+z$ contains the triad $T(x,e)$ in Ω_x. However Lemma 3.11(ii) implies that $|(x+z) \cap \mathrm{END}| = 3$ for each $z \in \{e,x\}^{\perp} \backslash \{d_1, d_2\}$ and $x+z$ contains the unique element of Ω_x fixed by G_{ax} for precisely one $z \in \{e,x\}^{\perp} \backslash \{d_1, d_2\}$. Therefore we may assume $\{e,x\}^{\perp} \backslash \{d_1, d_2\} = \{z_1, z_2, z_3\}$ where $x+z_1 \in \alpha_3(x, \mathrm{END}, +)$ and $x+z_2, x+z_3 \in \alpha_3(x, \mathrm{END}, -)$.

Let $\Gamma_0(e+d_1) = \{e, d, d_1\}$. Then $d \in \Delta_3^3(a)$ by Lemma 4.15(i) and $d \in \Delta_1(f)$ for each $f \in \bigcup\{\Gamma_0(x+y) \backslash \{x,y\} \mid y \in \{e,x\}^{\perp} \backslash \{d_1\}\}$ using Lemma 3.10. Since $\Gamma_3(a,x) = \emptyset$, $X(d,a) \notin \Gamma_3(T(d,x))$. Considering the 5 heptads in Γ_d incident with $T(d,x)$ together with (2.15)(xii) and (xiv) yields that $\Delta_3^3(a) \cap \{d,x\}^{\perp} = \{d_1\}$, $|\Delta_4^4(a) \cap \{d,x\}^{\perp}| = 3$ and $|\Delta_4^6(a) \cap \{d,x\}^{\perp}| = 1$. We know that $\Gamma_0(x+d_2) \backslash \{x, d_2\} \subseteq \Delta_4^4(a)$ by Lemma 8.8. This, together with the fact that $\alpha_3(x, \mathrm{END}, -)$ and $\alpha_3(x, \mathrm{END}, +)$ are G_{ax}-orbits forces $\Gamma_0(x+z_1) \backslash \{x, z_1\} \subseteq \Delta_4^6(a)$ and $\Gamma_0(x+z_i) \backslash \{x, z_i\} \subseteq \Delta_4^4(a)$ for $i = 2, 3$. This proves parts (i) and (ii).

Part (iii) follows because $\{e\} = \Delta_2^1(x) \cap \Delta_3^1(a) \cap \Delta_1(z)$ for each $z \in \Delta_4^1(a) \cap \{e,x\}^{\perp}$ by Lemma 6.1, together with the fact that $\alpha_3(x, \mathrm{END}, -)$ and $\alpha_3(x, \mathrm{END}, +)$ are G_{ax}-orbits on $\Gamma_1(x)$.

□

In the next lemma we reconsider a point in $\Delta_4^1(a)$.

Lemma 8.10. *Let $y \in \Delta_4^1(a)$, $l \in \Gamma_1(y)$ and $\{d\} = \Delta_1(y) \cap \Delta_3^1(a)$. Suppose there exists $z \in \Gamma_0(l) \cap \Delta_4^4(a)$.*

(i) If $l \in \alpha_3(z, \mathrm{END}, +)$, then $l \in \alpha_{3,0}^{\mathcal{L}^c}(y, y+d, \mathrm{DUAD})$ and $|\Gamma_0(l) \cap \Delta_4^1(a)| = |\Gamma_0(l) \cap \Delta_4^4(a)| = |\Gamma_0(l) \cap \Delta_4^6(a)| = 1$.

(ii) If $l \in \alpha_3(z, \mathrm{END}, -)$, then $l \in \alpha_{3,1}(y, y+d, \mathrm{DUAD})$ with $|\Gamma_0(l) \cap \Delta_4^1(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_4^4(a)| = 2$.

