Strangely Dispersed Minimal Sets in the Quasiperiodically Forced Arnold Circle Map

Glendinning, Paul and Jager, Tobias and Stark, Jaroslav

2008

MIMS EPrint: **2008.101**

ISSN 1749-9097

Strangely Dispersed Minimal Sets in the Quasiperiodically Forced Arnold Circle Map

P.A. Glendinning[*], T. Jäger[†] and J. Stark[‡]

July 29, 2008

Abstract

We study quasiperiodically forced circle endomorphisms, homotopic to the identity, and show that under suitable conditions these exhibit uncountably many minimal sets with a complicated structure, to which we refer to as 'strangely dispersed'. Along the way, we generalise some well-known results about circle endomorphisms to the uniquely ergodically forced case. Namely, all rotation numbers in the rotation interval of a uniquely ergodically forced circle endomorphism are realised on minimal sets, and if the rotation interval has non-empty interior then the topological entropy is strictly positive. The results apply in particular to the quasiperiodically forced Arnold circle map, which serves as a paradigm example.

1 Introduction

Quasiperiodically forced circle (QPF) maps such as the QPF forced Arnold map $f : \mathbb{T}^1 \times \mathbb{T}^1 \to \mathbb{T}^1 \times \mathbb{T}^1$

$$f(\theta,\varphi) \;=\; \left(\theta+\omega, \varphi+\tau+\frac{\alpha}{2\pi}\sin(2\pi\varphi)+\beta\sin(2\pi\theta) \bmod 1\right) \,, \tag{1.1}$$

where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ denotes the circle and $\omega \notin \mathbf{Q}$, have been studied by a number of authors. The motivation for this comes from two related directions. First, Grebogi *et al* [1] showed that it is possible to have strange (i.e. geometrically complicated) nonchaotic attractors (SNAs) over a range of parameter values with positive measure, and later (e.g. [2, 3, 4, 5]) that maps such as 1.1 are good candidates for simple invertible examples of such behaviour. This aspect has been followed up in the work of Feudel and Pikovsky and ourselves amongst others [6, 7, 8, 9]. Secondly, from a different perspective Herman [10] had already proved the existence of SNA in certain parameter families of QPF circle diffeomorphisms that are induced by the projective action of $SL(2,\mathbb{R})$-cocycles over an irrational rotation.

Despite considerable interest over subsequent years, rigourous mathematical results remained rare. The original Grebogi *et al* example [1] was a non-invertible map with a special structure. This structure was abstracted by Keller, who proved the existence of SNAs under simple conditions in this class of maps [11]. Jäger [12] further analysed the structure of such invariant sets. Subsequently, Glendinning *et al* [13] proved that although non-chaotic in the sense of Lyapunov exponents, such systems did exhibit sensitive dependence on initial conditions. Meanwhile, Stark [14] showed that SNAs in QPF maps could not be non-smooth graphs, but had to have a more complex structure, and an extension of this approach by Sturman and Stark [15] showed that the normal Lyapunov exponents of a SNA could not all be negative. Finally, new methods were established quite recently by Bjerklöv and Jäger, which allow to prove the existence of SNA in much broader classes of quasiperiodically forced maps than the two mentioned above [16, 17, 18].

[*]University of Manchester. Email: p.a.glendinning@manchester.ac.uk
[†]Collège de France, Paris. Email: tobias.jager@college-de-france.fr
[‡]Imperial College London. Email: j.stark@imperial.ac.uk

Additional properties of invertible circle maps were derived in [19, 20], and used together with results in [21] to give a generalization of the Poincare classification of circle homeomorphisms [22]. Further, Jäger and Keller [21] showed that if a QPF circle homeomorphism, homotopic to the identity, with appropriate conditions on its rotation number, was topologically transitive then any minimal set was 'strangely dispersed' (see below for definition). Dynamics of this type are constructed in [23]. However, it is also known that the minimal set is unique in this situation ([24] or [23]), such that there is no co-existence as in Theorem 2.6. Further, the examples given in [23] only have low regularity, and it is still completely open whether the same phenomenon can occur in smooth systems as well (e.g. in QPF analytic circle diffeomorphisms).

Here we turn to examine the behaviour of QPF maps such as the forced Arnold map (1.1) above. This is motivated both by the considerable volume of numerical work, and the fact that the unforced Arnold map has a rich and interesting structure has been described in some detail by MacKay and Tresser [25]. This gave a beautiful description of the transition to chaotic behaviour in the unforced case. Numerical experiments have suggested that in the QPF map the appearance of strange nonchaotic structures occurs at the complex boundary between the regular and chaotic parameter regions.

