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2003

MIMS EPrint: **2005.39**

ISSN 1749-9097

Published in *Topology* **42** (2003), 833–844.

OPENNESS OF MOMENTUM MAPS AND PERSISTENCE OF EXTREMAL RELATIVE EQUILIBRIA

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Abstract. We prove that for every proper Hamiltonian action of a Lie group G in finite dimensions the momentum map is locally G-open relative to its image (i.e. images of G-invariant open sets are open). As an application we deduce that in a Hamiltonian system with continuous Hamiltonian symmetries, extremal relative equilibria persist for every perturbation of the value of the momentum map, provided the isotropy subgroup of this value is compact. We also demonstrate how this persistence result applies to an example of ellipsoidal figures of rotating fluid. We also provide an example with plane point vortices which shows how the compactness assumption is related to persistence.

Mathematics Subject Classification: 53D20, 37J15

1. Introduction

In a Hamiltonian system, nondegenerate equilibria are isolated; in particular, they do not *persist* from one energy level to nearby levels. In this paper we prove that in a symmetric Hamiltonian system, every extremal relative equilibrium persists to nearby levels of the momentum map, provided the isotropy subgroup of its momentum value is compact. The crucial ingredient in the proof is a generalisation of Reyer Sjamaar's result on the openness of momentum maps.

Let M be a symplectic manifold with a proper and symplectic action of a connected Lie group G and let h be a G-invariant Hamiltonian. We suppose that the action of G is Hamiltonian, in that it is infinitesimally generated by a momentum map $\Phi : M \to \mathfrak{g}^*$. A trajectory of the Hamiltonian vector field X_h of h is a *relative equilibrium* if its image in the orbit space M/G is a single point; such a relative equilibrium with momentum value μ is *extremal* if its image in the reduced space $\Phi^{-1}(\mu)/G_\mu$ is a local extremum for the reduced Hamiltonian. G_μ denotes the isotropy subgroup of μ for the possibly *modified coadjoint action* of G on $\mathfrak{g}^*$ (Section 2).

Let $f : X \to Y$ be a G-equivariant map. We say that f is *G-open* if the image of any G-invariant open set is open in Y. This is equivalent to the orbit map $\bar{f} : X/G \to Y/G$ being open.

Suppose that γ is an extremal relative equilibrium with momentum value μ. We wish to prove that under certain hypotheses such a relative equilibrium persistes to *all* nearby values of the momentum map. However, since this is really a local result in the phase space M, the question arises as to what is meant by all nearby values. If the momentum map Φ is proper then a result of Sjamaar [20] (see also [7]) says that the momentum map is G-open, so that images of G-invariant neighbourhoods of γ are open in the image of Φ, and the phrase 'all nearby values' means just that: a full G-invariant neighbourhood of μ in $\Phi(M)$. On the other hand, if Φ is not proper, then it may not be G-open (for an example, see [5, Example 3.10]), so the image of a G-invariant neighbourhood of γ may not be open in $\Phi(M)$. Our first result shows that there is always a G-invariant neighbourhood U_0 of γ restricted to which

The research of JM was partially supported by the European Union through the Research Training Network MASIE.

the momentum map $U_0 \to \Phi(U_0)$ *is* G-open relative to its image. This neighbourhood U_0 is a tubular neighbourhood of the group orbit containing γ whose existence follows from the Marle-Guillemin-Sternberg normal form for symplectic group actions (Section 3). Of course, if Φ is proper one can take $U_0 = M$.

Theorem 1. *Let M be a symplectic manifold with a proper Hamiltonian action of a connected Lie group G and a momentum map $\Phi : M \to \mathfrak{g}^*$. Suppose that $x \in M$ has momentum value $\mu = \Phi(x)$ whose isotropy subgroup G_μ for the modified coadjoint action is compact. Then there exists a G-invariant neighbourhood U_0 of x such that the restriction $\Phi\big|_{U_0} : U_0 \to \Phi(U_0)$ is G-open, where $\Phi(U_0)$ is given the subspace topology induced from $\mathfrak{g}^*$.*

This result then allows us to state the persistence theorem for extremal relative equilibria.

Theorem 2. *Let M, G, Φ, x, μ and U_0 be as in Theorem 1, let $h \in C^\infty(M)$ be a G-invariant Hamiltonian, and suppose that γ is an extremal relative equilibrium for the given Hamiltonian system, with $x \in \gamma$. Then there exists a G-invariant neighbourhood V of μ in $\Phi(U_0)$ such that for every $\mu' \in V$, there is a relative equilibrium in $\Phi^{-1}(\mu') \cap U_0$.*

Remarks. (a) When the G-action is trivial, $\Phi^{-1}(\mu') = M$, so the theorem becomes trivial, too. Theorems in Hamiltonian systems often have natural generalisations to those in symmetric Hamiltonian systems, when a group action is thrown in. Theorem 2 is an instance of a theorem in the latter that has no nontrivial specialisation in the former.

(b) In [13], it was shown that extremal relative equilibria are Lyapunov-stable relative to G. Also in [13] appeared a version of persistence, but the proof is incomplete. The present version is stronger, first because it does not require G to be compact, but just G_μ, which actually suffices to reduce to the compact case (Section 3), and second because it proves persistence to a full neighbourhood of μ. To our knowledge, Theorem 2 is the first application of this topological property of the G-openness of momentum maps to problems of Hamiltonian dynamical systems.

