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Structure sheaves of definable additive categories

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1 Introduction

A category of modules is said to be definable if it is closed under direct products, direct limits and pure submodules. More generally we make the same definition for subcategories of Mod-$\mathcal{R}$, the category of additive contravariant functors from a skeletally small preadditive category $\mathcal{R}$ to the category **Ab** of abelian groups. Such definable categories are precisely the exactly definable categories of [26]: those equivalent to one of the form $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ where $\mathcal{A}$ is a skeletally small

abelian category and where $\mathrm{Ex}(\mathcal{C}, \mathcal{B})$ denotes the category of exact additive functors from $\mathcal{C}$ to $\mathcal{B}$.

We prove (2.3) that the 2-category with objects the small abelian categories and arrows the exact functors between them is equivalent to the 2-category whose objects are the definable additive categories and whose arrows are the functors which preserve direct products and direct limits. In each case the 2-arrows are the natural transformations. This is an additive analogue of the kinds of 2-equivalences seen in [33], [22].

On objects, this equivalence of 2-categories takes a small abelian category $\mathcal{A}$ to $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ and takes a definable category to its "(finitely presented) functor category", $\mathrm{fun}(\mathcal{D})$. This functor category may be defined in a number of equivalent ways, most directly as the category of those additive functors from $\mathcal{D}$ to $\mathbf{Ab}$ which commute with direct products and direct limits ([42, §§11, 12]).

Let $\mathrm{pinj}(\mathcal{D})$ denote the set of isomorphism types of indecomposable pure-injective objects of a definable category $\mathcal{D}$. This set may be equipped with the rep-Zariski topology, $\mathrm{Zar}(\mathcal{D})$, by declaring that the sets $[F] = \{N \in \mathrm{pinj}(\mathcal{D}) : FN = 0\}$, for $F \in \mathrm{fun}(\mathcal{D})$, form a basis of open sets.

Over this space there is a sheaf of categories: to a basic open set $[F]$ as above is associated the localisation of $\mathrm{fun}(\mathcal{D})$ at the Serre subcategory generated by F. This presheaf-over-a-basis, denoted $\mathrm{Def}(\mathcal{D})$, is separated, hence embeds in its sheafification, which we denote $\mathrm{LDef}(\mathcal{D})$. We identify the stalks of this sheaf (3.3 and comments following that). We also show that the duality between a definable category and its elementary dual, a relation which generalises that between Mod-R and R-Mod, extends to this "categoried locale" (3.4). Furthermore, any functor between definable categories which preserves direct products and direct limits induces a morphism of categoried locales (4.1).

We could treat the above presheaf-on-a-basis as a fibred category and then replace the sheafification process by the formation of the associated stack. For some purposes moving to this more general context may be unavoidable but, at least initially, we have enough control over the restriction functors to make this unnecessary.

Any locally coherent additive category $\mathcal{C}$ is definable. Let $\mathrm{inj}(\mathcal{C})$ denote the set of isomorphism types of indecomposable injective objects of such a, necessarily Grothendieck ([10, 2.4]), category $\mathcal{C}$. One may define the Gabriel-Zariski topology on $\mathrm{inj}(\mathcal{C})$ by declaring, for A running over the category $\mathcal{C}^{\mathrm{fp}}$, the $[A] = \{E \in \mathrm{inj}(\mathcal{C}) : (A, E) = 0\}$ to form a basis of open sets; here $\mathcal{C}^{\mathrm{fp}}$ denotes the full subcategory of finitely presented objects of $\mathcal{C}$. Again there is a presheaf, $\mathrm{FT}(\mathcal{C})$, of categories, obtained by associating to $[A]$ the localisation of $\mathcal{C}^{\mathrm{fp}}$ at the hereditary torsion theory generated by A, equivalently the quotient of $\mathcal{C}^{\mathrm{fp}}$ by the Serre subcategory generated by A. We prove that this is a restriction, both of base and of sections, of the presheaf over $\mathrm{pinj}(\mathcal{C})$ defined above (5.1).

We recall ([19], see [23], [43]) that the full subcategory, $\mathrm{Pinj}(\mathcal{D})$, of all pure-injective objects of a definable category $\mathcal{D}$ is equivalent to the category of injective objects of the associated, locally coherent, functor category, $\mathrm{Fun}(\mathcal{D})$ (we

write $\mathrm{fun}(\mathcal{D}) = \big(\mathrm{Fun}(\mathcal{D})\big)^{\mathrm{fp}}$), and the Gabriel-Zariski topology on $\mathrm{inj}\big(\mathrm{Fun}(\mathcal{D})\big)$ induces a topology which coincides with the rep-Zariski topology on $\mathrm{pinj}(\mathcal{D})$ (see [43, 14.1.7]). This also induces an equivalence of "categoried spaces" between the sheafification of $\mathrm{FT}\big(\mathrm{Fun}(\mathcal{D})\big)$ and $\mathrm{LDef}(\mathcal{D})$ (this is direct from [43, 12.3.20]). Theorem 5.1 in a sense complements this, restricting the rep-Zariski topology on $\mathrm{pinj}(\mathcal{D})$ to $\mathrm{inj}(\mathcal{D})$ in the case that $\mathcal{D}$ is abelian and locally coherent.

In the case that $\mathcal{D}$ is a module category, Mod-R, over a right coherent ring, it makes sense to look at that part of a sheaf which corresponds to "localisations" of R and, in this case, 5.1 implies that the presheaf of definable scalars restricted to inj_R (meaning $\mathrm{inj}(\text{Mod-}R)$) coincides with that of finite type torsion-theoretic localisations of R (see [43, 6.1.17]). In this, fifth, section we also show how this contains the classical equivalence between noetherian commutative rings and affine varieties. In particular we show that a commutative ring may be recovered from the finitely presented functor category of its module category (5.2).

For any ring R the finitely presented functor category also has a realisation as the free abelian category $\mathrm{Ab}(R)$ (actually $\mathrm{Ab}(R^{\mathrm{op}})$) of R, and evaluation at R is an exact functor from this to the category of R-modules. If R is right coherent then the image of this functor is the category of finitely presented modules. We identify the image of this functor from $\mathrm{Ab}(R)$ to Mod-R in the general case: as the, non-full, subcategory of modules which occur as the kernel of a morphism between finitely presented modules (6.4).

The Ziegler topology on $\mathrm{pinj}(\mathcal{D})$, where $\mathcal{D}$ is a definable category, and the rep-Zariski topology are "dual". In the final section we describe, for any ring R, a simple basis for the restriction of the Ziegler topology to the set, inj_R of isomorphism types of indecomposable injective R-modules (7.3, 7.5). Model-theoretically, this is an elimination of imaginaries result.

We assume some acquaintance with the relevant background. Much of this can be found in [42], or [43], which we will often use as references in favour of the original sources. A great deal of the relevant background can be found also in [23], [21], [28].

2 Definable additive categories

Suppose that $\mathcal{D}$ is a **definable** subcategory of Mod-$\mathcal{R} = (\mathcal{R}^{\mathrm{op}}, \mathbf{Ab})$ where $\mathcal{R}$ is a skeletally small preadditive category: that is, $\mathcal{D}$ is a full subcategory closed under arbitrary products, direct limits and pure submodules. Recall that an embedding $f : M \to N$ is **pure** if for every $L \in \mathcal{R}$-Mod the morphism $f \otimes 1_L : M \otimes_{\mathcal{R}} L \to N \otimes_{\mathcal{R}} L$ is monic; there are various equivalents, see, e.g., [42, 5.2]. Denote by Fun-$\mathcal{R} = \mathrm{Fun}(\text{Mod-}\mathcal{R})$ the category, $(\text{mod-}\mathcal{R}, \mathbf{Ab})$, of additive functors from the category, mod-$\mathcal{R}$, of finitely presented right $\mathcal{R}$-modules to **Ab**. Also set fun-$\mathcal{R} = (\text{Fun-}\mathcal{R})^{\mathrm{fp}}$ where, for any category $\mathcal{C}$ we denote by $\mathcal{C}^{\mathrm{fp}}$ the full subcategory of finitely presented objects. Recall that an object C is **finitely presented** if the representable functor $(C, -) : \mathcal{C} \longrightarrow \mathbf{Ab}$ commutes

with direct limits.

Every functor F in Fun-$\mathcal{R}$ has a unique extension to a functor, $\overrightarrow{F}$, which is defined on all of Mod-$\mathcal{R}$ and commutes with direct limits: for the definition just use that every module is a direct limit of finitely presented modules and check well-definedness. Usually we will identify F and $\overrightarrow{F}$ notationally and write FM for the value of $\overrightarrow{F}$ at $M \in$ Mod-$\mathcal{R}$.

Set $\mathcal{S}_{\mathcal{D}} = \{F \in \text{fun-}\mathcal{R} : F\mathcal{D} = 0\}$ (that is, $\overrightarrow{F}D = 0$ for every $D \in \mathcal{D}$). This is a Serre subcategory of fun-$\mathcal{R}$ and every Serre subcategory of fun-$\mathcal{R}$ arises in this way. The condition that a subcategory $\mathcal{S}$ of an abelian category $\mathcal{C}$ be **Serre** is that if $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{C}$ then $B \in \mathcal{S}$ iff $A, C \in \mathcal{S}$. The hereditary torsion theory on Fun-$\mathcal{R}$ whose torsion class is generated by $\mathcal{S}_{\mathcal{D}}$ is of finite type and is denoted $\tau_{\mathcal{D}}$. Every finite type torsion theory on the functor category arises in this way (we can take this - the torsion class being generated as such by the finitely presented torsion objects - as the definition of finite type; for background on torsion theories see, for instance, [50], [28] or [43]). We set $\text{Fun}(\mathcal{D}) = (\text{Fun-}\mathcal{R})_{\tau_{\mathcal{D}}}$, the localisation of Fun-$\mathcal{R}$ at $\tau_{\mathcal{D}}$. Since Fun-$\mathcal{R}$ is locally coherent and $\tau_{\mathcal{D}}$ is of finite type the localisation $\text{Fun}(\mathcal{D})$ also is locally coherent, so its subcategory $(\text{Fun}(\mathcal{D}))^{\text{fp}} = ((\text{Fun-}\mathcal{R})^{\text{fp}})_{\tau_{\mathcal{D}}} = (\text{fun-}\mathcal{R})/\mathcal{S}_{\mathcal{D}}$ is abelian and is denoted $\text{fun}(\mathcal{D})$. We refer to the latter as the "(finitely presented) functor category of $\mathcal{D}$". For all this, see [21], [28], [43] Although the definition of $\text{fun}(\mathcal{D})$ is given in terms of a representation of the abstract category $\mathcal{D}$ as a definable subcategory of a particular module category, it is the case, see [42, 12.2, 12.10], that this functor category depends only on $\mathcal{D}$ as a category. Evaluation of objects of $\text{fun}(\mathcal{D})$ at objects of $\mathcal{D}$ is, by definition of $\mathcal{S}_{\mathcal{D}}$, well-defined since $\text{fun}(\mathcal{D}) \simeq \text{fun-}\mathcal{R}/\mathcal{S}_{\mathcal{D}}$.