Unfortunately, MacKay and Tresser's analysis made heavy use of periodic orbits and doubling cascades. Since (1.1) has no periodic orbits (this follows immediately from the fact that ω is irrational) it is not immediately clear how to generalize their work to the QPF case. Indeed, almost all of our understanding of chaos is based on generalizations of the horseshoe (e.g. [26]) and horseshoes imply the existence of periodic orbits, so either horseshoes are irrelevant for the study of chaos in quasiperiodically forced systems or the chaos is essentially a suspension of a horseshoe. If the former is the case then it is natural to ask which orbits form the backbone of the chaos, i.e. which orbits play the role of the periodic orbits in the horseshoe? There are therefore at least three reasons for considering noninvertible quasiperiodically forced circle maps, $k > 1$ in (1.1). First in an attempt to obtain some rigorous results on complex invariant sets, second as an extension of the results for noninvertible circle maps, and third as a move towards understanding chaotic sets which are not modelled by horseshoes. Our main motivation has been the first of these. We shall prove that if k and β are sufficiently large then the the forced Arnold map (1.1) exhibits uncountably many minimal sets with a complicated structure, to which we refer to as 'strangely dispersed'. In the proof of this result it becomes necessary to prove analogues of a number of results for noninvertible circle maps in the context of noninvertible quasiperiodically forced circle maps. An appealing, albeit heuristic, interpretation of this result is that in moving along a path in parameter space from a nonchaotic state of an invertible quasiperiodically forced circle map to a chaotic noninvertible circle map of the type discussed below it is necessary to create strange nonchaotic invariant sets. One way of achieving this is to create this set as a stable set, which later loses stability. If this is the case then it goes some way towards explaining why nonchaotic strange attractors must exist in such families.

Acknowledgements. We would like to thank Sylvain Crovisier for pointing out to us the result by Bowen [27] and its consequences for the entropy of QPF monotone circle maps. Tobias Jäger was supported by a research fellowship of the German Research Foundation (DFG, Ja 1721/1-1).

2 Main Results

Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ denote the circle and suppose Θ is a compact metric space and $r : \Theta \to \Theta$ a continuous map. We consider skew-products on $\Theta \times \mathbb{T}^1$ given by continuous maps $f : \Theta \times \mathbb{T}^1 \to \Theta \times \mathbb{T}^1$ of the form

$$f(\theta, \varphi) \;=\; (r(\theta), f_\theta(\varphi)) \; . \tag{2.1}$$

The case we are primarily interested in is that of QPF circle endomorphisms, that is $\Theta = \mathbb{T}^1$ and $r(\theta) = \theta + \omega$ with $\omega \in \mathbb{T}^1$ irrational. However, some of the results we obtain naturally generalise to the uniquely ergodically forced (UEF) case (meaning that there exists a unique r-invariant probability measure μ on Θ).

The maps f_θ in (2.1) will be called *fibre maps*. Most of the time, we will assume in addition that f is homotopic to the map $(\theta, \varphi) \mapsto (r(\theta), \varphi)$. If this holds, we say f is *homotopic to the identity* (slightly abusing terminology in the case that r is not homotopic to the identity on Θ). Then all f_θ are circle endomorphisms of degree one, and further there exist a continuous lift $F : \Theta \times \mathbb{R} \to \Theta \times \mathbb{R}$ that satisfies $\pi \circ F = f \circ \pi$, where $\pi : \Theta \times \mathbb{R} \to \Theta \times \mathbb{T}^1$ denotes the natural projection. Moreover, if Θ is connected, then these lifts are always uniquely defined modulo an integer. In the same way we can define the continuous lifts $F_\theta : \mathbb{R} \to \mathbb{R}$ of fibre maps f_θ which satisfy $\pi \circ F_\theta = f_\theta \circ \pi$, where $\pi : \mathbb{R} \to \mathbb{T}^1$ denotes the natural projection, with the obvious abuse of notation on the projection operators π.

We define the *fibred rotation interval* of a lift F by

$$\rho_{\text{fib}}(F) \;:=\; \left\{ \limsup_{n\to\infty} \frac{1}{n}(F^n_\theta(x) - x) \,\middle|\, (\theta, x) \in \Theta \times \mathbb{R} \right\} . \tag{2.2}$$

where $F^n_\theta(x) = F_{r^{n-1}(\theta)} \circ \ldots \circ F_\theta(x)$. Observe that ρ_{fib} for two different lifts of the same UEF endomorphism of $\Theta \times \mathbb{T}^1$ will differ by an integer translation.