(c) It is natural to ask whether the hypothesis of extremality is necessary. The answer is affermative, an example of a non-extremal relative equilibrium which does not persist is given in [13] (Example 1.1). If the action is free, then a non-degeneracy hypothesis is sufficient for persistence (see [13], applications in [14] and extensions in [16]). George Patrick and Mark Roberts [17, 18] discuss the structure of the set of relative equilibria from a different point of view (not using the momentum value as a parameter).

(d) The proof in fact shows that the perturbed relative equilibria are also extremal, though possibly not isolated—see also Remarks 1.3(b) in [13].

After its proof in Section 5, Theorem 2 is applied in Section 6 to the problem of ellipsoidal figures of rotating fluid (affine rigid bodies). In Section 7 we check that the compactness hypothesis on the isotropy subgroup G_μ is essential in Theorem 2 by analysing point vortices on the plane. Finally, in Section 8 we explain how reduction by stages yields a partial persistence result even in the case of noncompact momentum isotropy. For complementary results on persistence of relative equilibria for noncompact group actions, see Wulff [24].

2. Modification of coadjoint action

Theorem 1 does not assume the equivariance of the momentum map Φ with respect to the standard coadjoint action. However, Souriau [22] showed that the momentum map can always be made equivariant by modifying the coadjoint action, as follows.

Let a Lie group G act in a Hamiltonian manner on a connected symplectic manifold M with a momentum map $\Phi : M \to \mathfrak{g}^*$. Define the cocycle $\theta : G \to \mathfrak{g}^*$ by

$$\theta(g) = \Phi(g \cdot x) - \mathrm{Coad}_g(\Phi(x))$$

(which is independent of the choice of $x \in M$); $\mathrm{Coad}_g = \mathrm{Ad}^*_{g^{-1}}$ is our notation for the standard coadjoint action of $g \in G$ on $\mathfrak{g}^*$. The *modified coadjoint action* is

$$\mathrm{Coad}^\theta_g(\mu) = \mathrm{Coad}_g(\mu) + \theta(g), \qquad \mu \in \mathfrak{g}^*.$$

With respect to this shifted affine action Φ becomes equivariant. All the usual properties of standard coadjoint action continue to hold for the modified actions [22]: for example the momentum map is Poisson for a suitably modified Poisson structure on $\mathfrak{g}^*$, and the symplectic leaves of the modified Poisson structure are the modified coadjoint orbits.

Throughout this paper the reduced space at $\mu \in \mathfrak{g}^*$ is understood to be $\Phi^{-1}(\mu)/G_\mu$, where G_μ is the isotropy subgroup of μ for the modified coadjoint action.

3. Reduction to actions of compact groups

Theorem 1 does not assume the compactness of the symmetry group G, but only the compactness of the isotropy subgroup G_μ. The reduction to compact group actions is based on the Marle-Guillemin-Sternberg normal form for symplectic actions and momentum maps, which we now recall [10, 4]. Let again a connected Lie group G act in a Hamiltonian manner on a symplectic manifold M with a momentum map $\Phi : M \to \mathfrak{g}^*$ and the corresponding cocycle $\theta : G \to \mathfrak{g}^*$ (Section 2). At $x \in M$, consider the four spaces

$$\begin{aligned} T_0 &= T_x(G\cdot x) \cap \ker d\Phi(x) = T_x(G_\mu \cdot x), \\ T_1 &= T_x(G\cdot x)/T_0, \\ N_1 &= \ker d\Phi(x)/T_0, \\ N_0 &= T_xM/(\,T_x(G\cdot x) + \ker d\Phi(x)). \end{aligned}$$

Since $\ker d\Phi_x$ is the symplectic complement to $T_x(G\cdot x)$, these spaces depend only on the G-action and not on the choice of Φ. Using the compactness of $G_x \subset G_\mu$, we can realise the quotients T_1, N_1, N_0 as G_x-invariant subspaces of T_xM satisfying

$$T_0 \oplus T_1 = T_x(G\cdot x), \quad T_0 \oplus N_1 = \ker d\Phi(x), \quad T_0 \oplus T_1 \oplus N_1 \oplus N_0 = T_xM$$

(the so-called Witt or Moncrief decomposition). N_1 is the *symplectic slice* to the action at x. With respect to such a decomposition, the symplectic form ω has the matrix

$$[\omega]_x = \begin{bmatrix} 0 & 0 & 0 & A \\ 0 & \omega_{T_1} & 0 & * \\ 0 & 0 & \omega_{N_1} & * \\ -A^t & * & * & * \end{bmatrix}, \tag{3.1}$$

where ω_{T_1} and ω_{N_1} are the restrictions of ω to T_1 and N_1, A is nondegenerate, and the $*$'s are of no interest. The Marle-Guillemin-Sternberg normal form theorem states that in a G-invariant neighbourhood U of x, the symplectic G-action is isomorphic to that on $G \times_{G_x} (\mathfrak{m}^* \times N_1)$, where $\mathfrak{m}^* = \mathfrak{g}_x^\circ \cap \mathfrak{g}_\mu^*$, so that $\mathfrak{g}_\mu \simeq \mathfrak{m} \oplus \mathfrak{g}_x$ and $\mathfrak{g}_\mu^* \simeq \mathfrak{m}^* \oplus \mathfrak{g}_x^*$. ($\mathfrak{g}_x^\circ \cap \mathfrak{g}_\mu^*$ is the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}_\mu^*$.) The momentum map has the explicit form