The next result is due, in varying degrees of generality, to Herzog, Crawley-Boevey, Krause, see [42, 10.8, 10.9] or [43, 18.1.4]. It may be obtained also from a theorem of Makkai, [33, 5.1, §6] (also see [48, 4.4]), though that is by a very different route.

Theorem 2.1 *Let $\mathcal{D}$ be a definable subcategory of* Mod-$\mathcal{R}$. *Then evaluation defines an equivalence of categories $\mathcal{D} \simeq \text{Ex}(\text{fun}(\mathcal{D}), \mathbf{Ab})$.*

Given $\mathcal{D}$, the equivalence $\mathcal{D} \simeq \text{Ex}(\mathcal{B}, \mathbf{Ab})$ determines the abelian category $\mathcal{B}$ up to natural equivalence (as the functor category, $\text{fun}(\mathcal{D})$, of $\mathcal{D}$). Suppose also that $\mathcal{C} \simeq \text{Ex}(\mathcal{A}, \mathbf{Ab})$. An exact functor $E : \mathcal{B} \longrightarrow \mathcal{A}$ induces, by composition, a functor $E^* : \mathcal{C} \longrightarrow \mathcal{D}$ which, one may check, commutes with direct products and direct limits. The converse also holds, and follows from the next result, due to Krause [26, 7.2] in the case that $\mathcal{D}$ is finitely accessible and Prest [42, 11.2, 12.10] in general.

Theorem 2.2 *Suppose that $\mathcal{D} \simeq \text{Ex}(\mathcal{B}, \mathbf{Ab})$ is a definable category. Then $\mathcal{B}$ is equivalent to the category $(\mathcal{D}, \mathbf{Ab})^{\Pi\to}$, of functors on $\mathcal{D}$ which commute with direct products and direct limits.*

This also can obtained, again through a very different route, from a rather general theorem, namely Hu's [22, 5.10(ii)].

It follows that any functor $I : \mathcal{C} \longrightarrow \mathcal{D}$ which commutes with direct products and direct limits induces, by composition, a functor $I_0 : \mathcal{B} \longrightarrow \mathcal{A}$ which, one may check, is exact. Furthermore, $(E^*)_0 \simeq E$ and $(I_0)^* \simeq I$. Indeed, one has an equivalence between the category $\mathbb{DEF}$ whose objects are definable additive categories and whose morphisms are those which preserve direct products and direct limits, and the category $\mathbb{ABEX}$ of skeletally small abelian categories and exact functors. We show that this is, in fact, an equivalence of 2-categories (and, in the process, give some more details of what we have referred to above).

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Theorem 2.3 *The assignments $\mathcal{D} \mapsto \mathrm{fun}(\mathcal{D})$ and $\mathcal{A} \mapsto \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ on objects, $I \mapsto I_0$ and $E \mapsto E^*$ on functors, extend to inverse natural equivalences of the 2-categories $\mathbb{DEF}$ and $\mathbb{ABEX}$.*

Proof. The 2-category structure on each category is the usual one, with natural transformations being the 2-arrows.

Note, for reference, that if $I : \mathcal{C} \to \mathcal{D}$ is a functor between definable categories $\mathcal{C}$ and $\mathcal{D}$ which preserves direct products and direct limits then $I_0 : \mathrm{fun}(\mathcal{D}) \to \mathrm{fun}(\mathcal{C})$ is defined as follows. On an object $G \in \mathrm{fun}(\mathcal{D})$ (that is, G is a functor from $\mathcal{D}$ to $\mathbf{Ab}$ which commutes with direct products and direct limits) the functor $I_0G \in \mathrm{fun}(\mathcal{C}) = (\mathcal{C}, \mathbf{Ab})^{\prod \to}$ is defined on objects by $I_0G.C = GI.C$ for $C \in \mathcal{C}$, and $I_0G.f = GI.f$. Furthermore, if $\tau : G \to G'$ is a natural transformation in $\mathrm{fun}(\mathcal{D})$ then $I_0\tau : I_0G \to I_0G'$ has component at $C \in \mathcal{C}$ defined by $(I_0\tau)_C = \tau_{IC}$.

So suppose that $\eta : I \to J$ is a natural transformation between $I, J : \mathcal{C} \to \mathcal{D}$ in $\mathbb{DEF}$: we must define the corresponding natural transformation $\eta' : I_0 \to J_0$. The component of η' at $G \in \mathrm{fun}(\mathcal{D})$ is $\eta'_G : I_0G \to J_0G$ and so we have to define the component of a morphism between functors in $\mathrm{fun}(\mathcal{C})$ at $C \in \mathcal{C}$. That will be a map from $I_0G.C$ to $J_0G.C$, that is from GIC to GJC, so we set $(\eta'_G)_C = G\eta_C$. It must be checked that η'_G is a natural transformation.

So let $f : C \to C'$ be in $\mathcal{C}$. Then the relevant diagram is

$$\begin{array}{ccc} I_0GC & \xrightarrow{(\eta'_G)_C} & J_0GC \\ {\scriptstyle I_0Gf}\downarrow & & \downarrow{\scriptstyle J_0Gf} \\ I_0GC' & \xrightarrow[(\eta'_G)_{C'}]{} & J_0GC' \end{array}$$

that is

$$\begin{array}{ccc} GIC & \xrightarrow{G\eta_C} & GJC \\ {\scriptstyle GIf}\downarrow & & \downarrow{\scriptstyle GJf} \\ GIC' & \xrightarrow[G\eta_{C'}]{} & GJC' \end{array}$$

the commutativity of which follows by applying G to the commutative diagram

$$\begin{array}{ccc} IC & \xrightarrow{\eta_C} & JC \\ {\scriptstyle If}\downarrow & & \downarrow{\scriptstyle Jf} \\ IC' & \xrightarrow[\eta_{C'}]{} & JC' \end{array} .$$

Therefore η'_G is a natural transformation. Next we have to check that η' is a natural transformation. So suppose that $\tau : G \to G'$ is in fun($\mathcal{D}$). Consider the diagram

$$\begin{array}{ccc} I_0G & \xrightarrow{\eta'_G} & J_0G \\ {\scriptstyle I_0\tau}\downarrow & & \downarrow{\scriptstyle J_0\tau} \\ I_0G' & \xrightarrow[\eta'_{G'}]{} & J_0G' \end{array} .$$

This is a diagram of natural transformations so, to check that it commutes, it is enough to check at each object $C \in \mathcal{C}$, that is, consider

$$\begin{array}{ccc} I_0GC & \xrightarrow{(\eta'_G)_C} & J_0GC \\ {\scriptstyle (I_0\tau)_C}\downarrow & & \downarrow{\scriptstyle (J_0\tau)_C} \\ I_0G'C & \xrightarrow[(\eta'_{G'})_C]{} & J_0G'C \end{array}$$

which is the diagram

$$\begin{array}{ccc} GIC & \xrightarrow{G\eta_C} & GJC \\ {\scriptstyle \tau_{IC}}\downarrow & & \downarrow{\scriptstyle \tau_{JC}} \\ G'IC & \xrightarrow[G'\eta_C]{} & G'JC \end{array}$$

commutativity of which follows since τ is a natural transformation.

Now consider the other direction. Note for reference that if $I_0 : \mathcal{A} \to \mathcal{B}$ is an exact functor between the (skeletally small) abelian categories $\mathcal{A}$ and $\mathcal{B}$ then $I_0^* : \mathcal{D} = \mathrm{Ex}(\mathcal{B}, \mathbf{Ab}) \to \mathcal{C} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ is defined as follows. If $D \in \mathcal{D}$ then I_0^*D is defined on objects by $I_0^*D.A = D.I_0A$ and if $f : A \to A'$ is in $\mathcal{A}$ then $I_0^*D.f = D.I_0f$. Furthermore, if $\tau : D \to D'$ is a natural transformation in $\mathrm{Ex}(\mathcal{B}, \mathbf{Ab})$ then the component of $I_0^*\tau$ at $A \in \mathcal{A}$ is given by $(I_0^*\tau)_A = \tau_{I_0A}$.

Suppose that $\theta : I_0 \to J_0$ is a natural transformation between $I_0, J_0 : \mathcal{A} \to \mathcal{B}$, so at $A \in \mathcal{A}$ we have the component $\theta_A : I_0A \to J_0A$ such that if $f : A \to A'$ is in $\mathcal{A}$ then the diagram

$$\begin{array}{ccc} I_0A & \xrightarrow{\theta_A} & J_0A \\ {\scriptstyle I_0f}\downarrow & & \downarrow{\scriptstyle J_0f} \\ I_0A' & \xrightarrow[\theta_{A'}]{} & J_0A' \end{array}$$

commutes. We define $\theta^* : I_0^* \to J_0^*$ by defining its component at $D \in \mathrm{Ex}(\mathcal{B}, \mathbf{Ab})$,

that is $\theta^*_D : I^*_0 D \to J^*_0 D$, to be the natural transformation between functors on $\mathcal{A} = \mathrm{Ex}(\mathcal{C}, \mathbf{Ab})$ which has component at $A \in \mathcal{A}$ given by $(\theta^*_D)_A = D\theta_A$; that is $(\theta^*_D)_A : I^*_0 DA \to J^*_0 DA$ is defined to be $D\theta_A : DI_0 A \to DJ_0 A$. It must be checked that θ^*_D is a natural transformation.
So let $f : A \to A'$ be in $\mathcal{A}$. Then the relevant diagram is

$$\begin{array}{ccc} I^*_0 DA & \xrightarrow{(\theta^*_D)_A} & J^*_0 DA \\ {\scriptstyle I^*_0 Df}\downarrow & & \downarrow{\scriptstyle J^*_0 Df} \\ I^*_0 DA' & \xrightarrow[(\theta^*_D)_{A'}]{} & J^*_0 DA' \end{array}$$

that is

$$\begin{array}{ccc} DI_0 A & \xrightarrow{D\theta_A} & DJ_0 A \\ {\scriptstyle DI_0 f}\downarrow & & \downarrow{\scriptstyle DJ_0 f} \\ DI_0 A' & \xrightarrow[D\theta_{A'}]{} & DJ_0 A' \end{array}$$

which is commutative (apply D to the relevant "θ-diagram").

