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Abstract

We give a complete characterization of Priest's Finite Inconsistent Arithmetics observing that his original putative characterization included arithmetics which cannot in fact be realized.

Introduction

In [2] Priest investigates finite models of true arithmetic based, not on classical logic, where of course there can be no finite models, but on the paraconsistent logic LP standing for 'logic of paradox' (see [3], [4]). In this paper he aims to give a complete characterization of all such models. However he includes there some models (the 'clique models' on pages 232-233 of [2]) which cannot in fact be realized. Our purpose in this note is to tidy up the characterization and make some few comments and evident generalizations.

We shall borrow heavily from [2] and Priest's earlier papers. An LP interpretation (or structure) for a language $\mathcal{L}$ is a pair $\langle D, I \rangle$, where D is a non empty set and I assigns denotations to the non-logical symbols of the language in the following way.

- For any constant symbol c, $I(c)$ is a member of D
- For every n-ary function symbol f, $I(f)$ is an n-ary function on D.
- For every n-ary predicate symbol P, $I(P)$ is the pair $\langle I^+(P), I^-(P) \rangle$ where $I^+(P)$ and $I^-(P)$ are respectively the extension and anti-extension of P.

We furthermore require that equality really is equality, or more formally,

$$I^+(=) = \{\, \langle x, x\rangle \mid x \in D\},$$

and that for every n-ary predicate P, $I^+(P) \cup I^-(P) = D^n$.

We do not however require that $I^+(P) \cap I^-(P) = \emptyset$, if we did of course LP structures would just be classical structures.

For a term $t(\vec{x})$, a formula $\theta(\vec{x})$ of $\mathcal{L}$, an LP structure $\mathcal{A} = \langle D, I\rangle$ and an assignment v from the free variables of the language into D we define $t^{\mathcal{A},v}(\vec{x})$ and $\mathcal{A}, v \vDash \theta(\vec{x})$ inductively as follows:

- If $t(\vec{x}) = c$ then $t^{\mathcal{A},v}(\vec{x}) = I(c)$, if $t(\vec{x}) = x$ then $t^{\mathcal{A},v}(\vec{x}) = v(x)$.
- If $t(\vec{x}) = f(t_1(\vec{x}), \ldots, t_m(\vec{x}))$ then $t^{\mathcal{A},v}(\vec{x}) = I(f)(t_1^{\mathcal{A},v}(\vec{x}), \ldots, t_m^{\mathcal{A},v}(\vec{x}))$.
- For an n-ary predicate symbol P,

$$\begin{aligned} \mathcal{A}, v \vDash P(t_1(\vec{x}), \ldots, t_n(\vec{x})) &\iff \langle t_1^{\mathcal{A},v}(\vec{x}), \ldots, t_n^{\mathcal{A},v}(\vec{x})\rangle \in I^+(P), \\ \mathcal{A}, v \vDash \neg P(t_1(\vec{x}), \ldots, t_n(\vec{x})) &\iff \langle t_1^{\mathcal{A},v}(\vec{x}), \ldots, t_n^{\mathcal{A},v}(\vec{x})\rangle \in I^-(P). \end{aligned}$$