$$\begin{array}{rcl} \Phi : G \times_{G_x} (\mathfrak{m}^* \times N_1) & \longrightarrow & \mathfrak{g}^* \\ {[g,\, \nu,\, v]} & \longmapsto & \mathrm{Coad}^\theta_g(\mu + (\nu \oplus \Phi_{G_x}(v)). \end{array} \tag{3.2}$$

Now we reduce the problem to the case where the whole group G is compact. (This part of the argument is similar to the beginning of Section 2 in [9].) Since the isotropy subgroup G_μ of μ is compact, we can choose a momentum map so that the restriction of θ to G_μ vanishes (essentially by averaging [13]), so that for $g \in G_\mu$ we have $\Phi(g\cdot x) = \mathrm{Coad}_g \Phi(x)$. There is therefore an inner product on $\mathfrak{g}^*$, invariant under $\mathrm{Coad}(G_\mu)$, inducing a G_μ-equivariant splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{h}$. Then a small enough G_μ-invariant neighbourhood B of μ in the affine plane $\mu + \mathfrak{h}^\circ$ is transverse to the momentum map. ($\mathfrak{h}^\circ$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$). Hence

$R := \Phi^{-1}(B)$ is a G_μ-invariant submanifold of M containing the given relative equilibrium γ.

We claim that in some neighbourhood of γ, R is a *symplectic* submanifold of M. Should the momentum map be equivariant already with respect to the standard coadjoint action, this is a consequence of the symplectic cross-section theorem of Guillemin and Sternberg (cf. [3], Corollary 2.3.6). In general, we resort to the Witt-Moncrief decomposition described above. As R is complementary to T_1 by construction, the restriction of ω to R is obtained by eliminating the second row and the second column of $[\omega]_x$ in (3.1). The resulting matrix is nondegenerate, hence R is symplectic in a neighbourhood of γ, as claimed.

The action of G_μ on R is Hamiltonian, and its momentum map is the restriction of Φ to R followed by the natural projection $\mathfrak{g}^* \to \mathfrak{g}_\mu^*$. Since $\mathfrak{g}^* \to \mathfrak{g}_\mu^*$ restricted to $\mu + \mathfrak{h}^\circ$ is an isomorphism, the restriction $\Phi|_R$ is a momentum map for the action of G_μ up to this isomorphism. It follows that

$$\ker d\Phi(y) = \ker d(\Phi|_R)(y) \qquad \forall y \in R.$$

Moreover, because h is G-invariant, the flow of X_h preserves the fibres of the momentum map, and so the flow preserves R. It follows that

$$(X_h)|_R = X_{(h|_R)}.$$

Another question that requires attending to is whether Φ inherits openness from $\Phi|_R$. The answer is affermative in view of

Lemma 3. *Let K be a closed subgroup of a Lie group G and H be a closed subgroup of K. Let A be an H-space and B a K-space, and let $f : A \to B$ be an H-equivariant map. Then the map*

$$\begin{aligned} F : G \times_H A &\longrightarrow G \times_K B \\ ([g,a]_H) &\mapsto [g, f(a)]_K \end{aligned}$$

is well-defined and G-equivariant. Furthermore, if f is H-open, then F is G-open.

Proof. The only nontrivial conclusion is the G-openness of F. The diagram below commutes:

$$\begin{array}{ccc} G \times A & \xrightarrow{\mathrm{id}\times f} & G \times B \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ G \times_H A & \xrightarrow{F} & G \times_K B. \end{array}$$

Here π_1 and π_2 are the orbit (quotient) maps, open by definition of the topology on orbit spaces. Let $U \subset G \times_H A$ be G-invariant and open. Then $\pi_1^{-1}(U) = G \times U'$, with U' open and H-invariant in A. Then $F(U)$ is open, since $F(U) = \pi_2 \circ (\mathrm{id} \times f)(\pi_1^{-1}(U)) = \pi_2(G \times f(U'))$. $\square$

Thus, we have found a Hamiltonian sub-system $(R, G_\mu, \Phi|_R, h|_R)$ for which the symmetry group G_μ is compact. Passing to this sub-system, we may and shall assume without loss of generality that G is compact and $G = G_\mu$.

4. Openness of momentum maps

In this Section we establish Theorem 1. By the result of Section 3, we may focus our attention on $\Phi : M \to \mathfrak{g}^*$, a momentum map for a Hamiltonian action of a *compact* Lie group. Sjamaar [20] proved that if Φ is proper then it is G-open relative to its image: that is, if $U \subset M$ is a G-invariant open subset, then $\Phi(U)$ is open in $\Phi(M)$, where $\Phi(M)$ is given the subspace topology induced from $\mathfrak{g}^*$. In this section we deduce from Sjamaar's theorem that Φ is *locally G-open* even when it is not proper.