Then it has to be checked that θ^* is a natural transformation. So let $\tau : D \to D'$ be in $\mathcal{D} = \mathrm{Ex}(\mathcal{B}, \mathbf{Ab})$. The diagram to be proved commutative is

$$\begin{array}{ccc} I^*_0 D & \xrightarrow{\theta^*_D} & J^*_0 D \\ {\scriptstyle I^*_0 \tau}\downarrow & & \downarrow{\scriptstyle J^*_0 \tau} \\ I^*_0 D' & \xrightarrow[\theta^*_{D'}]{} & J^*_0 D' \end{array}$$

which at $A \in \mathcal{A}$ is

$$\begin{array}{ccc} I^*_0 DA & \xrightarrow{(\theta^*_D)_A} & J^*_0 DA \\ {\scriptstyle (I^*_0 \tau)_A}\downarrow & & \downarrow{\scriptstyle (J^*_0 \tau)_A} \\ I^*_0 D'A & \xrightarrow[(\theta^*_{D'})_A]{} & J^*_0 D'A \end{array}$$

that is,

$$\begin{array}{ccc} DI_0 A & \xrightarrow{D\theta_A} & DJ_0 A \\ {\scriptstyle \tau_{I_0 A}}\downarrow & & \downarrow{\scriptstyle \tau_{J_0 A}} \\ D'I_0 A & \xrightarrow[D'\theta_A]{} & D'J_0 A \end{array}$$

which does commute since τ is a natural transformation.

Then we have to show that $((-)_0)^*$ and $((-)^*)_0$ are equivalent to the respective identities. For $\mathcal{C} \in \mathbb{DEF}$ we define the component of the relevant natural transformation at $\mathcal{C}$ to be the functor $\epsilon_C : \mathcal{C} \to \mathrm{Ex}(\mathrm{fun}(\mathcal{C}), \mathbf{Ab})$ which takes $C \in \mathcal{C}$ to the functor ev_C (evaluation at C) and which has the obvious effect on morphisms $f : C \to C'$ (namely $(\mathrm{ev}_f)F = Ff$ for $F \in \mathrm{fun}(\mathcal{C})$.) It is a theorem

due to Herzog and Krause, see, e.g., [42, 10.8], that this does give an equivalence between $\mathcal{C}$ and $\mathrm{Ex}(\mathrm{fun}(\mathcal{C}), \mathbf{Ab})$. Then it must be checked that if $I : \mathcal{C} \to \mathcal{D}$ is in $\mathbb{DEF}$ then the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\epsilon_{\mathcal{C}}} & \mathrm{Ex}(\mathrm{fun}(\mathcal{C}), \mathbf{Ab}) \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle (I_0)^*} \\ \mathcal{D} & \xrightarrow{\epsilon_{\mathcal{D}}} & \mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathbf{Ab}) \end{array}$$

commutes, that is, that $(I_0)^*\mathrm{ev}_C = \mathrm{ev}_{IC}$. Now, for $B \in \mathrm{fun}(\mathcal{D})$ we have $(I_0)^*\mathrm{ev}_C B = \mathrm{ev}_C I_0 B = B(IC) = \mathrm{ev}_{IC} B$ and similarly for morphisms. So these are the components on a natural equivalence between the identity functor on $\mathbb{DEF}$ and the composition $((-)_0)^*$. Similarly for the other way round, bearing in mind that $\mathcal{A} = \mathrm{fun}(\mathcal{C} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})) = (\mathcal{C}, \mathbf{Ab})^{\Pi\to}$ so that objects of $\mathcal{A}$ may be regarded as functors on $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ and hence evaluation of these at objects of $\mathcal{A}$ makes sense. Thus the result is proved. □

The 2-category $\mathbb{ABEX}$ has the obvious involution, defined on objects by taking an abelian category $\mathcal{A}$ to its opposite. It follows from 2.3 that there is a corresponding involution on $\mathbb{DEF}$: a definable category $\mathcal{D} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ is taken to $\mathcal{D}^{\mathrm{d}} = \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$, its (**elementary**) **dual**. Other descriptions of, and further information on, this category are given in [42, §§9, 10]. In particular, if $\mathcal{D}$ is finitely accessible, hence can be represented as the category, Flat-$\mathcal{R}$, of flat right $\mathcal{R}$-modules for some small preadditive category $\mathcal{R}$ then $\mathcal{D}^{\mathrm{d}}$ is equivalent to the category, $\mathcal{R}$-Abs, of absolutely pure left $\mathcal{R}$-modules. Now, any definable category $\mathcal{D}$ is a definable subcategory of some finitely accessible category $\mathcal{C}$ and there is a natural bijection between definable subcategories of $\mathcal{C}$ and of $\mathcal{C}^{\mathrm{d}}$, which takes $\mathcal{D}$ to $\mathcal{D}^{\mathrm{d}}$. It is also the case that $\mathcal{C}$ may be taken to be a functor category, Mod-$\mathcal{R} = (\mathcal{R}^{\mathrm{op}}, \mathbf{Ab})$ for some skeletally small preadditive category $\mathcal{R}$; so it makes sense to form the tensor product, $D \otimes_{\mathcal{R}} D'$, over $\mathcal{R}$, of objects $D \in \mathcal{D}$ and $D' \in \mathcal{D}^{\mathrm{d}}$. Also recall, e.g. [42, 10.12], that the bijection between definable subcategories of $\mathcal{C}$ and $\mathcal{C}^{\mathrm{d}}$ restricts to one between definable subcategories of $\mathcal{D}$ and $\mathcal{D}^{\mathrm{d}}$ and, since the topology on the Ziegler spectrum may be defined in terms of such subcategories (see, e.g., [43, §5.1.1]), the rep-Zariski spectrum, $\mathrm{Zar}(\mathcal{D})$, of $\mathcal{D}$ and that of $\mathcal{D}^{\mathrm{d}}$ are isomorphic as locales ("homeomorphic at the level of topology").

3 The structure sheaf

Let $\mathcal{D}$ be a definable category. An object $D \in \mathcal{D}$ is **pure-injective** if every pure embedding with domain D is split. Here purity may be defined with respect to any representation of $\mathcal{D}$ as a definable subcategory but also is defined purely internally because a morphism is a pure embedding iff some ultraproduct of it is split (and ultraproducts, being certain direct limits of direct products, need

only the assumed structure on $\mathcal{D}$). Let $\mathrm{pinj}(\mathcal{D})$ denote the set (it is a set) of (isomorphism types of) (direct-sum) indecomposable pure-injective objects of $\mathcal{D}$. We equip this with the rep-Zariski (=dual-Ziegler, [38] or e.g. [43, §5.3]) topology which has, for a basis of open sets, the

$$[F] = \{N \in \mathrm{pinj}(\mathcal{D}) : FN = 0\}$$

where $F \in \mathrm{fun}(\mathcal{D})$ (and FN really means $\overrightarrow{F}N$). Since $[F] \cap [G] = [F \oplus G]$ this is a basis of open sets for a topology, called the **rep-Zariski** topology on $\mathrm{pinj}(\mathcal{D})$. We write $\mathrm{Zar}(\mathcal{D})$ for this space. It does generalise the usual Zariski spectrum of a commutative noetherian ring ([38], [41] or e.g. [43, Chpt. 14]) but, despite the name it shares few properties with the spectrum of a commutative ring, in particular, it need not be a spectral space.

If $\mathcal{D}$ is represented as a definable subcategory of Mod-$\mathcal{R}$ say (for instance one may take $\mathcal{R}$ to be $\big(\mathrm{fun}(\mathcal{D})\big)^{\mathrm{op}}$) we denote by $\mathcal{S}_{\mathcal{D}}$ the Serre subcategory of $(\text{mod-}\mathcal{R}, \mathbf{Ab})^{\mathrm{fp}}$ consisting of all those finitely presented functors F which vanish on $\mathcal{D}$ (more accurately, those F whose unique extension $\overrightarrow{F}$ to a functor on all of Mod-$\mathcal{R}$ which commutes with direct limits, satisfies $\overrightarrow{F}D = 0$ for all $D \in \mathcal{D}$). Let $\tau_{\mathcal{D}}$ denote the finite type torsion theory on $(\text{mod-}\mathcal{R}, \mathbf{Ab})$ which $\mathcal{S}_{\mathcal{D}}$ generates.

There is a duality d between the categories, $(\mathcal{R}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ and $(\text{mod-}\mathcal{R}, \mathbf{Ab})^{\mathrm{fp}}$, of finitely presented functors and this induces a natural bijection $\mathcal{S} \mapsto d\mathcal{S} = \{dF : F \in \mathcal{S}\}$ between Serre subcategories and hence a natural bijection, $\tau \mapsto \tau^{\mathrm{d}}$, of finite type torsion theories on the whole functor categories.

The localisation of $(\text{mod-}\mathcal{R}, \mathbf{Ab})$ at $\tau_{\mathcal{D}}$ is the full functor category $\mathrm{Fun}(\mathcal{D})$ of $\mathcal{D}$ and we denote the localisation of $(\mathcal{R}\text{-mod}, \mathbf{Ab})$ at $\tau^{\mathrm{d}}_{\mathcal{D}}$ by $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$. All these (localised) functor categories are locally coherent. We have, as discussed at the end of the previous section, $\mathrm{fun}^{\mathrm{d}}(\mathcal{D}) = \mathrm{fun}(\mathcal{D}^{\mathrm{d}}) \simeq \big(\mathrm{fun}(\mathcal{D})\big)^{\mathrm{op}}$.