- For formulae $\theta_1(\vec{x}), \theta_2(\vec{x})$ of $\mathcal{L}$,

$$\begin{aligned} \mathcal{A}, v \vDash \neg\neg\theta_1(\vec{x}) &\iff \mathcal{A}, v \vDash \theta_1(\vec{x}), \\ \mathcal{A}, v \vDash \theta_1(\vec{x}) \wedge \theta_2(\vec{x}) &\iff \mathcal{A}, v \vDash \theta_1(\vec{x}) \text{ and } \mathcal{A}, v \vDash \theta_2(\vec{x}), \\ \mathcal{A}, v \vDash \neg(\theta_1(\vec{x}) \wedge \theta_2(\vec{x})) &\iff \mathcal{A}, v \vDash \neg\theta_1(\vec{x}) \text{ or } \mathcal{A}, v \vDash \neg\theta_2(\vec{x}), \\ \mathcal{A}, v \vDash \theta_1(\vec{x}) \vee \theta_2(\vec{x}) &\iff \mathcal{A}, v \vDash \theta_1(\vec{x}) \text{ or } \mathcal{A}, v \vDash \theta_2(\vec{x}), \\ \mathcal{A}, v \vDash \neg(\theta_1(\vec{x}) \vee \theta_2(\vec{x})) &\iff \mathcal{A}, v \vDash \neg\theta_1(\vec{x}) \text{ and } \mathcal{A}, v \vDash \neg\theta_2(\vec{x}), \\ \mathcal{A}, v \vDash \theta_1(\vec{x}) \to \theta_2(\vec{x}) &\iff \mathcal{A}, v \vDash \neg\theta_1(\vec{x}) \text{ or } \mathcal{A}, v \vDash \theta_2(\vec{x}), \\ \mathcal{A}, v \vDash \neg(\theta_1(\vec{x}) \to \theta_2(\vec{x})) &\iff \mathcal{A}, v \vDash \theta_1(\vec{x}) \text{ and } \mathcal{A}, v \vDash \neg\theta_2(\vec{x}). \end{aligned}$$

- For a formula $\theta(y, \vec{x})$,

$$\begin{aligned} \mathcal{A}, v \vDash \exists y\, \theta(y, \vec{x}) &\iff \text{for some } a \in D,\ \mathcal{A}, v' \vDash \theta(y, \vec{x}), \\ \mathcal{A}, v \vDash \neg\exists y\, \theta(y, \vec{x}) &\iff \text{for all } a \in D,\ \mathcal{A}, v' \vDash \neg\theta(y, \vec{x}), \\ \mathcal{A}, v \vDash \forall y\, \theta(y, \vec{x}) &\iff \text{for all } a \in D,\ \mathcal{A}, v' \vDash \theta(y, \vec{x}), \\ \mathcal{A}, v \vDash \neg\forall y\, \theta(y, \vec{x}) &\iff \text{for some } a \in D,\ \mathcal{A}, v' \vDash \neg\theta(y, \vec{x}), \end{aligned}$$

  where v' agrees with v on all variables except possibly y when $v'(y) = a$.

As usual we shall occasionally write $\mathcal{A} \vDash \theta(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in D$, in place of $\mathcal{A}, v \vDash \theta(x_1, \ldots, x_n)$, where v is some (equivalently any) assignment such that $v(x_i) = a_i$ for $i = 1, \ldots, n$.

We say that $\mathcal{A}$ is an LP model of a set of sentences T if for all $\theta \in T$, $\mathcal{A} \vDash \theta$ (as usual the choice of v does not matter here).

Henceforth we shall restrict ourselves to the language $\mathcal{LA}$ of arithmetic. Let T be a complete theory (in the classical sense) in $\mathcal{LA}$ extending Peano's Axioms, $\mathcal{PA}$[1]. In what follows we appear to need to work with a complete theory because we do not necessarily have soundness for classical entailment. In particular Modus Ponens is no longer sound because we can have $\mathcal{A}, v \vDash \theta$ and $\mathcal{A}, v \vDash (\theta \rightarrow \phi)$ without having $\mathcal{A}, v \vDash \phi$ (because we also had $\mathcal{A}, v \vDash \neg\theta$ and it was this that justified $\mathcal{A}, v \vDash (\theta \rightarrow \phi)$).

In [2] Priest gives a method (which is much more general than its application here) for constructing finite LP models of T. Namely, let M be a classical, non-standard, model of T and $\sim$ a congruence relation[2] on M with only finitely many equivalence classes. Now define $D_\sim$ to be the set of equivalence classes, say $[a]$ is the equivalence class containing $a \in M$, and define $I_\sim(0) = [0]$, $I_\sim(')([a]) = [a']$, $I_\sim(+)([a],[b]) = [a+b]$, $I_\sim(\times)([a],[b]) = [ab]$ and

$$\begin{aligned} I^+_\sim(=) &= \{\, \langle [a],[b] \rangle \mid a \sim b \,\}, \\ I^-_\sim(=) &= \{\, \langle [a],[b] \rangle \mid a \neq b \,\}. \end{aligned}$$

Then $\langle D_\sim, I_\sim \rangle$ is an LP model of T.