Besides Lemma 3 of Section 3, we need two more lemmas.

Lemma 4. *Let Δ_1 be a compact convex polytope in $\mathbb{R}^N$, and for $r > 0$ let*

$$\Delta_{<r} = \{a\delta \mid 0 \leqslant a < r,\ \delta \in \Delta_1\}.$$

Then $\Delta_{<r}$ is an open subset of $\Delta = \Delta_{<\infty}$, where the latter has the topology induced from $\mathbb{R}^N$.

Proof. There are two cases to examine depending on whether or not $0 \in \Delta_1$. Only one ($0 \in \Delta_1$) is needed for the proof of Theorem 1, but we include the other for completeness.

Case $0 \in \Delta_1$. Let Δ_1^j be the open faces of the polytope Δ_1, and let

$$d = \min_j \{\mathrm{dist}(0, \Delta_1^j) \mid 0 \notin \overline{\Delta_1^j}\}.$$

Note that $d > 0$. Then $S_d \cap \Delta_1 = S_d \cap \Delta$, where S_d is the sphere in $\mathbb{R}^N$ of radius d.

Let $\{x_n\}$ be any sequence in Δ converging to 0, and suppose that all $x_n \neq 0$. Write $y_n = dx_n/\|x_n\|$. Then $y_n \in S_d \cap \Delta$ and so $y_n \in \Delta_1$. Therefore $x_n = (\|x_n\|/d)\, y_n \in \Delta_{<r}$ provided n is sufficiently large for $\|x_n\| < rd$ to hold.

If $x_n \to x \neq 0$, we can write $x_n = a_n\delta_n$ and $x = a\delta$, with $\delta, \delta_n \in S_d \cap \Delta_1$. Then $a_n \to a$ and $\delta_n \to \delta$. Rescaling δ and δ_n if necessary the result ensues.

Case $0 \notin \Delta_1$. Let $x \in \Delta_{<r}$, so that $x = a\delta$ with $0 \leqslant a < r$ and $\delta \in \Delta_1$. Let $\{x_n\}$ be a sequence in Δ converging to x. Write $x_n = a_n\delta_n$, with $\delta_n \in \Delta_1$. If $x = 0$, then $a_n \to 0$ as δ_n is bounded away from 0, so that $x_n \in \Delta_{<r}$ for n sufficiently large. If on the other hand $x \neq 0$, then both sequences $\{a_n\}$ and $\{\delta_n\}$ are bounded and bounded away from 0 for sufficiently large n. By rescaling δ_n if necessary we can arrange for a_n to converge to a, and so $x_n \in \Delta_{<r}$ for n sufficiently large. □

Lemma 5. *Let V be a symplectic representation of a compact Lie group G, and let $\Phi : V \to \mathfrak{h}^*$ be the homogeneous quadratic momentum map. Then Φ is G-open relative to its image.*

Proof. Take a G-invariant Hermitian metric whose imaginary part is the given symplectic structure on V. G acts as unitary transformations on V seen as a complex vector space.

Consider the unit sphere S in V with respect to the real part of the Hermitian metric, and the symplectic action of the circle group $\mathrm{U}(1)$. As $\mathrm{U}(1)$ is the centre of the unitary group, the actions of G and $\mathrm{U}(1)$ commute. The G-action descends to the symplectic manifold $\mathbb{P}V := S/\mathrm{U}(1)$ (diffeomorphic to $\mathbb{C}P^{\dim V/2-1}$). Denote its momentum map by $\Phi_1 : \mathbb{P}V \to \mathfrak{g}^*$. Since the actions of G and $\mathrm{U}(1)$ commute, Φ_1 can be chosen so that $\Phi_1(\mathrm{U}(1)\cdot x) = \Phi(x)$.

Now let U be a G-invariant open subset of V. We want to show that $\Phi(U)$ is open in $\Phi(V)$, and to do so we examine two cases separately: (i) $0 \notin U$ and (ii) $U = B(0,\varepsilon)$; indeed a general open set containing the origin is the union of sets of these types.

Case (i): We exploit a basis for the topology of $V\setminus\{0\} \simeq S\times\mathbb{R}^+$ (diffeomorphic) consisting of 'product sets'. Thus let $U = U_1 \times (a,b) \subset S \times \mathbb{R}^+$, where U_1 is a G-invariant open subset of S. Then by the homogeneity of Φ,

$$\Phi(U) = \{r^2\nu \mid r \in (a,b),\ \nu \in \Phi(U_1)\}.$$

By Sjamaar's theorem [20, 7], $\Phi(U_1)$ is open in $\Phi_1(\mathbb{P}V) = \Phi(S)$, and it follows that $\Phi(U)$ is open in $\Phi(V)$. To see this, let $\mu \in \Phi(U)$, and let $\{\mu_n\}$ be a sequence in $\Phi(V\setminus\{0\})$ converging to μ; we may suppose $\mu_n \neq 0$ (otherwise it is trivial). Then $\mu = \Phi(v,r) = r^2\Phi_1(v)$ for some $(v,r) \in S \times (a,b)$. Since $\mu/r^2 \in \Phi_1(\mathbb{P}V)$, there is a sequence $r_n \to r$ satisfying, $\mu_n/r_n^2 \in \Phi_1(\mathbb{P}V)$ for all n. Φ_1 being open by Sjamaar's theorem, there is a sequence

(v_n) in S converging to v such that $\mu_n/r_n^2 = \Phi_1(v_n)$; in other words $\mu_n = \Phi(v_n, r_n)$, and $(v_n, r_n) \to (v, r)$. Consequently $\Phi(U)$ is open, as required.