Consider the embedding of Mod-$\mathcal{R}$ into $(\mathcal{R}\text{-mod}, \mathbf{Ab})$ which is given on objects by $M \mapsto M \otimes_{\mathcal{R}} -$. This is a full and faithful embedding and is such that $M \in \text{Mod-}\mathcal{R}$ is pure-injective iff $M \otimes -$ is injective ([19], or see any of the background reference texts). Thus $\mathrm{pinj}_{\mathcal{R}}$ may be identified with $\mathrm{inj}\big((\mathcal{R}\text{-mod}, \mathbf{Ab})\big)$ and this restricts to an identification of $\mathrm{pinj}(\mathcal{D})$ with the set of indecomposable $\tau^{\mathrm{d}}_{\mathcal{D}}$-torsionfree injectives in the functor category $(\mathcal{R}\text{-mod}, \mathbf{Ab})$ which, in turn, may be identified with the set of indecomposable injectives of the corresponding localisation, $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$, of $(\mathcal{R}\text{-mod}, \mathbf{Ab})$. Using the formula $\overrightarrow{F}N \simeq (dF, N \otimes -)$ it follows directly that, under this identification, the Gabriel-Zariski topology on $\mathrm{inj}\big((\mathcal{R}\text{-mod}, \mathbf{Ab})\big)$ restricts to the rep-Zariski topology on $\mathrm{pinj}_{\mathcal{R}}$, hence similarly for $\mathcal{D}$.

zeroonpi

Proposition 3.1 *Let $\mathcal{D}$ be a definable category and let $F \in \mathrm{fun}(\mathcal{D})$. If $FN = 0$ for every indecomposable pure-injective $N \in \mathcal{D}$ then $F = 0$.*

Proof. Since the functor category $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$, is locally coherent the set of indecomposable injectives objects is cogenerating (in the sense that the only object with only zero morphisms to all these indecomposables is 0). The $N \otimes -$

for $N \in \mathrm{pinj}(\mathcal{D})$ are the exactly these indecomposable injectives and so, since $FN \simeq (dF, N \otimes -)$, the result follows. $\square$

For $F \in \mathrm{fun}(\mathcal{D})$ let $\mathcal{S}(F)$ denote the Serre subcategory of $\mathrm{fun}(\mathcal{D})$ generated by F: thus $G \in \mathcal{S}(F)$ iff G has a finite composition series consisting of subquotients of F.

Lemma 3.2 *Let $F, G \in \mathrm{fun}(\mathcal{D})$. Then $[G] \subseteq [F]$ iff $\mathcal{S}(G) \supseteq \mathcal{S}(F)$.*

Proof. Since $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$ is locally coherent the Serre subcategories $d\mathcal{S}$ of $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$ generate the torsion classes of finite type on $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$; these, in turn, are determined by the indecomposable injective torsionfree objects. That is $d\mathcal{S}$, hence $\mathcal{S}$ is determined by those indecomposable injective functors E such that $(d\mathcal{S}, E) = 0$ equivalently, see the proof above, by the set of indecomposable pure-injectives N such that $FN = 0$ for every $F \in \mathcal{S}$. $\square$

We define a presheaf on the above basis for $\mathrm{Zar}(\mathcal{D})$ by assigning to $[F]$ the localisation $\mathrm{fun}(\mathcal{D})/\mathcal{S}(F)$ and to an inclusion $[G] \subseteq [F]$ the localisation $\mathrm{fun}(\mathcal{D})/\mathcal{S}(F) \longrightarrow \mathrm{fun}(\mathcal{D})/\mathcal{S}(G) = \big(\mathrm{fun}(\mathcal{D})/\mathcal{S}(F)\big)/\big(\mathcal{S}(G)/\mathcal{S}(F)\big)$. Note that if $\mathcal{D}_F = \{D \in \mathcal{D} : FD = 0\}$ is the definable subcategory of $\mathcal{D}$ corresponding to F then $\mathrm{fun}(\mathcal{D})/\mathcal{S}(F) \simeq \mathrm{fun}(\mathcal{D}_F)$ so the restriction maps/functors of this presheaf can literally be read as restrictions of functors to definable subcategories of $\mathcal{D}$. It follows immediately that this presheaf-on-a-basis is separated. For suppose that $H \in \mathrm{fun}(\mathcal{D})/\mathcal{S}(F)$ and $[F] = \bigcup_\lambda [F_\lambda]$ is such that each localisation, H_λ, of H at $\mathcal{S}(F_\lambda)/\mathcal{S}(F)$ is 0. Then, regarding H as a functor on $\mathcal{D}_F$, the hypothesis is that each restriction $H \restriction \mathcal{D}_{F_\lambda}$ is 0. In particular, for each $N \in \mathcal{D}_{F_\lambda}$ we have $HN = 0$. Thus $HN = 0$ for every $N \in [F]$. But, by 3.1, that implies that $H = 0$ on $\mathcal{D}_F$ as required. This presheaf-on-a-basis we denote by $\mathrm{Def}(\mathcal{D})$ and its sheafification is denoted $\mathrm{LDef}(\mathcal{D})$: the **sheaf of locally definable scalars** on $\mathcal{D}$.

Note that, at least at this point, it is not necessary to move to fibred categories and stacks: the issue is that, typically, functors between categories are unique only up to natural equivalence so when one tries to define a presheaf of categories one can expect the restriction maps to compose only up to (specified) natural equivalences; the resulting notion is that of a fibred category over the base (glueing morphisms gives a prestack and then glueing objects gives a stack - the general notion of "sheaf of categories"). In this case, all the categories that appear are localisations of a certain category. There is one definition of localised category which leaves the objects fixed while changing the morphism groups so, if we adopt that definition, we can have restriction maps (i.e. localisations) composing "on the nose". However, the language of fibred categories and stacks is the natural one in this context.

defstalk

Proposition 3.3 *Let $N \in \mathrm{pinj}(\mathcal{D})$. The stalk of $\mathrm{LDef}(\mathcal{D})$ at N is the localisation of $\mathrm{fun}(\mathcal{D})$ at the Serre subcategory $\mathcal{S}_N = \{F \in \mathrm{fun}(\mathcal{D}) : FN = 0\}$ of functors which annihilate N.*

Proof. First, if $\mathcal{S} = \varinjlim \mathcal{S}_\lambda$ is a directed union of Serre subcategories of some abelian category $\mathcal{A}$ then it is easily checked that $\mathcal{A}/\mathcal{S}$ is naturally equivalent to $\varinjlim(\mathcal{A}/\mathcal{S}_\lambda)$ where the direct limit of categories should be understood as being taken in a suitable 2-category of categories; in our situation it can be taken in $\mathbb{ABEX}$. Then, just from the definitions, we deduce $\mathrm{LDef}(\mathcal{D})_N = \varinjlim_{[F]\ni N} \mathrm{fun}(\mathcal{D})/\mathcal{S}(F)$. □

The terminology derives from the case $\mathcal{D} = \text{Mod-}R$ and from just that part of fun-R and its localisations which are the endomorphism rings of the forgetful functor and its localisations. That is, if $\mathcal{E}$ is a definable subcategory of Mod-R then we set $R_\mathcal{E}$ to be the endomorphism ring of the image of $(R, -)$ in fun-$R/\mathcal{S}_\mathcal{E}$. This ring has a model-theoretic interpretation as the ring of all pp-definable functions ("definable scalars") on modules in $\mathcal{E}$ ([8], see [43, §12.8]). Indeed, the terminology "definable category" derives from the same source, such categories being exactly the subcategories of module categories which are axiomatisable and closed under direct summands and (finite, hence arbitrary) direct sums. And the functors between definable categories which commute with direct products and direct limits also have a model-theoretic meaning, being exactly the interpretation functors ([43, §18.2.1] or [42]). Furthermore the functor category $\mathrm{fun}(\mathcal{D})$ is equivalent to the category of "pp-imaginaries" (another notion from model theory, [29] or see, e.g., [42] or [43]). Examples of these (pre)sheaves of definable scalars are worked out in [41] (or see [43, §14.2]).

Let T be any topological space. By $\mathcal{O}(T)$ we denote the locale of open subsets of T. This is the set of all open subsets of T regarded first as a lattice, indeed a complete Heyting algebra but, rather than thought of as an object of the category of complete Heyting algebras, it is regarded as an object of the opposite category - the category of **locales** (see, e.g., [24] or [32]). Thus a continuous map $T \to T'$ of topological spaces induces a map $\mathcal{O}(T) \to \mathcal{O}(T')$ of locales. We will use the terminology **abelian space** to refer to a "categoried space" or "categoried locale" of the form $(\mathrm{Zar}(\mathcal{D}), \mathrm{LDef}(\mathcal{D}))$.

dualcatshf

Proposition 3.4 *Let $\mathcal{D}$ be a definable category and let $\mathcal{D}^\mathrm{d}$ denote its elementary dual. Then the duality $\mathrm{fun}(\mathcal{D}^\mathrm{d}) \simeq \big(\mathrm{fun}(\mathcal{D})\big)^\mathrm{op} = \mathrm{fun}^\mathrm{d}(\mathcal{D})$ induces an isomorphism of abelian spaces (at the locale level), $(\mathrm{Zar}(\mathcal{D}), \mathrm{LDef}(\mathcal{D})) \simeq (\mathrm{Zar}(\mathcal{D}^\mathrm{d}), \mathrm{LDef}(\mathcal{D}^\mathrm{d}))^\mathrm{op}$.*

Proof. What we mean by the statement is, first, that there is an isomorphism, $U \mapsto DU$ say, of locales, so of complete lattices of open sets, between $\mathrm{Zar}(\mathcal{D})$ and $\mathrm{Zar}(\mathcal{D}^\mathrm{d})$ - that is described at the end of Section 2 - and, second, that for each open subset, U say, of $\mathrm{Zar}(\mathcal{D})$ the category of sections is opposite to that over the corresponding open subset, DU, of $\mathrm{Zar}(\mathcal{D}^\mathrm{d})$. It can be checked that it is enough to establish that categories of sections are opposite on a basis; that this is so is direct from the duality d, between the (localised) functor categories, which is described near the beginning of this section.