In particular if we take $p_0, p_1, \ldots, p_m \in \mathbb{N}$, with $p_1 > 0$, $p_0 > 0$ or $m = 1$,[3] $p_i | p_j$ for $i \geq j > 0$, and $C_1, \ldots, C_m$ increasing proper cuts in M (so closed under successor, addition and multiplication) with $C_m = M$, and define for $a, b \in M$,

$$a \sim b \iff \begin{cases} a = b < p_0 \text{ or } p_0 \leq a, b \in C_i - C_{i-1} \\ \text{for some } i \quad (\text{take } C_0 = \emptyset) \text{ and } a = b \bmod p_i, \end{cases}$$

then $\sim$ is a congruence relation and the resulting *Finite Linear LP* model, $\mathcal{A}_\sim$ has universe

$$\begin{aligned} 0, 0^{(1)}, 0^{(2)}, \ldots, 0^{(p_0-1)},\ & b_1, b_1^{(1)}, \ldots, b_1^{(p_1-1)},\ b_2, b_2^{(1)}, \ldots, b_2^{(p_2-1)}, \\ & b_3, b_3^{(1)}, \ldots, b_3^{(p_3-1)}, \ldots, b_m, b_m^{(1)}, \ldots, b_m^{(p_m-1)}, \end{aligned}$$

for some $b_1, \ldots, b_m$ where $a^{(i)}$ is the i-th successor of a according to $I_\sim(')$, and successor, and (commutative) addition and multiplication are as follows.

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[1] Priest restricts T to being the theory of true arithmetic but that will not be necessary for our purposes.

[2] I.e. $\sim$ is an equivalence relation and satisfies that if $a_1 \sim a_2$, $b_1 \sim b_2$ then $a_1' \sim a_2'$, $a_1 + b_1 \sim a_2 + b_2$ and $a_1 b_1 \sim a_2 b_2$.

[3] Unfortunately this condition was omitted from the original published version of this paper. The necessity of this follows because if $p_0 = 0$ and $m > 0$ then, in the notation of that paper, $b_1 = 0$ so

$$b_1 = 0 = b_2 b_1 = b_2 b_1^{(p_1)} = b_2 b_1 + p_1 b_2 = p_1 b_2 = b_2,$$

contradicting the non-equivalence of b_1, b_2.

- Successor is the next term in the above sequence except that $I_\sim(')(b_1^{(p_1-1)}) = b_1,\ I_\sim(')(b_2^{(p_2-1)}) = b_2,\ I_\sim(')(b_3^{(p_3-1)}) = b_3, \ldots, I_\sim(')(b_m^{(p_m-1)}) = b_m$.

- $0^{(i)} + 0^{(j)} = 0^{(i+j)}$ if $i+j < p_0$, otherwise $0^{(i)} + 0^{(j)} = b_1^{(i+j-p_0)}$.

- $0^{(i)} + b_k^{(j)} = b_h^{(i)} + b_k^{(j)} = b_k^{(i+j)}$ for $h \leq k$.

- $0^{(i)} \times b_k^{(j)} = b_h^{(i)} \times b_k^{(j)} = b_k^{(ij)}$ for $h \leq k$.

Finally $I_\sim^+(=)$ is just equality whilst

$$I_\sim^-(=) = D_\sim \times D_\sim - \{\, \langle 0^{(i)}, 0^{(i)} \rangle \mid i = 0, 1, \ldots, p_0 - 1 \,\}.$$

By Priest's Extension Lemma (see [2]) we also obtain an LP model of T if we enlarge this $I_\sim^-(=)$ to any superset of $D_\sim \times D_\sim - \{\, \langle 0^{(i)}, 0^{(i)} \rangle \mid i = 0, 1, \ldots, p_0 - 1 \,\}$ whilst keeping everything else the same. In the next section we shall show that these Linear Plus (as we shall call them) LP models are the only finite LP models of T.