Case (ii): Let $U = B(0, \varepsilon)$, the open ball in V with centre 0 and radius ε. Because G is compact, we can and do identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and the adjoint action with the coadjoint action. Let $\mathfrak{t}^+$ be a positive Weyl chamber in $\mathfrak{g} = \mathfrak{g}^*$, and let

$$\Delta_1 = \Phi_1(\mathbb{P}V) \cap \mathfrak{t}^+,$$
$$\Delta = \Phi(V) \cap \mathfrak{t}^+.$$

By the convexity theorem of Atiyah-Guillemin-Sternberg-Kirwan, Δ_1 is a convex polytope, and by the homogeneity of Φ, $\Delta = \mathbb{R}^+\Delta_1$. U is G-invariant, and

$$\Phi(U) \cap \mathfrak{t}^+ = \Delta_{<\varepsilon^2} = \bigcup_{0 \leqslant r < \varepsilon^2} r\Delta_1.$$

By Lemma 4, $\Delta_{<\varepsilon^2}$ is open in Δ.

To finish the proof that $\Phi(U)$ is open in $\Phi(V)$, note that U is G-invariant, so that both $\Phi(U)$ and $\Phi(V)$ are G-invariant subsets of $\mathfrak{g}^*$. Since $\mathfrak{t}^+$ and $\mathfrak{g}^*/G$ are homeomorphic, we have that $\Phi(U)/G$ is open in $\Phi(V)/G$, and the result ensues. □

Remark This lemma illustrates why it is important to consider G-openness rather than openness, for momentum maps are not in general open. For example the momentum map for the SO(3) action on $T^*\mathbb{R}^3$, namely $\Phi(q, p) = q \times p$, is not open. For example, the images of sufficiently small neighbourhoods of $(q, p) = (e_1, e_1)$ are not neighbourhoods of 0: in particular they do not contain nonzero points of the e_1-axis.

Equipped with these lemmas, we are in a position to prove Theorem 1.

Proof. Let $x \in M$ and $\mu = \Phi(x)$, and let U_1 be the G-invariant neighbourhood of x whose existence is guaranteed by the Marle-Guillemin-Sternberg normal form (Section 3). Let V_1 be a G-invariant tubular neighbourhood of μ, so that

$$V_1 \simeq G \times_{G_\mu} \mathcal{O},$$

where $\mathcal{O}$ is a neighbourhood of 0 in the slice $\mathfrak{g}_\mu^\sharp$ (the fixed-point set of the action of the centre of G_μ on $\mathfrak{g}^*$, which is isomorphic to $\mathfrak{g}_\mu^*$). Finally, let $U = U_1 \cap \Phi^{-1}(V_1)$. Thus, as symplectic G-spaces, $U \simeq G \times_{G_x} (\mathfrak{m}^* \times Y)$, where Y is a G_x-invariant neighbourhood of 0 in N_1, and it therefore suffices to show that the momentum map (3.2) is open. Since G is compact, we can take the cocycle θ in (3.2) to vanish.

By Lemma 5, the quadratic momentum map $\Phi_{G_x} : N_1 \to \mathfrak{g}_x^*$ is open relative to its image. It follows that the restriction of Φ to the slice $\mathfrak{m}^* \times Y \to \mathfrak{g}_\mu^*$ is open relative to *its* image. Lemma 3 now applies, with f replacing the restriction of Φ to the slice and F replacing Φ. □

5. Persistence of extremal relative equilibria

In this section we establish Theorem 2, using Theorem 1 which was proved in Section 4. By the results of Section 3 we may assume G to be compact.

We treat the case when γ is a minimal relative equilibrium; the maximal case is similar. Let U_0 be the G-invariant neighbourhood of γ guaranteed by Theorem 1. The minimality of γ means that there is a precompact G-invariant neighbourhood $U \subset U_0$ of γ such that

$$h|_{\Phi^{-1}(\mu) \cap \overline{U}} \geqslant h(\gamma) \quad \text{with equality only on} \quad G \cdot \gamma \cap U.$$

Suppose Theorem 2 is false. Let $\{\mu_n\}$ be a sequence of points in $\Phi(U)$ (which is open in $\Phi(U_0)$) converging to μ, such that the restriction of h to $\Phi^{-1}(\mu_n) \cap U$ has no minimum. However, by compactness, the restriction of h to $\overline{\Phi^{-1}(\mu_n) \cap U}$ has a minimum, say at $y_n \in \overline{U} \setminus U$. Also by compactness, $y_n \to y$, with $y \in \overline{U} \setminus U$ (possibly after passing to a subsequence).

We claim that there is a sequence $\{x_n\}$ converging to some $x \in G \cdot \gamma$, with $\Phi(x_n) = \mu_n$. Granted that claim, we have $h(x) = h(G \cdot \gamma) < h(y)$ by construction of y. On the other hand, for each n, $h(y_n) < h(x_n)$. In the limit we get $h(y) \leqslant h(x)$, which is a contradiction.