Note (cf. proof of 3.3) that, in the case that we actually have a homeomorphism of spaces (i.e. at the level of points) that will imply that if $DN \in \mathrm{Zar}(\mathcal{D}^{\mathrm{d}})$ corresponds to $N \in \mathrm{Zar}(\mathcal{D})$, then the stalk at DN will be the opposite category to the stalk at N. $\square$

4 Functors between definable categories

interpinduce

Theorem 4.1 *Suppose that $\mathcal{C}$, $\mathcal{D}$ are definable categories and that $I : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor which preserves direct products and direct limits. Then I induces a morphism of locales $\mathcal{O}(\mathrm{Zar}(\mathcal{C})) \longrightarrow \mathcal{O}(\mathrm{Zar}(\mathcal{D}))$, which we also denote by I, and there is a corresponding morphism of presheaves $\mathrm{Def}(\mathcal{D}) \longrightarrow I_*\mathrm{Def}(\mathcal{C})$, where the latter denotes the direct image of the sheaf $\mathrm{Def}(\mathcal{C})$ under I. Thus there is induced a morphism of abelian spaces $(\mathrm{Zar}(\mathcal{C}), \mathrm{LDef}(\mathcal{C})) \to (\mathrm{Zar}(\mathcal{D}), \mathrm{LDef}(\mathcal{D}))$.*

Proof. In general I will not induce a map of sets from $\mathrm{pinj}(\mathcal{C})$ to $\mathrm{pinj}(\mathcal{D})$ since although I preserves pure-injectivity, it need not preserve indecomposability. Nevertheless, if we take $[F]$ with $F \in \mathrm{fun}(\mathcal{D})$ a basic open subset of $\mathrm{Zar}(\mathcal{D})$ then, with notation as in 2.3, $I^{-1}[F] = [I_0F]$ (since $C \in \mathcal{C}$ satisfies $I_0F.C = 0$ iff $F.IC = 0$) and $I_0F \in \mathrm{fun}(\mathcal{C})$ so $[I_0F]$ is a (basic) open subset of $\mathrm{Zar}(\mathcal{C})$. Certainly I^{-1} commutes with finite intersections and arbitrary unions of basic open sets and hence gives a map of algebras of open sets, that is, I induces a map of locales as stated.

It must be shown that I also induces a morphism of presheaves $\mathrm{Def}(\mathcal{D}) \to I_*\mathrm{Def}(\mathcal{C})$ where $I_*\mathrm{Def}(\mathcal{C})$ is defined to take $[F]$ (for $F \in \mathrm{fun}(\mathcal{D})$) to $\mathrm{Def}(\mathcal{C}) \cdot I^{-1}F = \mathrm{Def})(\mathcal{C}) \cdot [I_0F] = \mathrm{fun}(\mathcal{C})/\mathcal{S}(I_0F)$. Now, there is a natural functor from $\mathrm{fun}(\mathcal{D})/\mathcal{S}(F)$ to $\mathrm{fun}(\mathcal{C})/\mathcal{S}(I_0F)$ induced by $I_0 : \mathrm{fun}(\mathcal{D}) \to \mathrm{fun}(\mathcal{C})$. We check that the functors of this form give a morphism of presheaves. If $[F] \supseteq [F']$ then $\mathcal{S}(F) \subseteq \mathcal{S}(F')$ and also, since I_0 is exact from $F \in \mathcal{S}(F')$, we have $I_0F \in \mathcal{S}(I_0F')$, so there is a commutative diagram as shown.

$$\begin{array}{ccc} \mathrm{Def}(\mathcal{D})/\mathcal{S}(F) & \longrightarrow & \mathrm{Def}(\mathcal{C})/\mathcal{S}(I_0F) \\ \downarrow & & \downarrow \\ \mathrm{Def}(\mathcal{D})/\mathcal{S}(F') & \longrightarrow & \mathrm{Def}(\mathcal{C})/\mathcal{S}(I_0F') \end{array} .$$

This morphism of presheaves induces the morphism of sheaves referred to in the statement. $\square$

Example 4.2 *Let $\alpha : R \to S$ be a morphism of rings and consider the induced forgetful functor I : Mod-$S \longrightarrow$ Mod-R. Certainly I commutes with direct products and direct limits (it will be full exactly if α is an epimorphism of rings). We describe the corresponding exact functor $I_0 : \mathrm{fun}\text{-}R \to \mathrm{fun}\text{-}S$.*

If we regard the objects of fun-R *in terms of pp conditions then the description is simple: just replace every occurrence of an element $r \in R$ in a formula*

by its image $\alpha r \in S$. If ϕ is a pp condition for R-modules and we denote by ϕ^α the pp condition for S-modules which results from these replacements then we have $I_0(F_\phi/F_\psi) = F_{\phi^\alpha}/F_{\psi^\alpha}$.

In more algebraic terms the description is as follows. First note that the functor $- \otimes_R S_S$: Mod-$R \longrightarrow$ Mod-S restricts to a functor from mod-R *to* mod-S*: for $- \otimes S$ is right exact so if $A \in$ mod-R has presentation $R^m \to R^n \to A \to 0$ then $S^m \to S^n \to A \otimes S \to 0$ is exact so $A \otimes_R S_S$ is finitely presented. Then, since $- \otimes_R S_S$ is left adjoint to the restriction functor I, we have $(A \otimes S_S, N_S) \simeq (A_R, N_R)$ for all N_S. By definition I_0 is given on $(A, -)$ by $I_0(A,-).N_S = (A,-)IN = (A, N_R)$, so I_0 is given on representables by $I_0(A,-) = (A \otimes_R S_S, -)$. Every functor in* fun-$R$ *has a presentation by representables and I_0 is exact, so this is a complete description of I_0.*

The action on rep-Zariski locales is to take a basic open set $[F]$ of Zar_R *to the basic open set $[I_0F]$ of* Zar_S *(in terms of pp conditions, $[\phi/\psi]$ is taken to $[\phi^\alpha/\psi^\alpha]$).*

And the morphism of presheaves, from Def_R *to the direct image $\alpha_*\mathrm{Def}_S$, induced by α is given at a basic open $[F] = [\phi/\psi]$ by applying I_0 to morphisms in the corresponding localisation of* fun-R *or, in terms of pp conditions, if ρ defines a function on the Ziegler-closed set $[\phi/\psi]$ then the positive atomic diagram of R (expressed through axioms for modules) together with $\phi \leftrightarrow \psi$ proves "ρ is functional" (that is, certain formulas are equivalent), and applying $(-)^\alpha$ to the relevant deduction also gives a valid deduction, so the axioms for S-modules together with $\phi^\alpha \leftrightarrow \psi^\alpha$ also imply that ρ^α is functional.*

If R, S are commutative, so α induces a continuous map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$*, indeed a morphism of ringed spaces* $(\mathrm{Spec}(S), \mathcal{O}_S) \to (\mathrm{Spec}(R), \mathcal{O}_R)$ *then (cf. the following section) this morphism is the restriction of that in 4.1 to the appropriate subspaces and subcategories.*

More generally, we may take a bimodule ${}_SL_R$ with ${}_SL$ finitely presented; then $I = - \otimes_S L_R$: Mod-$S \longrightarrow$ Mod-R is a functor which commutes with direct products and direct limits (in the previous example L is ${}_SS_R$ where the action of R on S is given by α). The corresponding functor I_0 : fun-$R \longrightarrow$ fun-S is such that the action of I_0F on N_S is, in model-theoretic terms, to restrict $N^{\mathrm{eq}+}$ to the sort $N \otimes L$ and then act with R.

Example 4.3 *An example which illustrates why we need to use locales rather than spaces in 4.1 is the following. Let $S = kA_2$ be the path algebra of the quiver $A_2 = \bullet \to \bullet$ over a field k and let R be its subring $k \times k$ (the ring of diagonal matrices if we represent S as a triangular 2×2 matrix ring). Let N_S be the representation $k \xrightarrow{1} k$. Then N is a point of* Zar_S*. The restriction of N to R is the direct sum of two non-isomorphic simple modules so the corresponding subset of* Zar_R *consists of two points (indeed, is the whole of* Zar_R*). Since this set is not even irreducible it is clear that there is no sensible way of assigning a single point of* Zar_R *to be the "image" of N.*

The result 4.1 also applies to the tensor embedding of Mod-R into (R-mod, **Ab**) but we will say more about this in the next section.

5 Restricting to injectives

In this section we assume that the category $\mathcal{C}$ is locally coherent abelian, in particular Grothendieck, so has enough injectives. Also $\mathcal{C}^{\rm fp}$ is abelian. Any such category is definable: if $\mathcal{G}$ is a generating set of finitely presented objects then $\mathcal{C}$ is a definable subcategory of Mod-$\mathcal{G}$ ([36], see [43, 11.1.27, 11.1.21]).

Recall that we defined the Gabriel-Zariski topology on $\mathrm{inj}(\mathcal{C})$ by declaring the sets $[A] = \{E \in \mathrm{inj}(\mathcal{C}) : (A, E) = 0\}$ for $A \in \mathcal{C}^{\rm fp}$ to be open and then we defined a presheaf-on-a-basis by assigning, to a basic open set $[A]$, the localisation of $\mathcal{C}^{\rm fp}$ at the hereditary torsion theory with torsion class generated by A. This is a torsion theory of finite type, so is determined by the set of *indecomposable* torsionfree injective objects (see, e.g., [43, 11.1.29]); therefore the reasoning that showed $\mathrm{Def}(\mathcal{D})$ to be a separated presheaf also applies here and we deduce that this presheaf-on-a-basis embeds in its sheafification, which we denote $\mathrm{LFT}(\mathcal{C})$.

restrtop

Theorem 5.1 *Suppose that $\mathcal{C}$ is a locally coherent abelian category. Then the Gabriel-Zariski topology on* $\mathrm{inj}(\mathcal{C})$ *coincides with the restriction of the rep-Zariski topology on* $\mathrm{pinj}(\mathcal{C})$ *to* $\mathrm{inj}(\mathcal{C})$*. Furthermore, there is an induced full embedding* $\big(\mathrm{LFT}(\mathcal{C})\big)^{\rm op} \to \mathrm{LDef}(\mathcal{C}) \restriction \mathrm{inj}(\mathcal{C})$*, of categoried spaces over* $\mathrm{inj}(\mathcal{C})$*.*

Proof. A basic Gabriel-Zariski open subset of $\mathrm{inj}(\mathcal{C})$ has the form $[A]$ for some $A \in \mathcal{C}^{\rm fp}$. Since A is finitely presented it follows that so is the functor $(A, -)$ and clearly $[(A, -)] \cap \mathrm{inj}_R = [A]$, giving us one inclusion of topologies.