This corrects an error in [2] where, on pages 232-233, Priest claims the existence of a further family of finite LP models of arithmetic, the 'clique models', which again are to be formed by collapsing according to a congruence relation, $\approx$. Unfortunately the proof of the theorem on page 232 stating that this relation $\approx$ is a congruence relation is incorrect[4] and as our forthcoming analysis will show there are in fact no such finite LP models beyond the linear plus models.

The Structure of Finite LP Models of T

Let $\mathcal{A} = \langle D, I \rangle$ be a finite LP model of T. Since $\mathcal{PA} \subseteq T$ all the consequences of $\mathcal{PA}$ hold in $\mathcal{A}$, in particular addition and multiplication are commutative, associative, 0 is an additive identity, $a = a$ and either $a = b$ or $\neg(a = b)$ for all $a, b \in D$ etc. In what follows we shall largely assume these without further mention.

Following Priest consider the elements $0, 0^{(1)}, 0^{(2)}, \ldots$. Since D is finite, for some least p_0 $0^{(p_0)}$ must appear again in this list, say it appears for the second time as $0^{(p_0+p_1)}$, $p_1 > 0$. Then because $I(')$ is a function, $0^{(p_0+i)} = 0^{(p_0+j)}$ whenever $i = j \bmod p_1$. Indeed this goes both ways since suppose that $0^{(p_0+i)} = 0^{(p_0+j)}$ and $0 < i < j < p_1$ (clearly the case $i = 0$ is impossible by choice of p_1). Then $0^{(p_0+i+p_1-j)} = 0^{(p_0+j+p_1-j)} = 0^{(p_0)}$, contradicting the choice of p_1.

[4] In the notation of that proof take $C_1 < C_2 < C_3$ and $a \in C_1, b \in C_2, c \in C_3$ with $a \approx c$. Then $a + b \in C_2$ and $c + b \in C_3$ so $a + b \not\approx c + b$ and $\approx$ cannot be a congruence relation. For additional background to this and the results in this present paper see [1].

Since for $n, m \in \mathbb{N}$ the following are in T, they are true in $\mathcal{A}$,

$$\begin{aligned} x^{(n)} &= x + 0^{(n)}, \\ x + 0 &= x, \\ x^{(n)} + y^{(m)} &= (x+y)^{(n+m)}, \\ x0 &= 0, \\ x^{(n)} y^{(m)} &= x^{(n)} y + x0^{(m)}. \end{aligned} \tag{1}$$

It is now straightforward to show that as far as successor, addition and multiplication on

$$\{ 0, 0^{(1)}, 0^{(2)}, \ldots 0^{(p_0-1)}, 0^{(p_0)}, \ldots, 0^{(p_1+p_0-1)} \}$$

are concerned the picture is as in the linear LP model of the last section. Also if $r \geq p_0$ then $\mathcal{A} \vDash 0^{(r)} \neq 0^{(r)}$ since

$$T \vDash \forall x \, x^{(p_1)} \neq x \tag{2}$$

so

$$\mathcal{A} \vDash 0^{(r)} \neq 0^{(r+p_1)}$$

whilst in fact $0^{(r+p_1)} = 0^{(r-p_0+p_0+p_1)} = 0^{(r-p_0+p_0)} = 0^{(r)}$. Thus on these elements at least $I^+(=)$ and $I^-(=)$ have the required form for a Linear Plus LP model.

For $a, b \in D$ set $a \leq b$ if

$$\mathcal{A} \vDash \exists y \; a + y = b.$$

This ordering is reflexive and transitive, since if $\mathcal{A} \vDash a + e = b$ and $\mathcal{A} \vDash b + f = c$ then $a + e = b$, i.e. $a + e$ and b really are the same thing, etc. so $a + (e + f) = (a + e) + f = c$, giving $a \leq c$. Let $\equiv$ be the equivalence relation on D defined by

$$a \equiv b \iff a \leq b \text{ and } b \leq a.$$

Since for $a \in D$, $a + 0' = a'$, $a \leq a'$ and hence $a \leq a^{(n)}$ for all $n \in \mathbb{N}$. Furthermore if $a \leq b$, say $a + c = b$ then $a' + c = b'$ so $a^{(n)} \leq b^{(n)}$ for $n \in \mathbb{N}$. From these observations it follows that the $0^{(p_0)}, \ldots, 0^{(p_1+p_0-1)}$ are all equivalent.