An important special case will be the one of UEF (or QPF) *monotone circle maps*, by which we mean that each fibre map f_θ preserves the cyclic order on $\mathbb{T}^1$ (but we allow f to be non-injective). This is true if and only if the fibre maps F_θ of any lift F of f are monotonically increasing. It is a well-known result of Herman [10] that the fibred rotation interval of a UEF monotone circle map is always a single point (restated below as Theorem 3.3).

Remark 2.1. *Note that in the QPF case there are in general several ways of assigning a rotation set to a torus endomorphism which is homotopic to the identity, as discussed very concisely in [28]. However, if f has skew-product structure as in (2.1), then all these different notions coincide. This follows easily from Theorem 2.2 below, in combination with [28, Theorem 2.4 and Corollary 2.6]. The above definition is the one which is most convenient for our purposes, and we have adapted it to the fibred setting by projecting to the second coordinate, thus obtaining a subset of the real line instead of a subset of $\mathbb{R}^2$ for a general endomorphism of $\mathbb{T}^2$.*

Recall that a closed, f-invariant set M is *minimal* if it contains no proper f-invariant closed subset [26]. This is equivalent to the orbit of every point in M being dense in M. The *topological entropy* $h_{\text{top}}(f)$ of a map f is a common measure of the complexity of its dynamics, and indeed provides one of the standard definitions of chaotic behaviour [26]. A definition and brief overview is given below in Section 4. The following theorem is then a generalisation of well-known results on unforced circle endomorphisms (see, for example, [29]).

Theorem 2.2. *Suppose F is the lift of a UEF circle endomorphism $f : \Theta \times \mathbb{T}^1 \to \Theta \times \mathbb{T}^1$, homotopic to the identity. Then $\rho_{\text{fib}}(F)$ is a closed interval (including the possibility of a singleton $\rho_{\text{fib}}(F) = [\rho, \rho]$). For any $\rho \in \rho_{\text{fib}}(F)$ there exists a minimal set $M_\rho \subset \Theta \times \mathbb{T}^1$ with the following properties:*

(i) $\frac{1}{n}(F^n_\theta(x) - x)$ *converges uniformly to ρ on $\pi^{-1}(M_\rho)$ as $n \to \infty$.*

(ii) $h_{\text{top}}(f|_{M_\rho}) = 0$.

The proof of (i) is given in Section 3 and that of (ii) at the end of Section 4. Although the dynamics on each M_ρ is simple, if the rotation interval is non-trivial, the overall dynamics of the map is complex:

Theorem 2.3. *Suppose f is a UEF circle endomorphism, homotopic to the identity, with lift F. If $\rho_{\text{fib}}(F)$ has non-empty interior, then $h_{\text{top}}(f) > 0$.*

The proof is given in Section 4. We remark that for a QPF monotone circle map f the situation is quite different. As mentioned above, the rotation interval is reduced to a single point in this case, and the topological entropy is always zero. The latter is a more or less direct consequence of a result by Bowen [27], see Section 4 below.

Once these basic facts are established, we can turn to a new phenomenon which is specific to the quasiperiodically forced setting. In the case of unforced circle endomorphisms, minimal sets may be either periodic orbits or Cantor sets, corresponding to rational and irrational rotation numbers, respectively. In the quasiperiodically forced case however, they can have a much more complicated structure. In order to make this precise, we introduce the following notion.

Definition 2.4. *Suppose f is a QPF circle endomorphism, homotopic to the identity. We say a compact subset $M \subseteq \mathbb{T}^2$ is a strangely dispersed minimal set, if it has the following three properties:*

(i) M is a minimal set.

(ii) Every connected component C of M is contained in a single fibre, that is $\pi_1(C)$ is a singleton.

(iii) For any point $(\theta, x) \in M$ and any open neighbourhood U of (θ, x), the set $\pi_1(U \cap M)$ contains a non-empty open interval.

Remark 2.5.

(a) Property (iii) is a rather direct consequence of (i) (see Section 5). We have only included it here to emphasize the peculiarity of property (ii).

(b) It is actually not difficult to construct a set which has properties (ii) and (iii). Indeed, let $(a_\eta)_{\eta\in\mathbb{T}^1\cap\mathbb{Q}}$ be any sequence of strictly positive real numbers with $\sum_{\eta\in\mathbb{T}^1\cap\mathbb{Q}} a_\eta = 1$. For any $\theta \in \mathbb{T}^1$, let $\phi(\theta) := \sum_{\eta\in[0,\theta]\cap\mathbb{Q}} a_\eta$. Then the topological closure of the graph $\Phi := \{(\theta, \phi(\theta)) \mid \theta \in \mathbb{T}^1\}$ of ϕ is a compact set that has these two properties. Of course, the interesting point in the above definition is to have a set with this structure as the minimal set of a dynamical system.