For the other, let $F \in \mathrm{fun}(\mathcal{D})$ and let $g : B \to C$ in $\mathcal{C}^{\rm fp}$ be such that $(C, -) \xrightarrow{(g,-)} (B, -) \to F \to 0$ is a projective presentation of F. Factorise g as $B \xrightarrow{g'} B' \xrightarrow{g''} C$, yielding the factorisation $(C, -) \xrightarrow{(g'',-)} (B', -) \xrightarrow{(g',-)} (B, -)$ of $(g, -)$. Observe that since $\mathcal{C}$ is locally coherent, $B' \in \mathcal{C}^{\rm fp}$. Also note that the restriction of $(g'', -)$ to injective objects of $\mathcal{C}$ is an epimorphism and so $\mathrm{im}(g, -) = \mathrm{im}(g', -)$ and hence, for any $E \in \mathrm{inj}(\mathcal{C})$, the resulting sequence $0 \to (B', E) \to (B, E) \to FE \to 0$ is exact. But also, if $0 \to K \to B \to B' \to 0$ is exact (and note that $K \in \mathcal{C}^{\rm fp}$) then if E is injective the sequence $0 \to (B', E) \to (B, E) \to (K, E) \to 0$ is exact. Therefore $F \simeq (K, -)$ and $[F] \cap \mathrm{inj}(\mathcal{C}) = [K]$, as required.

For the second statement it is enough to compare the corresponding presheaves-on-a-basis. The section of $\mathrm{FT}(\mathcal{C})$ over $[A]$ is the quotient category $\mathcal{C}^{\rm fp}/\langle A\rangle$ and the section of $\mathrm{Def}(\mathcal{C}) \restriction \mathrm{inj}(\mathcal{C})$ over $[A] = [(A, -)] \cap \mathrm{inj}(\mathcal{C})$ is $\mathrm{fun}(\mathcal{C})/\langle (A, -)\rangle$ where, in each case, $\langle X\rangle$ denotes the Serre subcategory generated by X in the given category.

We have the Yoneda embedding of $(\mathcal{C}^{\rm fp})^{\rm op}$ into $\mathrm{fun}(\mathcal{C})$ and the composition of this with $\mathrm{fun}(\mathcal{C}) \to \mathrm{fun}(\mathcal{C})/\langle (A, -)\rangle$ clearly takes A to 0, hence induces a

morphism $\left(\mathcal{C}^{\rm fp}/\langle A\rangle\right)^{\rm op} \to {\rm fun}(\mathcal{C})/\langle (A,-)\rangle$. This will be an embedding provided the intersection of the Serre subcategory $\langle (A,-)\rangle$ with the image of $(\mathcal{C}^{\rm fp})^{\rm op}$ in ${\rm fun}(\mathcal{C})$ is no more than the image of $\langle A\rangle$; we show that this is so.

Suppose then that $C \in \mathcal{C}^{\rm fp}$ is such that $(C,-) \in \langle (A,-)\rangle$. It is easily seen (see the background references) that the duality between ${\rm fun}(\mathcal{D})$ and ${\rm fun}^{\rm d}(\mathcal{D})$ takes $(A,-)$ to $A\otimes -$. Also, if $F, G \in {\rm fun}(\mathcal{D})$ are such that $F \in \langle G\rangle$ then, because the closure conditions for a Serre subcategory are "self-dual", $dF \in \langle dG\rangle$. It follows that $(C\otimes -) \in \langle A\otimes -\rangle$. Let $E \in [A]$, that is $(A,E) = 0$, hence $(A\otimes -,\, E\otimes -) = 0$ and then it follows that $(C\otimes -,\, E\otimes -) = 0$. Therefore $(C,E) = 0$. This is true for each $E \in [A]$ so C belongs to the torsion class of the smallest finite type torsion theory on $\mathcal{C}$ generated by A. The intersection of that torsion class with $\mathcal{C}^{\rm fp}$ is exactly $\langle A\rangle$, as required. $\Box$

This is proved in [47, 2.4.2] for the case where $\mathcal{C} = $ Mod-R for R a right coherent ring; the proof there is considerably longer but does give explicitly the isomorphisms between definable scalars and elements of localisations of R.

In the above sense, then, the map which takes a definable category $\mathcal{D}$ (equivalently a small abelian category $\mathcal{R} = {\rm fun}(\mathcal{D})$) to the abelian space $({\rm Zar}(\mathcal{D}), {\rm LDef}(\mathcal{D}))$ extends the classical situation which takes a commutative coherent ring R to the affine variety $({\rm Spec}(R), \mathcal{O}_{{\rm Spec}(R)})$. For, given a commutative ring R, we assign to it the definable category Mod-R, equivalently the (opposite of the) free abelian category, ${\rm Ab}(R)$ on R, and, from that we obtain the corresponding abelian space, a subsheaf of which is isomorphic to $({\rm Spec}(R), \mathcal{O}_{{\rm Spec}(R)})$. To obtain this isomorphism, we identify a prime P of R with the (indecomposable) injective module $E(R/P)$ and use the fact that for a commutative coherent ring R, every point of the space ${\rm inj}_R$ is, in the Gabriel-Zariski topology, topologically indistinguishable from a point of ${\rm Spec}(R)$ ([41, 6.4]). Note also that R is recoverable from Mod-R, as the **centre** (the endomorphism ring of the identity functor) of this category. We observe that R may be recovered in the same way from ${\rm Ab}(R)$.

centrfreeab

Proposition 5.2 *Let R be any ring. Then the canonical morphism from the centre, $C(R)$, of R to the free abelian category,* ${\rm Ab}(R)$*, of R is an isomorphism. In particular a commutative ring R may be recovered from* ${\rm Ab}(R)$*.*

Proof. By the **centre** of a category is meant the set (ring if the category is additive) of natural transformations from the identity functor id to itself. Such a natural transformation τ is given by, for each object F of the category, an endomorphism τ_F of F such that for every morphism $f : F \to G$ of the category we have the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\tau_F} & F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ G & \xrightarrow[\tau_G]{} & G \end{array} .$$

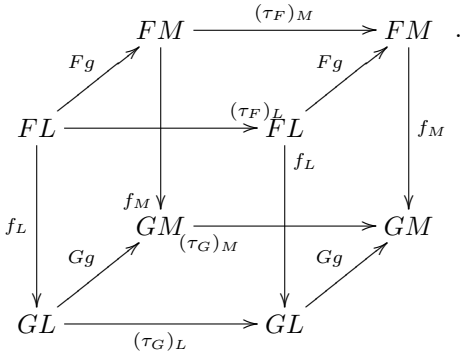
We identify $\mathrm{Ab}(R)$ with the category $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ of finitely presented functors from (left) R-modules to $\mathbf{Ab}$.

Given $r \in C(R)$, define the element τ_r of the centre of $\mathrm{Ab}(R)$ by setting $(\tau_r)_F$, for $F \in \mathrm{Ab}(R)$, to be multiplication by r. That is, the natural transformation $(\tau_r)_F$ has component at $M \in R$-mod the map $F((r \times -)_M)$ where $(r \times -)_M : M \to M$ is the endomorphism (since $r \in C(R)$) $m \mapsto rm$: $((\tau_r)_F)_M = F((r \times -)_M)$. If $f : F \to G$ is a morphism in $\mathrm{Ab}(R)$ then each component $f_M : FM \to GM$ is R-linear so it follows that τ_r is indeed a natural transformation. This gives a map $R \to \mathrm{Nat}(\mathrm{id}, \mathrm{id})$ which is clearly a ring homomorphism and which, on considering the component of τ_r at the forgetful functor $(R, -)$, evaluated say at R, is clearly monic.

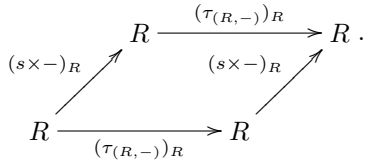
For the converse suppose that $\tau \in \mathrm{Nat}(\mathrm{id}, \mathrm{id})$. Then for every morphism $f : F \to G$ of $\mathrm{Ab}(R)$ and every $M \in R$-mod there is a commutative diagram

$$\begin{array}{ccc} FM & \xrightarrow{(\tau_F)_M} & FM \\ {\scriptstyle f_M}\downarrow & & \downarrow{\scriptstyle f_M} \\ GM & \xrightarrow[(\tau_G)_M]{} & GM \end{array} .$$

Indeed for every morphism $g : L \to M$ in R-mod there is a commutative diagram



Apply this with $F = (R, -)$, $M = L = R$, g being multiplication by any $s \in R$, so $Fg : FL \to FM$ is $(s \times -)_R : R \to R$, to obtain



That is, $(\tau_{(R,-)})_R$ commutes with multiplication by every element $s \in R$ and hence is multiplication by some $r \in C(R)$.

This identification of $\tau_{(R,-)})_R$ with multiplication by r must be extended, first to every component of $\tau_{(R,-)}$, then to every component of τ.

Let $M \in R$-mod, say $g : R^n \to M$ is surjective. Apply $(R, -)$ to obtain the commutative diagram below (where g means the image of g under the forgetful

functor and where we are making the (functorial) identification of (R, X) with X)

$$\begin{array}{ccc} (R^n = (R,R)^n & \xrightarrow{g} & M = (R,M) \\ \downarrow{\scriptstyle \tau_{(R,-)})^n_R = \tau_{(R,-)})_{R^n}} & & \downarrow{\scriptstyle (\tau_{(R,-)})_M} \\ R^n = (R,R)^n & \xrightarrow{g} & M = (R,M) \end{array} \quad .$$

Thus (choose $m \in M$, choose a preimage in R^n and follow it round) $(\tau_{(R,-)})_M : M = (R,M) \to M = (R,M)$ is also multiplication by r. So $\tau_{(R,-)}$ is multiplication by r.

Now choose $M \in R$-mod and a surjection $R^n \to M$, hence an injection $i : (M,-) \to (R,-)^n$. Therefore we have the commutative diagram

$$\begin{array}{ccc} (M,-) & \xrightarrow{\tau_{(M,-)}} & (M,-) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ (R,-)^n & \xrightarrow[\tau_{(R,-)^n}]{} & (R,-)^n \end{array}$$

and hence, at each $N \in R$-mod, the commutative diagram

$$\begin{array}{ccc} (M,N) & \xrightarrow{(\tau_{(M,-)})_N} & (M,N) \\ \downarrow{\scriptstyle i_N} & & \downarrow{\scriptstyle i_N} \\ (R,N)^n & \xrightarrow[(\tau_{(R,-)^n})_N]{} & (R,N)^n \end{array}$$

where the lower map is, by what has been proved already, just multiplication by r on N. Thus the effect of $(\tau_{(M,-)})_N$ is the restriction to (M,N) of multiplication by r on $(R,N) = N^n$ and hence is just multiplication by r.