The existence of the sequence $\{x_n\}$ is a consequence of the openness property. Indeed, we can choose a nested sequence $\{U_n\}$ of G-invariant neighbourhoods of $G \cdot \gamma$ whose intersection is $G \cdot \gamma$, such that $\mu_n \in \Phi(U_n)$. Choosing $x_n \in \Phi^{-1}(\mu_n) \cap U_n$ gives (after passing to a subsequence if necessary) a sequence converging to a point $x \in G \cdot \gamma$, as claimed.

6. An example: the affine rigid body

The problem of affine rigid bodies (*alias* Riemann ellipsoids) has a long and important history, dating back perhaps to when Newton correctly suggested that the Earth was an oblate spheroid. Since then, it has been studied by such illustrious figures as Maclaurin, Jacobi, Dirichlet, Riemann, and Poincaré. A classical discussion can be found in the book of Chandrasekhar [2]; for a recent account from the symmetry perspective, we refer to Roberts and Sousa Dias [19].

An affine rigid body models a mass of ideal fluid evolving in time in such a manner that it always remains an ellipsoid. This is a Hamiltonian system whose configuration space is either $\mathcal{Q} = \mathrm{SL}(3, \mathbb{R})$ or $\mathrm{GL}(3, \mathbb{R})$ depending on whether one is modelling incompressible or compressible fluids. The matrix $Q \in \mathcal{Q}$ represents the configuration that is the image of a sphere under Q, an ellipsoid whose semi-axes are given by the singular values of Q. It is supposed that the potential energy depends only on the shape of the ellipsoid, and so is invariant under the symmetry group $G = \mathrm{SO}(3) \times \mathrm{SO}(3)$, the first copy of $\mathrm{SO}(3)$ acting by multiplication on the left, and the second by multiplication on the right.

The phase space is then the cotangent bundle $T^*\mathcal{Q}$, and the group G acts by cotangent lift on $T^*\mathcal{Q}$. Accordingly, the momentum map has two 'components'

$$\Phi_L, \Phi_R : T^*\mathcal{Q} \to \mathfrak{so}(3)^*$$

given by

$$\Phi_L(Q, P) = \frac{1}{2}\left(PQ^T - QP^T\right), \qquad \Phi_R(Q, P) = \frac{1}{2}\left(P^T Q - Q^T P\right).$$

The particular example of the potential energy function used by Dirichlet, Riemann and others is the *self-gravitating* potential. Other potentials arise in linear elasticity theory. In most of these examples the potential energy $V(Q)$ has a minimum at the round sphere $Q = I$. It then follows that this point is an equilibrium and indeed an *extremal* (relative) equilibrium. From Theorem 2 we deduce

Corollary 6. *Suppose the potential energy has a minimum at the point $(I, 0) \in T^*\mathcal{Q}$. Then there exist $\varepsilon_L, \varepsilon_R > 0$ and a G-invariant neighbourhood U of $(I, 0)$ in $T^*\mathcal{Q}$ such that for all $(\mu_L, \mu_R) \in \mathfrak{so}(3)^* \times \mathfrak{so}(3)^*$ with $\|\mu_L\| < \varepsilon_L$, $\|\mu_R\| < \varepsilon_R$ there is an extremal relative equilibrium of the affine rigid body in U with momentum $(\Phi_L, \Phi_R) = (\mu_L, \mu_R)$.*

Recall [13] that extremal relative equilibria are Lyapunov-stable relative to G, and by [8] they are then Lyapunov-stable relative to G_μ as well.

7. A counter-example: plane point vortices

In this section we give an example to the effect that the hypothesis on the compactness of the momentum isotropy subgroup G_μ is necessary in Theorem 2. Consider the symplectic manifold

$$M = \mathbb{C}^N \setminus \bigcup_{k\neq l} \{z_k = z_l\}, \quad \omega = \frac{i}{2}\sum_{k=1}^{N} \Gamma_k dz_k \wedge d\overline{z}_k, \quad \Gamma_1, \ \ldots, \Gamma_N \in \mathbb{R}$$

on which the Euclidean group $G = \mathrm{SE}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ acts diagonally. This action is free, proper, and Hamiltonian, and has a momentum map $\Phi : M \to \mathfrak{g}^*$. We use the identification $\mathfrak{g}^* \simeq \mathbb{C} \times \mathbb{R}$ and denote the components of Φ by

$$(\Phi_{\mathbb{C}}, \Phi_{\mathbb{R}}) : (z_1, \ \ldots, z_N) \mapsto \left(i\sum_{k=1}^{N} \Gamma_k z_k, \sum_{k=1}^{N} \Gamma_k \frac{|z_k|^2}{2} \right).$$

It can be shown that Φ is equivariant with respect to the standard (unmodified) coadjoint action if and only if $\sum_{k=1}^{N} \Gamma_k = 0$. As the G-invariant Hamiltonian we take

$$h(z_1, \ \ldots, z_N) = -\frac{1}{2\pi}\sum_{k<l} \Gamma_k \Gamma_l \log|z_k - z_l|.$$

Hamilton's equation reads

$$\frac{dz_k}{dt} = \frac{2}{i}\frac{\partial h}{\partial(\Gamma_k \overline{z}_k)} = -\frac{1}{2\pi i}\sum_{l\neq k} \frac{\Gamma_l}{\overline{z}_k - \overline{z}_l} \qquad (k = 1, \ \ldots, N).$$

This system describes the motion of N interacting plane point vortices with vorticities $\Gamma_1, \ \ldots, \Gamma_N$. See for example [1].