Proof. Since $\Gamma_0(l) \cap \Delta_4^4(a) \neq \emptyset$, $l \in \alpha_{1,0}(y, y+d, \mathrm{DUAD}) \cup \alpha_{3,0}^{\mathcal{L}^c}(y, y+d, \mathrm{DUAD}) \cup \alpha_{3,1}(y, y+d, \mathrm{DUAD})$ by (2.9) and Lemmas 6.6 and 6.8. Assume $l \in \alpha_3(z, \mathrm{END}, +) \cup \alpha_3(z, \mathrm{END}, -)$. Then by Lemma 8.9(iii), $d \in \Delta_2^1(z)$. Hence, since $y \in \{d,z\}^{\perp}$, $y+$

$z = l \in \alpha_3(y, y+d)$. Therefore we must have $l \in \alpha^{\mathcal{L}^c}_{3,0}(y, y+d, \mathrm{DUAD}) \cup \alpha_{3,1}(y, y+d, \mathrm{DUAD})$.

Let $\mathrm{DUAD} = \{X_1, X_2\}$ where $X_1, X_2 \in \Gamma_3(y)$. Then $X_1, X_2 \in \Gamma_3(b)$ where $\{b\} = \Delta_1(a) \cap \Delta^1_3(y)$ by Lemma 6.5. Since $X_1, X_2 \notin \Gamma_3(a)$, Lemma 3.2 yields that $\tau := \tau(X_1)\tau(X_2) \in G_a \cap Q(y)$. If $l \in \alpha_{3,1}(y, y+d, \mathrm{DUAD})$, then $z^\tau \neq z$ by Lemma 3.2 because exactly one of X_1 and X_2 lies in $\Gamma_3(z)$. Thus $\Gamma_0(l)\backslash\{y\} \subseteq \Delta^4_4(a)$. Therefore $l \in \alpha_3(z, \mathrm{END}, +)$ implies that $l \in \alpha^{\mathcal{L}^c}_{3,0}(y, y+d, \mathrm{DUAD})$ by Lemma 8.9(ii) and part (i) follows from Lemma 8.9(ii) again.

So we may assume that $l \in \alpha_3(z, \mathrm{END}, -)$. If $l \in \alpha^{\mathcal{L}^c}_{3,0}(y, y+d, \mathrm{DUAD})$, then appealing to part (i) and Lemma 5.5 we get that $l \in \alpha_3(z, \mathrm{END}, +)$. This contradicts (2.12). Therefore we must have $l \in \alpha_{3,1}(y, y+d, \mathrm{DUAD})$ and now Lemma 8.9(i) yields (ii). □

In our next two lemmas we consider $y \in \Delta^2_4(a)$ and $\alpha_2(y, O(y,a))$, the line orbit of G_{ay} that has yet to receive our attention.

Lemma 8.11. *If $l \in \Gamma_1(y)$ and $l \in \alpha_2(y, O(y,a))$, then $\Gamma_0(l) \cap \Delta^6_4(a) = \emptyset$.*

Proof. We assume the result is false and seek a contradiction. Thus we have $f \in \Delta^6_4(a)$ such that $y + f \in \alpha_2(y, O(y,a))$. Further, from (2.10), we have that

(8.11.1) G_{ayf} contains a subgroup A isomorphic to A_5.

Let $d \in \Delta_1(f) \cap \Delta^3_3(a)$; by definition of $\Delta^6_4(a)$ such a d exists.

(8.11.2) $|G_{af}| = 2^3.3^2.5, 2^4.3^2.5$ or $2^5.3^2.5$.

Set $n = |\Delta_1(f) \cap \Delta^3_3(a)|$. Combining Lemmas 5.4 and 8.4(v) gives $n = [G_{af} : G_{adf}]$. From Lemmas 4.8(iii) and 8.4(v) we have that $|G_{adf}| = 2^3.3^2$. Hence, using (8.11.1) and the order of M_{23}, we deduce that $5|n$ and $3 \nmid n$. Consequently $n = 5, 10$ or 20 by Lemma 5.3(ii). Since $|G_{af}| = |G_{adf}|.n = 2^3.3^2.n$, this yields the possibilities listed in (8.11.2).

(8.11.3) $|Q(f) \cap G_{fa}| \leq 2$.

