First/second order transformation system. Tentative proof

Chahlaoui, Younes and Van Dooren, Paul

2008

MIMS EPrint: **2008.77**

ISSN 1749-9097

First/second order transformation system
Tentative proof

Y. Chahlaoui, P. Van Dooren

Theorem
The generalized state space system (of state dimension $2n$)

$$G(\lambda) \sim \left[\begin{array}{c|c} \lambda\hat{E}-\hat{A} & \hat{B} \\ \hline \hat{C} & 0 \end{array}\right] \tag{1}$$

is system equivalent to the so-called second order form

$$G(\lambda) \sim \left[\begin{array}{cc|c} \lambda I_n & -I_n & 0 \\ K & \lambda M + D & B \\ \hline C & 0 & 0 \end{array}\right] \tag{2}$$

if and only if there exists an $n \times 2n$ matrix R such that

$$\operatorname{rank}\left[\begin{array}{c} R\hat{E} \\ R\hat{A} \end{array}\right] = n, \quad R\hat{B} = 0, \quad \operatorname{rank}\left[\begin{array}{c} R\hat{E} \\ \hat{C} \end{array}\right] = \operatorname{rank}\left[\begin{array}{c} R\hat{E} \end{array}\right]. \tag{3}$$

Proof. The only if part is trivial since if both generalized state space systems are equivalent, then there exist invertible matrices S and T such that

$$\left[\begin{array}{c|c} S(\lambda\hat{E}-\hat{A})T & S\hat{B} \\ \hline \hat{C}T & 0 \end{array}\right] = \left[\begin{array}{cc|c} \lambda I_n & -I_n & 0 \\ K & \lambda M + D & B \\ \hline C & 0 & 0 \end{array}\right].$$

But then clearly the matrix R made from the first n rows of S satisfies

$$\left[\begin{array}{cc} I_n & 0 \\ 0 & -I_n \end{array}\right]\left[\begin{array}{c} R\hat{E} \\ R\hat{A} \end{array}\right] T = I_{2n}, \quad R\hat{B} = 0, \quad \left[\begin{array}{c} R\hat{E} \\ \hat{C} \end{array}\right] T = \left[\begin{array}{cc} I_n & 0 \\ C & 0 \end{array}\right] \tag{4}$$

which clearly satisfies (3). The if part follows the converse reasoning. Construct the inverse of the matrix T from the given matrix R, and partition it as follows

$$T^{-1} := \left[\begin{array}{cc} I_n & 0 \\ 0 & -I_n \end{array}\right]\left[\begin{array}{c} R\hat{E} \\ R\hat{A} \end{array}\right], \quad \left[\begin{array}{c|c} T_1 & T_2 \end{array}\right] := T.$$

Now choose $S_1 = R$ and S_2 such that $S_2 \hat{E} T_1 = 0$ where

$$S := \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

is of full rank. This can always be done since $S_1 \hat{E} T_1 = I_n$ implies that none of the rows of S_1 are orthogonal to $\hat{E} T_1$ while $S_2 \hat{E} T_1 = 0$ implies that all the rows of S_2 are orthogonal to $\hat{E} T_1$. We now obtain with this construction the required equivalence (3) by putting

$$C := \hat{C} T_1, \quad B := S_2 \hat{B}, \quad K := S_2 \hat{A} T_1, \quad M := S_2 \hat{E} T_2, \quad D := S_2 \hat{A} T_2.$$

□

Notice that we have not assumed any special properties of the systems (1) and (2). It easily follows that

1. system (2) is regular iff system (1) is regular
2. system (2) is minimal iff system (1) is minimal
3. M in system (2) is invertible iff $\hat{E}$ in system (1) is invertible

since equivalence transformations do not change these properties. Moreover, if the system is regular, then the transfer function is also given by

$$G(\lambda) = C(\lambda^2 M + \lambda D + K)^{-1} B.$$

References

[1] Antoulas. A. C, "Approximation of Large-Scale Dynamical Systems", *Society for Industrial and Applied Mathematic, 2005, ISBN: 0898715296.*

[2] Chahlaoui. Y, "Low-rank approximation and model reduction", *PhD thesis number 14/2003, Louvain-la-Neuve : UCL*, 2003.

[3] Chahlaoui. Y, K. Gallivan, A. Vandendorpe and P. Van Dooren, "Model reduction of second order systems", *Model Reduction of Dynamical Systems , Eds. P. Benner et al., LNCSE Vol. 45, pp. 149–172, Springer Verlag, 2005.*

[4] Chahlaoui. Y, D. Lemonnier, A. Vandendorpe and P. Van Dooren, "Second Order Balanced Truncation", *accepted for Lin. Alg. Appl., 2003.*

[5] Salimbahrami. B, R. Eid, B. Lohmann, E. B. Rudnyi, J. G. Korvink, "Order Reduction of Second Order Systems with Proportional Damping", *ALA 2006, Joint GAMM-SIAM Conference on Applied Linear Algebra, University of Düsseldorf, Germany, July 24-27, 2006.*

[6] Van Dooren. P, "Gramian based model reduction of large-scale dynamical systems", *in Numerical Analysis 1999* pp. 231–247, 2000.