First/second order transformation system. Tentative proof

Chahlaoui, Younes and Van Dooren, Paul

2008

MIMS EPrint: 2008.77

Manchester Institute for Mathematical Sciences
School of Mathematics
The University of Manchester

Reports available from: http://eprints.maths.manchester.ac.uk/
And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097
First/second order transformation system

Tentative proof

Y. Chahlaoui, P. Van Dooren

Theorem

The generalized state space system (of state dimension \(2n\))

\[
G(\lambda) \sim \begin{bmatrix}
\lambda \hat{E} - \hat{A} & \hat{B} \\
\hat{C} & 0
\end{bmatrix}
\]

(1)

is system equivalent to the so-called second order form

\[
G(\lambda) \sim \begin{bmatrix}
\lambda I_n & -I_n & 0 \\
K & \lambda M + D & B \\
C & 0 & 0
\end{bmatrix}
\]

(2)

if and only if there exists an \(n \times 2n\) matrix \(R\) such that

\[
\text{rank}\begin{bmatrix}
R \hat{E} \\
R \hat{A}
\end{bmatrix} = n, \quad R \hat{B} = 0, \quad \text{rank}\begin{bmatrix}
R \hat{E} \\
\hat{C}
\end{bmatrix} = \text{rank}\begin{bmatrix}
R \hat{E}
\end{bmatrix}.
\]

(3)

Proof. The only if part is trivial since if both generalized state space systems are equivalent, then there exist invertible matrices \(S\) and \(T\) such that

\[
\begin{bmatrix}
S(\lambda \hat{E} - \hat{A})T \\
C T
\end{bmatrix}
\begin{bmatrix}
S \hat{B}
\end{bmatrix} =
\begin{bmatrix}
\lambda I_n & -I_n & 0 \\
K & \lambda M + D & B \\
C & 0 & 0
\end{bmatrix}.
\]

But then clearly the matrix \(R\) made from the first \(n\) rows of \(S\) satisfies

\[
\begin{bmatrix}
I_n & 0 \\
0 & -I_n
\end{bmatrix}
\begin{bmatrix}
R \hat{E} \\
R \hat{A}
\end{bmatrix}
T = I_{2n}, \quad R \hat{B} = 0, \quad \begin{bmatrix}
R \hat{E} \\
\hat{C}
\end{bmatrix} T = \begin{bmatrix}
I_n & 0 \\
0 & C
\end{bmatrix}
\]

(4)

which clearly satisfies (3). The if part follows the converse reasoning. Construct the inverse of the matrix \(T\) from the given matrix \(R\), and partition it as follows

\[
T^{-1} := \begin{bmatrix}
I_n & 0 \\
0 & -I_n
\end{bmatrix}
\begin{bmatrix}
R \hat{E} \\
R \hat{A}
\end{bmatrix}, \quad [T_1 | T_2] := T.
\]
Now choose $S_1 = R$ and $S_2$ such that $S_2 \hat{E} T_1 = 0$ where
\[
S := \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},
\]
is of full rank. This can always be done since $S_1 \hat{E} T_1 = I_n$ implies that none of the rows of $S_1$ are orthogonal to $\hat{E} T_1$ while $S_2 \hat{E} T_1 = 0$ implies that all the rows of $S_2$ are orthogonal to $\hat{E} T_1$. We now obtain with this construction the required equivalence (3) by putting
\[
C := \hat{C} T_1, \quad B := S_2 \hat{B}, \quad K := S_2 \hat{A} T_1, \quad M := S_2 \hat{E} T_2, \quad D := S_2 \hat{A} T_2.
\]

Notice that we have not assumed any special properties of the systems (1) and (2). It easily follows that
1. system (2) is regular iff system (1) is regular
2. system (2) is minimal iff system (1) is minimal
3. $M$ in system (2) is invertible iff $\hat{E}$ in system (1) is invertible
since equivalence transformations do not change these properties. Moreover, if the system is regular, then the transfer function is also given by
\[
G(\lambda) = C(\lambda^2 M + \lambda D + K)^{-1} B.
\]

References