It will follow from our arguments in Section 5 that the appearance of strangely dispersed minimal sets is a rather general phenomenon for QPF circle endomorphisms, provided that the quasiperiodic forcing has a certain strength. However, for simplicity we will formulate the results only for a particular example, namely for the QPF Arnold Circle Map (1.1).

Theorem 2.6. *Suppose f is given by (1.1), with driving frequency $\omega \in \mathbb{T}^1 \setminus \mathbb{Q}$ and real parameters τ, α and β.*

(a) If $\alpha > 1$ and $|\beta| \geq \frac{3}{2}$, then for any $\rho \in \rho_{\mathrm{fib}}(f)$ there exists a strangely dispersed minimal set M_ρ which satisfies properties (i) and (ii) in Theorem 2.2 .

(b) If $|\alpha| \geq \frac{5}{2}\pi$, then $\rho_{\mathrm{fib}}(f)$ has length ≥ 1, in particular its interior is non-empty.

Remark 2.7.

(a) The bounds given here are not optimal and may surely be improved by a more careful analysis. Further, part (b) of this theorem is rather trivial, but it is important for the interpretation of (a). Namely, if both conditions in (a) and (b) are satisfied simultaneously, we obtain the existence of uncountably many pairwise disjoint and strangely dispersed minimal sets, one for each rotation number in the rotation interval. Albeit most likely superficial, this presents an intriguing analogy to the theory of twist maps, where at suitable parameter values the standard map exhibits uncountably many Aubry-Mather sets, again one for each rotation number in the rotation interval.

(b) As indicated in the introduction above, the existence of strangely dispersed minimal sets is already known in QPF circle homeomorphisms [21, 23, 24] though existing constructions only work in maps of low regularity.

3 Plateau Maps and Rotation Numbers: Proof of Theorem 2.2

In order to prove Theorem 2.2, we will first be concerned with unforced circle endomorphisms and their lifts. The basic idea, which is the use of plateau maps to identify orbits with a given rotation number, was first introduced by Boyland [30] (see also [29] for a survey). Let $\mathcal{E}$ denote the space of continuous maps $G : \mathbb{R} \to \mathbb{R}$ which are the lift of a circle endomorphism of degree one. The latter just amounts to saying that

$$G(x+k) \;=\; G(x)+k \quad \forall k \in \mathbb{Z}\ . \tag{3.1}$$

We equip $\mathcal{E}$ with the topology of uniform convergence. Further, we denote by $\mathcal{E}_{\mathrm{mon}}$ the space of all maps in $G \in \mathcal{E}$ which are monotonically increasing. Then $G \in \mathcal{E}_{\mathrm{mon}}$ if and only if it is the lift of a monotone circle map of degree one. Note that we explicitly allow for $G \in \mathcal{E}_{\mathrm{mon}}$ to be non-injective, in which case there exist intervals that are mapped to a single point by G. We refer to such intervals as *plateaus*, and call maps in $\mathcal{E}_{\mathrm{mon}}$ *plateau maps* (including the case when there are no plateaus, for simplicity).

For any $G \in \mathcal{E}_{\mathrm{mon}}$, let $\mathcal{U}(G)$ denote the union of the interiors of all plateaus of G. In other words

$$\mathcal{U}(G) \;=\; \{x \in \mathbb{R} \mid \exists \varepsilon > 0 : G([x-\varepsilon, x+\varepsilon]) = \{G(x)\}\}\ .$$

Now suppose $G \in \mathcal{E}$. We assign to G a pair of plateau maps $G^- \leq G \leq G^+$ (Figure 1) by

$$G^+(x) \;:=\; \sup_{\xi \leq x} G(\xi) \quad \text{and} \quad G^-(x) \;:=\; \inf_{\xi \geq x} G(\xi)\ .$$