Clearly $Q(f) \cap G_{af}$ fixes the line $f + y$ and $[Q(f) \cap G_{af} : Q(f) \cap G_{afy}] \leq 2$. Now $Q(f) \cap G_{afy}$ is a subgroup of $G_{ay} \cong A_8$ (by Lemma 7.3(i)) which, by

(8.11.1), is normalized by $A \cong A_5$. Therefore $Q(f) \cap G_{afy} = 1$ by the structure of A_8, whence we have (8.11.3).

Put $\overline{G_{af}} = G^{*f}_{af}$, and use the usual bar notation for the subgroups of G_{af}. Let $\overline{N}$ be a minimal normal subgroup of $\overline{G_{af}}$. Then we must have either $\overline{A} \leq \overline{N}$ or $\overline{A} \cap \overline{N} = 1$. Suppose the former holds. Then $\overline{N}$ must be a direct product of isomorphic non-abelian simple groups. The possible orders for G_{af} in (8.11.2) then force $\overline{N}$ to be isomorphic to either A_5 or A_6. If $\overline{N} \cong A_6$, then $\overline{N}$ contains all the elements of $\overline{G_{af}}$ of order 3. So, together (8.11.3) and (2.3) imply that $\overline{N}$ ($\cong A_6$) contains a subgroup isomorphic to $3 \times A_4$, which is impossible. Thus $\overline{A} = \overline{N} \cong A_5$ and, since $3^2 || \overline{G_{af}}|$, we conclude that an element of $\overline{G_{af}}$ of order 3 centralizes $\overline{A}$.

Now we consider the possibility $\overline{N} \cap \overline{A} = 1$. By (8.11.2) $|\overline{N}| = 3$ or $\overline{N}$ is a 2-group of order at most 2^3. Hence $[\overline{A}, \overline{N}] = 1$. Since $C_{M_{23}}(\vartheta) = 15$ for ϑ an element of M_{23} of order 5, we deduce that $\overline{N} = 3$. Therefore we conclude that, in either case, $\overline{A} \leq N_{\overline{G_{af}}}(\langle\overline{\zeta}\rangle)$ where $\overline{\zeta} \in \overline{G_{af}}$ has order 3. Consulting (2.14) we observe that $\overline{A}$ does not fix any line in $\Gamma_1(f)$, yet we have that $\overline{A}$ fixes $f + y$. With this contradiction, the proof of Lemma 8.11 is complete.

$\square$

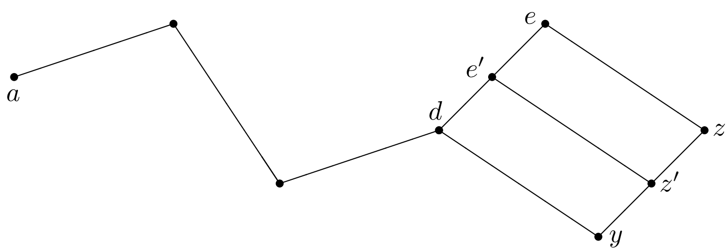
Lemma 8.12. *Let $l \in \alpha_2(y, O(y, a))$. Then $|\Gamma_0(l) \cap \Delta^4_4(a)| = 2$.*

Proof. Let $d \in \Delta_1(y) \cap \Delta^2_3(a)$ and choose $z \in \Delta_1(y)$ such that $y+z \in \alpha_2(y, O(y, a)) \cap \alpha_3(y, y+d)$ (by examining the MOG in [2] we see that such a heptad $y+z$ exists). Let $\Gamma_0(y + z) = \{y, z', z\}$. Since $d + y \in \alpha_{0,0}(d, O(d, a), X(d, a))$ by Corollary 7.4(ii), we have $T(d, z) \cap O(d, a) = \emptyset$ in Ω_d. Thus either,

(1) there exists $c \in \Delta_1(d) \cap \Delta^1_2(a)$ with $d + c \in \Gamma_1(T(d, z))$; or

(2) $\{d, z\}^\perp \backslash \{y\}$ consists of one point in $\Delta^3_3(a)$ and three points in $\Delta^1_4(a)$.