We now investigate further the equivalence classes of these initial elements. Suppose $p_0 > 0$ and $c \neq 0$, $c \leq 0$, so $c \equiv 0$ since certainly $0 \leq c$. Say $b + c = 0$. Since

$$\mathcal{A} \vDash \forall x \, (x = 0 \vee \exists y \; y' = x) \tag{3}$$

we must have $c = d^{(1)}$ for some $d \in D$, so

$$b + d + 0^{(1)} = b + d^{(1)} = b + c = 0.$$

Thus

$$0^{(p_0-1)} = b + d + 0^{(p_0)} = b + d + 0^{(p_1)} = 0^{(p_1-1)}$$

contradicting the choice of p_0. We conclude that if $p_0 > 0$ then no such c can exist and $[0] = \{0\}$. Exactly similarly if $p_0 > j$ there can be no $c \leq 0^{(j)}$ different from each of that $0, 0^{(1)}, \ldots, 0^{(j)}$, otherwise there would exist an a such that $0^{(j)} = a^{(j+1)} = a + 0^{(j+1)}$ so $p_0 - 1 \geq j$ and

$$0^{(p_0-1)} = a + 0^{(p_0)} = a + 0^{(p_1)} = 0^{(p_1-1)},$$

again contradicting the choice of p_0.

We conclude that $[0^{(j)}] = \{0^{(j)}\}$ for $j < p_0$.

Having sorted out the initial part of $\mathcal{A}$ let the equivalence classes with respect to $\equiv$ be

$$\{0\}, \{0^{(1)}\}, \{0^{(2)}\}, \ldots, \{0^{(p_0-1)}\}, [b_1], [b_2], \ldots, [b_m]$$

where $b_1 \leq b_2 \leq b_3 \leq \ldots \leq b_m$. Notice that since these are distinct equivalence classes $b_i \not\leq b_j$ for $j < i \leq m$.

We now show by induction on j that these $[b_j]$ are closed under successor and addition and multiplication. Starting with successor, since $b_j \neq 0$, $b_j = c'$ for some c, which we may assume is not $0^{(p_0-1)}$, otherwise replace c by $0^{(p_0+p_1-1)}$. Since $c \leq b_j$ and the union of the earlier equivalence classes $[b_1], [b_2], \ldots, [b_{j-1}]$ is closed under successor it must be that $c \equiv b_j$. Hence $b_j = c' \equiv b_j'$, as required.

To show that $[b_j]$ is closed under addition let b be such that $b + b = b_j$ or $b + b + 1 = b_j$, we know that some such b must exist since

$$\mathcal{A} \vDash \forall x \, \exists y \, (y + y = x \vee y + y + 1 = x).$$

If $b \in \{\, 0, 0^{(1)}. \ldots, 0^{(p_0-1)} \,\}$ we can replace it by $b^{(p_1(p_0+1))}$ and other b cannot be in an earlier equivalence class $[b_i]$ since their union is closed under addition and successor. Also b cannot be in a higher equivalence class since $b \leq b_j$. So we may assume that $b \in [b_j]$. But then $b_j \leq b$ so $b_j + c = b$ for some c and

$$b_j \leq b_j + b_j \leq b_j + b_j + c + c = b + b \leq b_j,$$

as required. Notice of course that $[b_j]$ is also closed under addition with an element of an earlier equivalence class.

To show that $[b_j]$ is also closed under multiplication let b be such that, with the obvious shorthand k for $0^{(k)}$,

$$b^2 \leq b_j \leq (b+1)^2 = b^2 + 2b + 1,$$

again we know that such a b must exist since

$$\mathcal{A} \vDash \forall x \, \exists y, w, v \, (y^2 + u = x \wedge x + w = (y+1)^2). \tag{4}$$

Since $b \leq b^2$ and we can, as above, assume that b is not in any the earlier equivalence classes this again leads to $b \in [b_j]$. Therefore $b_j + c = b$ for some c and

$$b_j \leq b_j^2 \leq (b_j + c)^2 = b^2 \leq b_j.$$

This shows that $[b_j]$ is closed under multiplication (and also with non-zero elements of earlier equivalence classes).