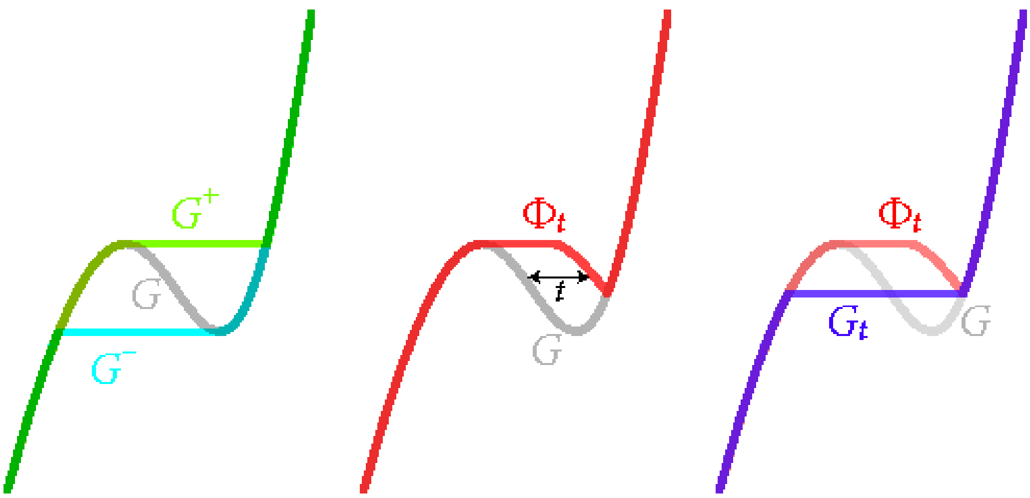


Figure 1: Illustration of the plateau maps G^+ and G^- and the functions Φ_t and G_t.

Note that if G is a plateau map itself, then $G^- = G^+ = G$. Further, it follows easily from (3.1) that

$$G^+(x) = \sup_{\xi \in [x-1,x]} \quad \text{and} \quad G^-(x) = \inf_{\xi \in [x,x+1]} G(\xi)\ . \tag{3.2}$$

The reason why plateau maps are such a convenient tool for the computation of rotation intervals is the fact that they always have a uniquely defined rotation number, and this remains true in the quasiperiodically forced setting (see Theorem 3.3 below). Furthermore, as the following proposition shows, there always exists a homotopy between the maps G^- and G^+ with some additional nice properties, and this will be the key ingredient in the proof of Theorem 2.2 .

Proposition 3.1. *There exists a continuous mapping $\mathbb{R} \times [0,1] \times \mathcal{E} \to \mathbb{R}$, $(x,t,G) \mapsto G_t(x)$, with the following properties:*

(i) The family $(G_t)_{t\in[0,1]}$ is a homotopy between G^- and G^+, that is $G_0 = G^-$ and $G_1 = G^+$.

(ii) For all $t \in [0,1]$ we have $G_t \in \mathcal{E}_{\mathrm{mon}}$.

(iii) For all $x \in \mathbb{R}$ the map $t \mapsto G_t(x)$ is monotonically increasing.

(iv) If $G_t(x) \neq G(x)$, then $x \in \mathcal{U}(G_t)$.

Note that due to the periodicity property (3.1) of $G \in \mathcal{E}$ and compactness, the induced mapping $[0,1] \times \mathcal{E} \to \mathcal{E}_{\mathrm{mon}},\ (t,G) \mapsto G_t$ is continuous as well.

Proof. The mappings $\mathbb{R} \times \mathcal{E} \to \mathbb{R},\ (x,G) \mapsto G^{\pm}(x)$ are clearly continuous and monotone in x. They are also continuous and monotonically increasing in G, the latter with respect to the partial ordering on $\mathcal{E}$ given by $G_1 \leq G_2$ if $G_1(x) \leq G_2(x)\ \forall x \in \mathbb{R}$. Similarly, the mapping (Figure 1)

$$\Phi : \mathbb{R} \times [0,1] \times \mathcal{E} \to \mathbb{R} \quad , \quad (x,t,G) \mapsto \Phi_t(x) := \sup_{\xi \in [x-t,x]} G(\xi) \ , \tag{3.3}$$

is continuous and monotonically increasing t and G. We define our required homotopy by

$$G_t(x) \ := \ (\Phi_t)^-(x) \ = \ \inf_{\zeta \geq x} \ \sup_{\xi \in [\zeta - t, \zeta]} G(\xi) \ . \tag{3.4}$$

For any given $x \in \mathcal{E}$ the function $(x,t) \mapsto G_t(x)$ is continuous as the composition of continuous functions. By definition $\Phi_0 = G$, and hence $G_0 = G^-$. Also, by (3.2) we have $\Phi_1 = G^+$ and hence $G_1 = (G^+)^- = G^+$. This proves (i). For any $t \in [0,1]$ the map $G_t = (\Phi_t)^-$ is a plateau map, since it is in the image of the mapping $\mathcal{E} \to \mathcal{E}_{\mathrm{mon}},\ G \mapsto G^-$. Thus (ii) holds. The monotonicity of the mapping Φ in t and of the mapping $G \mapsto G^-$ in G immediately implies (iii).