We study the case of 3 vortices with vorticities 1, 1, -2. Let us call the *axis* the subset $0 \times \mathbb{R}^+$ of $\mathbb{C} \times \mathbb{R} \simeq \mathfrak{g}^*$. A theorem of Synge, [23, Theorem 6], tells us that the relative equilibria $\gamma = (z_1, z_2, z_3)$ for h are of two types:

(1) $\Phi(\gamma)$ is on the axis, i.e. $\Phi_{\mathbb{C}}(\gamma) = 0$, in which case z_1, z_2, z_3 are collinear, with z_3 midway between z_1 and z_2;
(2) $\Phi(\gamma)$ is off the axis, i.e. $\Phi_{\mathbb{C}}(\gamma) \neq 0$, in which case z_1, z_2, z_3 form an equilateral triangle.

In an equilateral relative equilibrium (type 2), $z_2 - z_3 = e^{\pm i\pi/3}(z_1 - z_3)$, from which we calculate easily that

$$e^{2\pi h(\gamma)} = \frac{1}{3\sqrt{3}}|\Phi_{\mathbb{C}}(\gamma)|^3.$$

Proposition 7. *In a system of 3 vortices with vorticities 1, 1, -2, let γ be a collinear relative equilibrium for h (type 1), U the G-invariant neighbourhood of γ in M defined by $e^{2\pi h(U)} > e^{2\pi h(\gamma)}/3$, and $V = (D \times \mathbb{R} \setminus \text{axis})$ a punctured neighbourhood of $\mu = \Phi(\gamma)$ in $\mathfrak{g}^*$ where $D \subset \mathbb{C}$ is the disc of radius $(\sqrt{3}\, e^{2\pi h(\gamma)})^{1/3}$ centred at 0. Then γ is extremal, but $\Phi^{-1}(V) \cap U$ contains no relative equilibrium for h.*

Proof. Since the action of G is free, Φ is a submersion, and $\mathrm{codim}(\Phi^{-1}(\mu)) = \mathrm{codim}(\mu) = 3 = \dim(G) = \dim(G_\mu)$. This means that $\Phi^{-1}(\mu)/G_\mu$ is discrete (by explicit calculation, in fact a single point), hence γ is trivially extremal. Suppose a relative equilibrium γ' exists in $\Phi^{-1}(V) \cap U$. Since $\Phi(\gamma')$ is off the axis, γ' is an equilateral triangle (type 2) and $e^{h(\gamma')} = \frac{1}{3\sqrt{3}}|\Phi_{\mathbb{C}}(\gamma')|^3 < \frac{1}{3\sqrt{3}}|\sqrt{3}\, e^{h(\gamma)}|$. This is incompatible with $e^{h(\gamma')} > e^{h(\gamma)}/3$. $\square$

Remark. In Proposition 7, M, G, Φ, h, γ satisfy all the hypotheses in Theorem 2 except the compactness of the isotropy group of $\Phi(\gamma)$, which is $G = \mathrm{SE}(2)$ itself. The failure of γ to persist to nearby levels of the momentum map shows that this compactness hypothesis is essential. As observed in the proof, the momentum map is a submersion and the conclusion of Theorem 1 is still true.

Note, however, that γ persists when μ is perturbed along the axis $0 \times \mathbb{R}$, which is the annihilator of the noncompact part $\mathbb{R}^2$ of the Lie algebra and so in a natural way the dual of the Lie algebra of the compact subgroup $\mathrm{SO}(2)$ of G. This partial persistence in 'compact directions' is in fact also covered by Theorem 2, via reduction by stages and Corollary 8 below.

It is interesting that, in the model of N plane vortices, if the sum of the vorticities does not vanish, then the momentum map is equivariant with respect to a modified coadjoint action (with a nontrivial cocycle θ), and for this modified coadjoint action all the isotropy subgroups are isomorphic to $\mathrm{SO}(2)$ and so are compact. Hence Theorem 2 implies that the relative equilibrium persists to nearby values of momentum. This does not contradict the theorem of Synge quoted above, for in the case of nonvanishing total vorticity all nearby values of momentum are realisable by collinear configurations of 3 vortices.

8. Reduction by stages

Let K be a normal subgroup of a Lie group G with quotient $L = G/K$. Roughly speaking, we say that reduction by stages works if reduction by G coincides with reduction first by K and then by L. In detail, reduction by stages describes the following general procedure.

At the level of Lie algebras and their duals, we have $\mathfrak{k} \subset \mathfrak{g}$ and $\mathfrak{l}^* \subset \mathfrak{g}^*$. Moreover, the inclusion $\mathfrak{l}^* \hookrightarrow \mathfrak{g}^*$ naturally identifies $\mathfrak{l}^*$ with the annihilator $\mathfrak{k}^\circ$ of $\mathfrak{k}$ in $\mathfrak{g}^*$. Let $\pi : \mathfrak{g}^* \to \mathfrak{k}^*$ be the canonical projection.