We now turn to investigating these classes $[b_j]$ more fully. Let $1 \leq j \leq m$ and $c, d \in [b_j]$. Then since D is finite $c, c^{(1)}, c^{(2)}, \ldots$ cannot be all different, say k is least such that for some $s > k$, $c^{(k)} = c^{(s)}$. Then because $c^{(k)} \equiv d$, for some a, $d = a + c^{(k)}$. Hence

$$d^{(s-k)} = (a + c^{(k)})^{(s-k)} = a + c^{(s)} = a + c^{(k)} = d.$$

Since $d \in [b_j]$ was arbitrary it follows that for some $p_j \leq s - k$, and all $d \in [b_j]$, $d^{(p_j)} = d$ and

$$d, d^{(1)}, d^{(2)}, \ldots, d^{(p_j - 1)}$$

are all distinct. [Notice this agrees with the notation p_1 already introduced.]

Again for any $c, d \in [b_j]$, $cd = cd^{(p_j)} = cd + p_j c$. Since $d \equiv cd$ there is some a such that $a + cd = d$, hence

$$d = a + cd = a + cd + p_j c = d + p_j c.$$

It follows that $p_j c$ is an element x of $[b_j]$ such that $d + x = d$ for all $d \in [b_j]$. Indeed such an element must be unique since if we had two such, say x_1, x_2, then $x_1 = x_1 + x_2 = x_2 + x_1 = x_2$. We may assume that b_j is chosen to be this element. Then since

$$\mathcal{A} \vDash \forall x \, \exists y \, (x = p_j y \vee x = p_j y^{(1)} \vee x = p_j y^{(2)} \vee \ldots \vee x = p_j y^{(p_j - 1)}) \qquad (5)$$

every $d \in [b_j]$ must be of the form $p_j c^{(s)}$ for some c and $0 \leq s < p_j$. Since this c must be in $[b_j]$ (because these classes are closed under addition) this gives that $d = b_j^{(s)}$. In other words

$$[b_j] = \{ \, b_j, b_j^{(1)}, b_j^{(2)}, \ldots, b_j^{(p_j - 1)} \, \}. \qquad (6)$$

In particular some $[b_j]$ must equal $\{ \, 0^{(p_0)}, 0^{(p_0+1)}, 0^{(p_0+2)}, \ldots, 0^{(p_0+p_1-1)} \, \}$, indeed this must be $[b_1]$ since $0 \leq b_j$ so $0^{(p_0)} \leq b_j^{(p_0)} \equiv b_j$. Furthermore, as remarked by Priest in [2], if $1 \leq i \leq j \leq m$ then $b_j + b_i \in [b_j]$ and

$$b_j + b_i = (b_j + b_i)^{(p_i)} = b_j + b_i^{(p_i)} = b_j + b_i$$

so $p_j | p_i$.

Finally, to show that this LP model is Linear Plus it only remains to check the successor, addition and multiplication are of the required form. But using (1) that is now clear from the representation of the $[b_j]$ in (6) and the fact that, by our choice, $b_j + b_j = b_j$ for $j = 1, 2, \ldots, m$.

Concluding Remarks

One hope in investigating finite LP models of arithmetic is that it might somehow lead to independence results for Peano's Axioms. The above conclusions however appear to dash any such hopes. The resulting finite LP models would have been the same even if we had started with *any* theory containing the schemata (1), (2), (3), (4) (5), and the commutativity, associativity and distributivity of addition and multiplication. In particular since any sentence which is consistent with these schemata will hold in all these LP models, they really tell us nothing about the key axiom schema of induction.

In this paper we have concentrated on finite LP models and used the finiteness in an apparently non-trivial way to show the existence of p_0, p_1. In [5] Priest considers also infinite models and it is not clear to what extent they are now also restricted by our results. Indeed it remains an open question whether every countable (say) LP model of a complete extension T of PA arises by taking equivalence classes of a classical model of T with respect to some congruence relation.

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