It remains to prove (iv). We first show that if for a given $t \in [0,1]$ and $x \in \mathbb{R}$ we have $G_t(x) < \Phi_t(x)$ then G_t is constant in an open neighbourhood of x. Since $G_t = (\Phi_t)^-$, and Φ_t is continuous, then if $G_t(x) < \Phi_t(x)$ there must exist some $\xi_0 > x$ with $\Phi_t(\xi_0) = G_t(x)$. By the continuity of Φ_t we further have $\Phi_t(x') > G_t(x)$ for all x' in a small open neighbourhood U of x. Without loss of generality, we shrink U so that it does not contain ξ_0, which implies that $x' < \xi_0$ for all $x' \in U$. This means that $\inf_{\xi \geq x'} \Phi_t(\xi) \leq \Phi_t(\xi_0)$ for all $x' \in U$. If $x' \geq x$, then automatically $\inf_{\xi \geq x'} \Phi_t(\xi) \geq \inf_{\xi \geq x} \Phi_t(\xi) = \Phi_t(\xi_0)$, whereas if $x' < x$, then $\inf_{\xi \geq x'} \Phi_t(\xi) = \min\{\inf_{x' \leq \xi < x} \Phi_t(\xi), \inf_{\xi \geq x} \Phi_t(\xi)\} = \min\{\inf_{x' \leq \xi < x} \Phi_t(\xi), \Phi_t(\xi_0)\} = \Phi_t(\xi_0)$, since $\Phi_t(\xi) > G_t(x) = \Phi_t(\xi_0)$ for all $\xi \in U$. Hence $G_t(x') = \Phi_t(\xi_0)$ for all $x' \in U$, so that G_t is constant on U as required.

To now prove (iv), fix $t \in [0,1]$ and $x \in \mathbb{R}$ with $G_t(x) \neq G(x)$. First suppose $G_t(x) < G(x)$ (Figure 2a). Since $G(x) \leq \Phi_t(x)$ for all $x \in \mathbb{R}$, this implies that $G_t(x) < \Phi_t(x)$ and hence by the above G_t is constant on an open neighbourhood of x, as required.

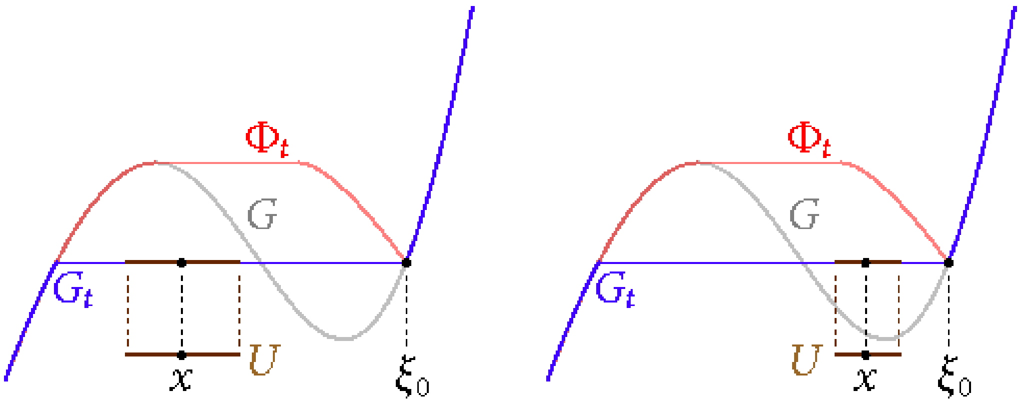
On the other hand, if $G_t(x) > G(x)$, we consider two cases. By definition, we always have $G_t(x) \leq \Phi_t(x)$. Hence either $G_t(x) < \Phi_t(x)$ (Figure 2b) or $G_t(x) = \Phi_t(x)$. In the former case we again apply the argument above.

In the latter case, $G_t(x) = \Phi_t(x)$, the definition of Φ_t implies that there exists some $\xi_0 \in [x-t, x)$ such that $\Phi_t(x) = G(\xi_0)$ (Figure 3). For any $x' \in [\xi_0, x]$ we have $\xi_0 \in [x'-t, x']$ and hence $\Phi_t(x') \geq G(\xi_0) = \Phi_t(x)$. Thus $G_t(x') = \inf_{\xi \geq x'} \Phi_t(\xi) = \inf_{\xi \geq x} \Phi_t(\xi) = G_t(x)$, so that G_t is constant on a left neighbourhood of x. Now, by definition $\Phi_t(x') \geq G_t(x)$ for all $x' \geq x$, and since $G_t(x) = \Phi_t(x)$, we have $\Phi_t(x') \geq \Phi_t(x)$ for all $x' \geq x$. Since $G(x) < G_t(x) = \Phi_t(x)$, the continuity of G implies that there exists an $\epsilon > 0$ such that $G(x') < \Phi_t(x)$ for all $x' \in [x, x+\epsilon)$ Furthermore, by the defintion of Φ_t we have $G(x') \leq \Phi_t(x)$ for all $x' \in [x-t, x]$. Hence for any $x' \in [x, x+\epsilon)$ we have $G(\xi) \leq \Phi_t(x)$ for all $\xi \in [x'-t, x']$. Thus $\Phi_t(x') \leq \Phi_t(x)$ for all $x' \in [x, x+\epsilon)$ and so Φ_t is constant on $[x, x+\epsilon)$. By definition G_t is non-decreasing, so $G_t(x') \geq G_t(x)$ for all $x' \in [x, x+\epsilon)$. But $G_t(x') \leq \Phi_t(x')$ for any x', and so in particular for $x' \in [x, x+\epsilon)$ we have $G_t(x') \leq \Phi_t(x') = \Phi_t(x) = G_t(x)$. Hence $G_t(x') = G_t(x)$ for all $x' \in [x, x+\epsilon)$, and so G_t is also constant on a right neighbourhood of x. This completes the proof of (iv).