If case (1) holds, then one of z or z' lies in $\Delta^2_3(a) \cup \Delta^6_3(a)$. Now $|\alpha_2(y, O(y, a))| = 168$ and Lemma 7.3(iii) rules out either of z and $z' \in \Delta^2_3(a)$. So we have $z \in \Delta^6_3(a)$. Hence, using Lemma 7.5, $|\Delta_1(y) \cap \Delta^6_3(a)| = 70 + 168i$ ($i = 1$ or 2). From Lemmas 4.11(ii) and 7.3(ii) $5 \mid |\Delta^6_3(a)|$ and $5 \nmid |\Delta^2_4(a)|$ and therefore, by counting edges between $\Delta^2_4(a)$ and $\Delta^6_3(a)$, $5 \mid |\Delta_1(y) \cap \Delta^6_3(a)|$. This is clearly impossible, so case (2) holds and we have



where $e \in \Delta_3^3(a)$ and $\Gamma_0(d+e) = \{d, e', e\}$. Applying Lemma 4.15 to $d+e$ we get that $e' \in \Delta_3^3(a)$ and $X(e,a) = X(d,a)$. Since $z \in \Delta_1(e)$ and $z' \in \Delta_1(e')$ by Lemma 3.10 we must have $z, z' \in \Delta_3^i(a) \cup \Delta_4^j(a)$ for $i \in \{1,2,3,4\}$ and $j \in \{4,6\}$. If $z \in \Delta_3^i(a)$ ($i \in \{1,2,3,4\}$), then $\Gamma_3(a,z,d) \neq \emptyset$ by Lemma 4.15. In this case $\Gamma_3(a,y) \neq \emptyset$ contrary to Lemma 5.1. Similarly we get a contradiction if $z' \in \Delta_3^i(a)$ ($i \in \{1,2,3,4\}$). Therefore $z, z' \in \Delta_4^4(a) \cup \Delta_4^6(a)$ and appealing to Lemma 8.11 we get $z, z' \in \Delta_4^4(a)$, as required.

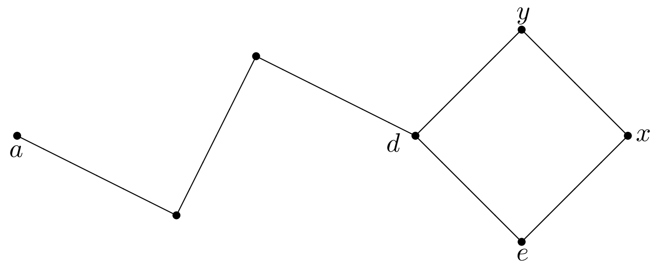
□

At this stage we remark that we have proved Theorem 12.

Recall that x is a fixed point in $\Delta_4^4(a)$.

Lemma 8.13. *Let $l \in \alpha_1(x, \mathrm{END}, +)$. Then $|\Gamma_0(l) \cap \Delta_4^2(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_4^4(a)| = 2$.*

Proof. Since $\Delta_4^4(a)$ is a G_a-orbit, there exists $y \in \Delta_1(x) \cap \Delta_4^2(a)$ with $y + x \in \alpha_2(y, O(Y,a))$ by Lemma 8.12. Appealing to Corollary 7.4(i) and the MOG in [2] we can choose $d \in \Delta_1(y) \cap \Delta_3^2(a)$ with $y + d \in \alpha_3(y, y+x)$. Thus $d \in \Delta_2^1(x)$. In Ω_d, $T(d,x) \cap O(d,a) = \emptyset$ because $d+y \in \alpha_{0,0}(d, O(d,a), X(d,a))$ by Corollary 7.4(ii). Since $x \in \Delta_4(a)$, $\{d,x\}^\perp \cap \Delta_2^1(a) = \emptyset$, whence $\{d,x\}^\perp \backslash \{y\}$ must consist of three points in $\Delta_4^1(a)$ and one point in $\Delta_3^3(a)$. Letting $\{\mathrm{e}\} = \{d,x\}^\perp \cap \Delta_3^3(a)$, we have