So now we have that each $\tau_{(M,-)}$ is multiplication by r. A general object $F \in \mathrm{Ab}(R)$ is a homomorphic image of some representable functor $\pi : (M,-) \to F$ so we have the commutative diagram

$$\begin{array}{ccc} (M,-) & \xrightarrow{\pi} & F \\ \downarrow{\scriptstyle \tau_{(M,-)}} & & \downarrow{\scriptstyle \tau_F} \\ (M,-) & \xrightarrow[\pi]{} & F \end{array}$$

which at $N \in R$-mod gives the commutative diagram

$$\begin{array}{ccc} (M,N) & \xrightarrow{\pi_N} & FN \\ \downarrow{\scriptstyle (\tau_{(M,-)})_N} & & \downarrow{\scriptstyle (\tau_F)_N} \\ (M,N) & \xrightarrow[\pi_N]{} & FN \end{array}$$

where the left-hand map is, by what has been proved, multiplication by r and it follows easily that $(\tau_F)_N$ also is multiplication by r.

Thus $\tau = \tau_r$ and so the isomorphism between the centre of R and the centre

of $\mathrm{Ab}(R)$ is established. □

6 The image of $\mathrm{Ab}(R)$ in Mod-R

Let $\mathcal{R}$ be a small preadditive category. The free abelian category on $\mathcal{R}$ is a functor $\mathcal{R} \longrightarrow \mathrm{Ab}(\mathcal{R})$ where $\mathrm{Ab}(\mathcal{R})$ is abelian such that every functor from $\mathcal{R}$ to an abelian category $\mathcal{B}$ factors through this functor *via* a unique (up to natural equivalence) exact functor from $\mathrm{Ab}(\mathcal{R})$ to $\mathcal{B}$. In the next result, existence is due to Freyd ([13, 4.1]) and the description is stated by Gruson in [16].

Theorem 6.1 *Given a small preadditive category $\mathcal{R}$ the free abelian category on $\mathcal{R}$ exists and is equivalent to the functor category $(\mathcal{R}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} = \text{fun-}\mathcal{R}^{\mathrm{op}} \simeq \text{fun}^{\mathrm{d}}\text{-}\mathcal{R}$.*

In particular, the functor from a ring R, regarded as a preadditive category with one point, to Mod-R which takes R to the free module R_R, factors through the free abelian category $\mathrm{Ab}(R) = (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ *via* the functor from $\mathrm{Ab}(R)$ which takes $F \in \mathrm{Ab}(R)$ to $FR \in \mathrm{Mod}$-R (for this functor, evaluation at ${}_RR$, certainly is exact, hence must be the required exact factorisation). The kernel of this functor from $\mathrm{Ab}(R)$ is $\mathcal{Z}_R = \{F : F({}_RR) = 0\}$ and the image, $\mathrm{Ab}(R)/\mathcal{Z}_R$, let us denote it $\mathcal{A}(R)$, is an abelian subcategory of Mod-R, in particular the inclusion functor from $\mathcal{A}(R)$ to Mod-R is exact. In general $\mathcal{A}(R)$ will not be a full subcategory of Mod-R.

Example 6.2 *Let k be a field and let $R = k[x_i\,(i \in \omega) : x_i x_j = 0\ \forall i, j\}$. This is a commutative non-coherent ring (the, 1-dimensional, ideal generated by x_0 is finitely generated but not finitely presented: the kernel of a surjection $R \to x_0R = x_0k$ is $J = \bigoplus_{i\in\omega} x_ik$ which is infinitely generated).*

The inclusion of J in R is in $\mathcal{A}(R)$ because J is defined by the pp condition $vx_0 = 0$ for instance (if ϕ is a pp condition then it defines a finitely presented functor F_ϕ, thus $J = F_{vx_0=0}(R)$).

The ideal J is a semisimple module of countably infinite rank so it has uncountably many endomorphisms and, at least if the field k is countable, these cannot each be the value of a pp-definable map at R (if k, hence R is countable then $\mathrm{Ab}(R)$ has only countably many objects and morphisms). Thus $\mathcal{A}(R)$ is not in general a full subcategory of Mod-R.

Lemma 6.3 *If R is right coherent then $\mathcal{A}(R) =$ mod-R. For any ring R, $\mathcal{A}(R)$ is the smallest abelian subcategory of* Mod-R *containing* mod-R.

Proof. In general if $f : M \to N$ is a morphism in mod-R then $(f \otimes -) : (M \otimes -) \to (N \otimes -)$ is a morphism in $\mathbf{Ab}(R)$ (that M finitely presented implies $M \otimes -$ finitely presented is [3, 6.1]). Evaluation at ${}_RR$ gives f, as well as M and N, in the image of ev_R, so mod-R is a subcategory of $\mathcal{A}(R)$.

If R is right coherent then mod-R is abelian so, in the definition of $\mathcal{A}(R)$ above, we may replace Mod-R by mod-R to get an exact functor $E' : \mathrm{Ab}(R) \to$ mod-R which, on composition with the inclusion of mod-R in Mod-R, must be equivalent to evaluation, ev_R, at R. So, in this case, we may take $\mathcal{A}(R)$ to be mod-R for, as we have seen, all of mod-R is in the image of ev_R.

If $\mathcal{A}$ is any abelian (not necessarily full) subcategory of Mod-R containing R_R, and hence mod-R, then this argument shows that $\mathcal{A}$ contains $\mathcal{A}(R)$ □

If R is not right coherent then $\mathcal{A}(R)$ strictly contains mod-R (since in this case mod-R is not abelian).

From now on we make free use of pp conditions and surrounding technology, see, for instance, either of [43], [37] or, for a short account, [45].

imfreeab

Theorem 6.4 *A right R-module K is isomorphic to an object of $\mathcal{A}(R)$ iff K is the kernel of a morphism between finitely presented modules.*

Proof. If K is such a kernel then, since $\mathcal{A}(R)$ is an exact subcategory of Mod-R, it must be that K is in $\mathcal{A}(R)$. We will, however, give a direct proof which exhibits explicitly (modulo the morphism being given explicitly) a finitely presented functor $F \in \mathbf{Ab}(R)$ such that $K \simeq F({}_R R)$.

($\Rightarrow$) Suppose that $f : M \to N$ is a morphism in mod-R. Suppose that $\bar{a} = (a_1, \ldots, a_n)$ is a generating set for M and that the columns of the matrix G generate the kernel of the corresponding surjection $R^n \to M$, so $(M, \bar{a})$ is a free realisation (see [43, §1.2.2])) of the pp condition $\bar{x}G = 0$, which we denote as $\theta(\bar{x})$. Similarly let $\bar{b} = (b_1, \ldots, b_m)$ generate N with matrix of relations H and denote the condition $\bar{y}H = 0$ by $\eta(\bar{y})$. Note that the kernel of the surjection $R^n \to M$ is $D\theta({}_R R)$ where D denotes elementary dual (defined in §7); for $(M, \bar{a})$ is a free realisation of θ and so in $M \simeq M \otimes_R R_R$ we have $\bar{a} \otimes \bar{r} = 0$ iff $\bar{r} \in D\theta({}_R R)$; similarly the kernel of the corresponding surjection $R^m \to N$ is $D\eta({}_R R)$.

Also let S be the matrix such that $f\bar{a} = \bar{b}S$. Then $\bar{a}\bar{r} \in \ker(f)$ ($\bar{r} \in R^n$) iff $\bar{b}S\bar{r} = 0$ iff $S\bar{r} \in D\eta({}_R R)$ iff $\bar{r} \in (D\eta : S)$. Here $(D\eta : S)$ denotes the pp condition $D\eta(\bar{r}S)$. It follows that $K = \ker(f) \simeq F({}_R R)$ where F is the functor $F_{(D\eta:S)}/F_{D\theta}$ and hence $K \in \mathcal{A}(R)$.

($\Leftarrow$) Suppose that $K = F_{D\psi}/F_{D\phi}$ for some pp conditions $\psi \leq \phi$ ($\leq (R_R, -)^n$) on right modules. We may suppose that ϕ is quantifier-free (since dK is a factor of a finitely presented projective functor, that is, a representable functor). Let $(C_\phi, \bar{c}_\phi)$ and $(C_\psi, \bar{c}_\psi)$ be free realisations of ϕ and ψ respectively. Since ϕ is quantifier-free we may suppose that $\bar{c}_\phi$ generates C_ϕ. Since ψ implies ϕ there is a morphism, $g : C_\phi \to C_\psi$ with $g\bar{c}_\phi = \bar{c}_\psi$. Consider the morphism $f : R^n \to C_\phi$ which takes a chosen basis of R^n to $\bar{c}_\phi$. By basic properties of free realisations and Herzog's Criterion ([20, 3.2]) $\ker(f) = D\phi({}_R R)$ and also $\ker(gf) = D\psi({}_R R)$. Since f is surjective it follows that $K \simeq \ker(g)$: a morphism between finitely presented modules, as required. □

Proposition 6.5 *Let I be a submodule of R^n_R. Then the inclusion $I \to R^n$ is in $\mathcal{A}(R)$ iff $I = \phi({}_RR)$ for some pp condition ϕ.*

Proof. If the inclusion is in $\mathcal{A}(R)$ then it is of the form $f_R : F'R \to FR$ for some functors F', F and natural transformation f. Since $FR \simeq R^n$, that is (by Yoneda) $((R,-),F) \simeq R^n$, there is a natural transformation $g : (R,-) \to F$ the component of which at R is an isomorphism, so we may as well take f_R to be an identification $\mathrm{id} : R^n \to FR = R^n$. Let F'' with $f' : F'' \to (R,-)$ and $g' : F'' \to F'$ be the pullback of f and g. Then the evaluation at R is a pullback and hence $f'_R : F''R \to R^n$ may be identified with the inclusion of I into R^n, as required.

The other direction is immediate from the definition of $\mathcal{A}(R)$. □

7 Simplified bases on injectives

Now we will investigate the Gabriel-Zariski topology on $\mathrm{inj}_{\mathcal{R}}$. If R is right coherent then, since $\mathcal{A}(R) = \text{mod-}R$, it coincides with that which has, for a basis of open sets, those of the form $[K] = \{E \in \mathrm{inj}_{\mathcal{R}} : (K,E) = 0\}$ with $K \in \mathcal{A}(\mathcal{R})$. In general, however, these two topologies - the Gabriel-Zariski topology and that defined by the $[K] \in \mathcal{A}(\mathcal{R})$ may differ, see the example of Puninski at [15, p. 402].