□

The following lemma will provide the link between the orbits of the original map and the plateau maps derived from it.

Lemma 3.2. *Suppose $(G_n)_{n \in \mathbb{N}_0}$ is a sequence of plateau maps and let $G^{(n)} := G_{n-1} \circ \ldots \circ G_0$. Then there exists $x \in \mathbb{R}$ with the property that $G^{(n)}(x) \notin \mathcal{U}(G_n)\ \forall n \in \mathbb{N}_0$.*



(a) The case $G_t(x) < G(x)$ (b) The case $G_t(x) > G(x)$

Figure 2: Proof of Proposition 3.1 (iv) when $G_t(x) < \Phi_t(x)$.

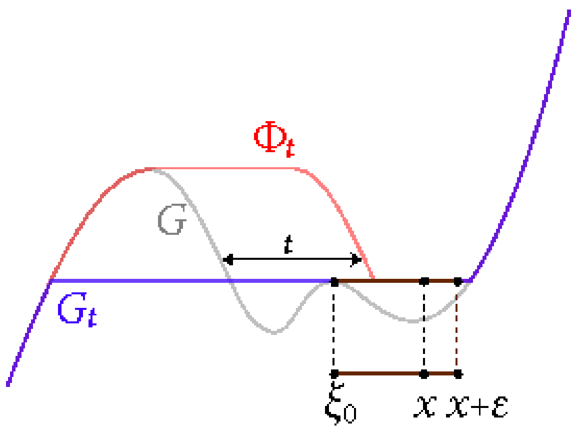


Figure 3: Proof of Proposition 3.1 iv) for the case $G(x) < G_t(x) = \Phi_t(x)$.

Proof. We argue by contradiction. Suppose for all $x \in \mathbb{R}$, there exists some $n \in \mathbb{N}_0$, such that $G^{(n)}(x) \in \mathcal{U}(G_n)$. Let $V_n := (G^{(n)})^{-1}(\mathcal{U}(G_n))$. Then the open sets $\pi(V_n)$ form an open cover of $\mathbb{T}^1$ and hence, since $\mathbb{T}^1$ is compact, there is a finite subcover. Thus $\mathbb{T}^1 \subseteq \pi(V_0) \cup \ldots \cup \pi(V_N)$ for some $N \in \mathbb{N}_0$ and hence $\mathbb{R} \subseteq V_0 \cup \ldots \cup V_N$. However, as every plateau is mapped to a single point and there are at most countably many plateaus, this implies that $G^{(N)}(\mathbb{R})$ is countable and therefore a strict subset of $\mathbb{R}$. Since all G_n are surjective, this yields the required contradiction.

□

Now we can turn to the forced setting. Recall that we say f is a UEF monotone circle map, if all of its fibre maps f_θ are circle maps of degree one which preserve the cyclic order on $\mathbb{T}^1$. This is true if and only if any lift $F : \Theta \times \mathbb{R} \to \Theta \times \mathbb{R}$ of f satisfies $F_\theta \in \mathcal{E}_{\mathrm{mon}}\ \forall \theta \in \Theta$. As mentioned before, the rotation number of UEF monotone circle maps is uniquely defined.

Theorem 3.3 (Herman [10, 19])**.** *Suppose f is a UEF monotone circle map, homotopic to the identity, with lift F. Then the limit*

$$\rho(F) \ := \ \lim_{n\to\infty} \frac{1}{n}(F^n_\theta(x) - x) \tag{3.5}$$

exists and is independent of (θ, x), and the convergence in (3.5) is uniform on $\Theta \times \mathbb{R}$. Furthermore, $\rho(F)$ depends continuously on F. We call $\rho(F)$ the fibred rotation number *of F.*

In fact, the result in [10] is only stated for UEF circle homeomorphisms, but the proof given there literally goes through in this slightly more general situation. Alternatively, [19] explicitly proves the existence of a unique rotation number for non-strictly monotone maps.