Suppose G acts in a Hamiltonian manner on a symplectic manifold M with momentum map $\Phi_G : M \to \mathfrak{g}^*$. This action restricts to an action of K, and the momentum map is just $\Phi_K = \pi \circ \Phi_G : M \to \mathfrak{k}^*$. G and K act on $\mathfrak{g}^*$ and $\mathfrak{k}^*$ in such a way as to make the momentum maps equivariant. For $\mu \in \mathfrak{g}^*$, we shall use the notation $M/\!\!/_\mu G$ to denote the reduced space $\Phi_G^{-1}(\mu)/G_\mu$.

The modified coadjoint action Coad^θ of G on $\mathfrak{g}^*$ descends to an action on $\mathfrak{k}^*$, and so L acts in a natural way on the set of K-orbits in $\mathfrak{k}^*$. Let L_ν be the subgroup of L that preserves the coadjoint orbit $K \cdot \nu$; one can show that $L_\nu \simeq G_\nu/K_\nu$. It follows that L_ν acts on $M/\!\!/_\nu K$ in a natural way, preserving the symplectic structure (compare [15]).

The affine subspace $\pi^{-1}(\nu)$ of $\mathfrak{g}^*$ can be identified with $\mathfrak{k}^\circ$, and hence with $\mathfrak{l}^*$, by translation in $\mathfrak{g}^*$. Let $\rho_\nu : \pi^{-1}(\nu) \to \mathfrak{l}^*_\nu$ be the composite of such an identification with the natural projection $\mathfrak{l}^* \to \mathfrak{l}^*_\nu$. Define

$$\Phi_\nu : M/\!\!/_\nu K \to \mathfrak{l}^*_\nu$$

by first restricting Φ_G to $\Phi_K^{-1}(\nu)$ (whose values lie in $\pi^{-1}(\nu)$), passing to the quotient $M/\!\!/_\nu K$ and finally applying the identification ρ_ν. If this map Φ_ν is well-defined and is a momentum map for the action of L_ν, then we say *reduction by stages works* in this context provided in addition that

$$M/\!\!/_\mu G \simeq (M/\!\!/_\nu K)/\!\!/_\sigma L_\nu$$

at least at the level of connected components, where $\nu = \pi(\mu)$ and $\sigma = \rho_\nu(\mu) = \Phi_\nu(\Phi_G^{-1}(\mu))$. The isomorphism between the two spaces should be as needed in the context; here a homeomorphism is sufficient, though more generally one might require an isomorphism as symplectic stratified spaces [21]. The papers [11, 12] explain the current state of the art on reduction by stages for free actions.

If the action of K is free, then the partially reduced space $M/\!\!/_\nu K$ is a smooth symplectic manifold, and one can apply Theorem 2 to the resulting L_ν-invariant system, provided $(L_\nu)_\sigma$ is compact. The result is that if the relative equilibrium γ in question is extremal in $M/\!\!/_\nu K$ then it persists to nearby values of the L_ν-momentum map Φ_ν.

It often happens that $L_\nu = L$, in which case ρ_ν is just a translation of $\mu + \mathfrak{k}^\circ$ to $\mathfrak{k}^\circ$, and one can ask for persistence to all $\mu' \in V \subset (\mu + \mathfrak{k}^\circ)$.

Corollary 8. *Let $G, M, \Phi, K, L, \mu, \nu$ and σ be as above, with K acting freely on M, and such that reduction by stages works. Suppose that $L_\nu = L$, and that L_σ is compact. Let h be a G-invariant Hamiltonian on M for which $\gamma \subset M$ is an extremal relative equilibrium with $\mu = \Phi(\gamma)$. Let U_0 be the L-invariant neighbourhood of γ in $M/\!\!/_\nu K$ guaranteed by Theorem 1. Then there is an L-invariant neighbourhood V of μ in $\Phi(U_0) \cap \mathfrak{k}^\circ$ such that for every $\sigma' \in V$ there is a relative equilibrium in $\Phi^{-1}(\sigma') \cap U_0$.*

For the vortex system of Section 7 the corollary applies as follows: $G = \mathrm{SE}(2)$, $K = \mathbb{R}^2$ the normal subgroup of $\mathrm{SE}(2)$ consisting of translations, $\mu = (0, r)$, and $\nu = 0$. Then $L_\nu = L = \mathrm{SE}(2)/\mathbb{R}^2 \simeq \mathrm{SO}(2)$, which is compact. The subspace $\mathfrak{k}^\circ$ is a line (the 'axis' of Section 7) consisting of the coadjoint orbits that are isolated points.

Remark. The freeness hypothesis on the K-action could easily be relaxed to local freeness. In the general setting where the action is not locally free, the same argument can be applied to the symplectic stratum in $M/\!\!/_\nu K$ containing the image of γ.

Acknowledgements. We would like to thank Eugene Lerman for many useful discussions, and for pointing out an error in a preliminary draft, and the referees for making some useful suggestions. Since completing this paper, we have become aware of the nice work of Friedrich Knop [6] in which he proves the local openness of the momentum map (among other things).

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