In this section we take R to be a ring but few changes would be required if it were a small preadditive catgory. We recall that every functor in $\mathrm{Ab}(R)$ has the form F_ϕ/F_ψ for some pp conditions with $\psi \leq \phi \leq (R,-)^n$ for some n. Every finitely generated (hence finitely presented since $\mathrm{Ab}(R)$ is locally coherent) subfunctor of $(R,-)^n$ has the form F_ϕ for some pp ϕ and we denote by $D\phi$ the pp condition such that the inclusion of $F_{D\phi}$ in $({}_RR,-)$ is the kernel of $d(F_\phi \to (R_R,-)^n) = ({}_RR,-)^n \to dF_\phi$. For each n, D is a duality between the lattice of finitely generated subfunctors of $(R_R,-)^n$ and those of $({}_RR,-)^n$. There is, moreover, an explicit recipe for computing $D\phi$ from ϕ (see for instance [43, §1.3.1]).

Since every finitely presented functor F has the form F_ϕ/F_ψ, an alternative form of the standard basis for the rep-Zariski topology is $[\phi/\psi] = \{N \in \mathrm{pinj}_R : \phi(N) = \psi(N)\}$ as $\phi > \psi$ ranges over pairs of pp conditions. Thus the restriction of the rep-Zariski topology to inj_R has a basis of open sets of the form $[\phi/\psi] \cap \mathrm{inj}_R$. We show that over right coherent rings this simplifies: that basic open sets of the form $[\phi]$ (that is $[\phi(\bar{x})/\bar{x} = 0]$) suffice. We will actually phrase things in terms of the Ziegler topology, which has for a basis of open sets the complementary sets $(\phi/\psi) = \{N \in \mathrm{pinj}_R : \phi(N) > \psi(N)\}$. We need the following result.

Proposition 7.1 *[46, 1.3] Let E_R be an absolutely pure (for instance an injective) module and let ϕ be any pp condition. Then $\phi(E) = \mathrm{ann}_E D\phi({}_RR)$.*

If ϕ has more than one free variable then annihilation for tuples of the same length is interpreted by $\bar{a}\bar{r} = 0$ meaning $\sum a_i r_i = 0$.

Corollary 7.2 *Let $E \in \text{inj}_R$ and let ϕ be a pp condition. Then $(\phi) \cap \text{inj}_R = \{E \in \text{inj}_R : \text{ann}_E D\phi({}_R R) \neq 0\} = \{E \in \text{inj}_R : (R/D\phi({}_R R), E) \neq 0\}$.*

Proof. We simplify by using the fact that, to have a basis, it is enough to take ϕ and ψ to be conditions with one free variable (i.e. to take subfunctors of $(R, -)$ rather than general $(R, -)^n$). The statement does, however, hold in general in the form given.

The first equality is immediate. For the second note that any pp-definable subgroup of ${}_R R$ is a right ideal, so the condition makes sense and then note that, if $f : R \to E$ is non-zero and has kernel containing $D\phi(R)$ then the image of 1 will be a non-zero element with annihilator containing $D\phi(R)$, hence by the proposition will be an element of $\phi(E)$ and, conversely, any element which annihilates $D\phi(R)$ will induce such a morphism. □

Theorem 7.3 *Let R be any ring and let $\phi \geq \psi$ be pp conditions (for right modules). Then $(\phi/\psi) \cap \text{inj}_R$ is a union of sets of the form $(\phi') \cap \text{inj}_R$, more precisely equals $\bigcup \{(\phi_r) \cap \text{inj}_R : r \in D\psi({}_R R) \setminus D\phi({}_R R)\}$ where $\phi_r(\bar{x})$ is the condition $\exists \bar{u}\, (\phi(\bar{u}) \wedge \bar{x} = \bar{u}r)$ (hence which is the functor $M \mapsto \phi(M) \cdot r$).*

Proof. By the previous result we have $(\phi/\psi) \cap \text{inj}_R = \{E : \text{ann}_E D\phi({}_R R) > \text{ann}_E D\psi({}_R R)\}$ (the inclusion $\psi \leq \phi$ gives $D\phi \leq D\psi$ hence the inclusion $\text{ann}_M D\phi({}_R R) \geq \text{ann}_M D\psi({}_R R)$ for any right module M).

Suppose that $E \in (\phi/\psi) \cap \text{inj}_R$ and choose $a \in \text{ann}_E D\phi({}_R R) \setminus \text{ann}_E D\psi({}_R R)$ and then choose $r \in \text{ann}_E D\psi(R)$ such that $ar \neq 0$. If $s \in (D\phi(R) : r) = \{t : tr \in D\phi(R)\}$ then we have $ar.s = a.rs = 0$ and so $ar \in \text{ann}_E(D\phi(R) : r)$. Then note that $(D\phi(R) : r)$ is a pp-definable subgroup of ${}_R R$, namely it is definable by the pp condition $D\phi(ry)$ and hence has the form $D\phi_r({}_R R)$ for some pp condition ϕ_r, the exact form of which we check at the end of this proof.

For the converse, suppose that E is such that $\text{ann}_E(D\phi(R) : r) \neq 0$ for some $r \in \psi({}_R R) \setminus \phi({}_R R)$, say $a \in E$ is non-zero, annihilates $D\phi(R)$ but $ar \neq 0$. Let $I = \text{ann}_R(a)$. Then $arR \simeq (rR + I)/I$, which is a homomorphic image of $(rR + D\phi(R))/D\phi(R)$. Since $aR \leq E$, which is injective, that isomorphism extends to a morphism, f say, from R/I to E. Then if $a' = f(1 + I)$ we have $a'I = 0$ so $a'D\phi(R) = 0$ and $a'r = f(r) = ar \neq 0$. Thus $a' \in \text{ann}_E D\phi({}_R R) \setminus \text{ann}_E D\psi({}_R R)$ and $E \in (\phi/\psi) \cap \text{inj}_R$, as required.

That proves the first statement and, to get the second part, we just need to compute $D\phi_r({}_R R)$. Say $\phi(\bar{x})$ (we drop the simplifying assumption that there is just one free variable) is the condition $\exists \bar{y}\, (\bar{x}H = \bar{y}K)$, that is, $\begin{pmatrix} \bar{x} & \bar{y} \end{pmatrix} \begin{pmatrix} H \\ -K \end{pmatrix} = 0$ for some matrices H, K with entries in R. The recipe for elementary duality D

gives that $D\phi(\bar{x})$ is the condition $\exists \bar{z} \begin{pmatrix} I & H \\ 0 & -K \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = 0$ (I an identity matrix of appropriate size) so $D\phi(r\bar{z})$ is the condition $\exists \bar{z} \begin{pmatrix} rI & H \\ 0 & -K \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = 0$. Then the dual of this, that is, what we have denoted ϕ_r, is the pp condition $\exists \bar{y}\bar{u} \begin{pmatrix} \bar{x} & \bar{u} & \bar{y} \end{pmatrix} \begin{pmatrix} I & 0 \\ rI & H \\ 0 & -K \end{pmatrix} = 0$, that is $\exists \bar{y}\bar{u} \, (\bar{x} = \bar{u}r \wedge \bar{u}H = \bar{y}K)$ which is $\exists \bar{u} \, (\phi(\bar{u}) \wedge \bar{x} = \bar{u}r)$, as stated. □

If R is right coherent then inj_R is a closed subset of pinj_R (which is compact) so, since each basic open set (ϕ/ψ) is compact (this is true over any ring), so is each relatively open set $(\phi/\psi) \cap \mathrm{inj}_R$. Therefore, for right coherent rings, the union given in the theorem reduces to a finite one: this is elimination of imaginaries for injectives over right coherent rings. It also shows that in this case the Gabriel-Zariski topology on inj_R has a basis of open sets of the form $[\phi]$.

Corollary 7.4 *For any ring R the Ziegler topology on inj_R has a basis of open sets of the form $(R/I) = \{E \in \mathrm{inj}_R : (R/I, E) \neq 0\}$ where I ranges over the right ideals of the form $\eta({}_R R)$ where η is a pp condition for left R-modules, that is over right ideals I such that the inclusion of I into R is in the category $\mathcal{A}(R)$.*

Proposition 7.5 *For any ring R the Ziegler topology on inj_R has a basis of open sets of the form $(K) = \{E \in \mathrm{inj}_R : (K, E) \neq 0\}$ where K ranges over objects of the category $\mathcal{A}(R)$. Indeed if $K = FR$ with $F \in \mathrm{Ab}(R)$ then $(K) = (dF) \cap \mathrm{inj}_R$. If R is right coherent then the sets $[K] = \{E \in \mathrm{inj}_R : (K, E) = 0\}$ form a basis of open subsets of the Gabriel-Zariski topology on inj_R.*

Proof. Say $K = F_{D\psi}/F_{D\phi}(R)$ where $\psi \leq \phi \leq (R_R, -)^n$ in $\mathrm{Ab}(R^{\mathrm{op}})$. If $E \in \mathrm{inj}_R$ and if f is a non-zero morphism from K to E then, by injectivity, there is a non-zero morphism f', extending f, from $R^n/D\phi(R)$ to E. Then $a = f'1 \in \mathrm{ann}_E D\phi(R) = \phi(E)$ and if also $a \in \mathrm{ann}_E D\psi(R)$ then the kernel of f' would contain $D\psi(R)$, contradicting that f' extends f (and that $f \neq 0$). Conversely, if $E \in (\phi/\psi)$ then $a \in \mathrm{ann}_E D\phi(R) \setminus \mathrm{ann}_E D\psi(R)$ gives that the map $R \to E$ taking 1 to a factors through $R/D\phi(R)$ and not through $D\psi(R)$. Thus there is a non-zero morphism from K to E. Thus $(K) = (\phi/\psi) \cap \mathrm{inj}_R$. □

For general rings, however, the Gabriel-Zariski topology will have open sets of the form $[\phi/\psi]$ and such might be an infinite intersection of sets of the form $[K]$. That is, as mentioned at the beginning of this section, the Gabriel-Zariski topology might be finer than that defined by $\mathcal{A}(R)$.

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