Lemma 8.8 implies that $x+e \in \alpha_1(x, \mathrm{END}, -)$. Also $|(x+y)\cap \mathrm{END}| = 1$ or 5 in Ω_x by (2.12) and Lemma 8.9. However $|(x+y)\cap(x+e)| = 3$ in Ω_x and so we must have $|(x+y)\cap \mathrm{END}| = 1$. Hence $x+y \in \alpha_1(x, \mathrm{END}, +)$ by (2.12) and Lemma 8.8. Since $y+x \in \alpha_2(y, O(y,a))$, Lemma 8.12 yields $|\Gamma_0(y+x)\cap \Delta_4^2(a)| = 1$ and $|\Gamma_0(y+x)\cap \Delta_4^4(a)| = 2$. The result now follows because $\alpha_1(x, \mathrm{END}, +)$ is a G_{ax}-orbit.

□

Finally in this section we prove a symmetry result.

Lemma 8.14. $a \in \Delta_4^4(x)$.

Proof. Let $d \in \Delta_1(x)\cap\Delta_3^3(a)$ and $e \in \Delta_3^1(a)\cap\Delta_2^1(x)\cap\Delta_1(d)$ (d and e exist by Lemmas 8.1, 8.7(i) and 8.9(iii)). We have $e+d \in \alpha_1(e, D(e,a))$ by Theorem 5. Since $\Gamma_3(a,x) = \emptyset$, $D(e,a)\cap T(e,x) = \emptyset$ in Ω_e. Let D be a duad of Ω_e contained in the triad $T(e,x)$ and let l be the unique heptad in Ω_e containing $D \cup D(e,a)$. Since $T(e,x) \subseteq e+d$ and $|(e+d)\cap D(e,a)| = 1$, $T(e,x) \nsubseteq l$ in Ω_e. By Theorem 5 again, there exists $c \in \Gamma_0(l)$ with $c \in \Delta_2^1(a)$. Since $D \subseteq T(e,x)$ in Ω_e, $|\Gamma_3(c,x)| \geq 2$ by Lemma 4.1. Hence $c \in \Delta_2^1(x)\cup\Delta_3^1(x)$ by Theorem 4.13. If $c \in \Delta_2^1(x)$, then , since $e \in \Delta_2^1(x)$ we must have $T(e,x) \in \Gamma_2(e+c)$, contrary to $T(e,x) \nsubseteq l$. Therefore $c \in \Delta_3^1(x)$ and $D = D(c,x)$.

In Ω_c, $T(c,a)\cap D(c,x) = \emptyset$ because $\Gamma_3(a,x) = \emptyset$. Therefore Lemma 3.11(ii) implies there exists $b \in \{a,c\}^{\perp}$ with $(c+b) \cap D(c,x) = \emptyset$ in Ω_c. Then $b \in \Delta_4^1(x)$ by definition (see (2.15)(ix)). Hence Lemmas 6.6(i),(iii) and 8.10(i),(ii) imply $a \in \Delta_3^i(x) \cup \Delta_4^j(x)$ for $i = 2$ or 6 and $j = 1, 4$ or 6 because G is transitive on Γ_0 and $b + a \in \alpha_3(b, b+c)$. Since $x \in \Delta_4(a)$, $a \notin \Delta_3(x)$. If $a \in \Delta_4^1(x)$ holds, then $x \in \Delta_4^1(a)$ by Lemma 6.4, which is impossible by Lemmas 6.2(i) and 8.7(ii). So $a \notin \Delta_4^1(x)$. If $a \in \Delta_4^6(x)$, then $|\Delta_4^4(a)| = |\Delta_4^6(a)|$ and $G_{ay} \cong L_2(11)$ for all $y \in \Delta_4^6(a)$. Appealing to Lemmas 8.4(v) and 8.7(ii) then yields

$$|\Delta_1(y) \cap \Delta_3^3(a)| = \frac{160.2^{10}.7.11.23}{2^{16}.3.7.23} \notin \mathbb{Z}$$

which is untenable. Therefore $a \in \Delta_4^4(x)$ and the lemma is proved.

□

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