Proof of Theorem 2.2 (i) . Suppose $f : \Theta \times \mathbb{T}^1 \to \Theta \times \mathbb{T}^1, (\theta, \varphi) \mapsto (r(\theta), f_\theta(\varphi))$ is a UEF circle endomorphism homotopic to the identity and $F \in \mathcal{E}$ is a lift of f. We define two UEF monotone maps F^-, F^+ by $F^-_\theta := (F_\theta)^-$ and $F^+_\theta := (F_\theta)^+$. Then F^- and F^+ are the lifts of two UEF monotone circle maps, and by Theorem 3.3 the fibred rotation numbers of F^- and F^+ are well-defined. Since $F^-_\theta(x) \leq F_\theta(x) \leq F^+_\theta(x)\ \forall (\theta, x) \in \Theta \times \mathbb{R}$ it follows easily that $\rho_{\mathrm{fib}}(F) \subseteq [\rho(F^-), \rho(F^+)]$.

We obtain a homotopy F_t from F^- to F^+ by defining $F_{t,\theta}(x) := (F_\theta)_t(x)$, where $(x, t, G) \mapsto G_t(x)$ is the mapping provided by Proposition 3.1 . Note that each F_t is continuous, because F_θ depends continuously on θ and the mapping $(x, t, G) \mapsto G_t(x)$ is continuous. Since $t \mapsto F_t$ is continuous and monotone (by property (iii) of the proposition), and as the fibred rotation number depends continuously on the system, the mapping $t \mapsto \rho(F_t)$ is a continuous and monotonically increasing function. Therefore, it maps the interval $[0, 1]$ surjectively onto $[\rho(F^-), \rho(F^+)]$. Consequently, for any fixed $\rho \in [\rho(F^-), \rho(F^+)]$ there exists some $t = t(\rho) \in [0, 1]$, such that $\rho(F_t) = \rho$. Fixing any $\theta_0 \in \Theta$ and applying Lemma 3.2 with $G_n = F_{t,r^n(\theta_0)}$, we obtain an $x_0 \in \mathbb{R}$ with $F^n_{t,\theta_0}(x_0) \notin \mathcal{U}(F_{t,r^n(\theta_0)})\ \forall n \in \mathbb{N}_0$. By property (iv) in Proposition 3.1, we have $\{x \in \mathbb{R} \mid F_{t,\theta}(x) \neq F_\theta(x)\} \subseteq \mathcal{U}(F_{t,\theta})\ \forall \theta \in \Theta$. Therefore $F^n_{t,\theta_0}(x_0) = F^n_{\theta_0}(x_0)\ \forall n \in \mathbb{N}_0$, which means that the orbits of (θ_0, x_0) under the maps F_t and F coincide. Hence

$$\lim_{n\to\infty} \frac{1}{n}(F^n_{\theta_0}(x_0) - x_0) \ = \ \lim_{n\to\infty} \frac{1}{n}(F^n_{t,\theta_0}(x_0) - x_0) \ = \ \rho(F_t) \ = \rho \ .$$

This shows that ρ is contained in $\rho_{\mathrm{fib}}(F)$, and since $\rho \in [\rho(F^-), \rho(F^+)]$ was arbitrary we obtain $\rho_{\mathrm{fib}}(F) = [\rho(F^-), \rho(F^+)]$.

Furthermore, by continuity it follows that $F_\theta(x) = F_{t,\theta}(x)$ for all (θ, x) in the set

$$A \ := \ \mathrm{cl}\left(\{F^n_{\theta_0}(x_0 + k) \mid n \in \mathbb{N}_0, k \in \mathbb{Z}\}\right) \ ,$$

where $\mathrm{cl}(\cdot)$ denotes the topological closure. If we define M_ρ as the omega limit set of $\pi(\theta_0, x_0)$, that is

$$M_\rho \ = \ \cap_{n\geq 0}\mathrm{cl}(\{f^k \circ \pi(\theta_0, x_0) \mid k \geq n\}) \ ,$$

then clearly $\pi^{-1}(M_\rho) \subseteq A$. Hence the restrictions of F and F_t to $\pi^{-1}(M_\rho)$ coincide. It follows that the quantities $\frac{1}{n}(F^n_\theta(x) - x)$ converge uniformly to ρ on $\pi^{-1}(M_\rho)$ as n tends to infinity, since this is true for the quantities $\frac{1}{n}(F^n_{t,\theta}(x) - x)$ by Theorem 3.3 . □

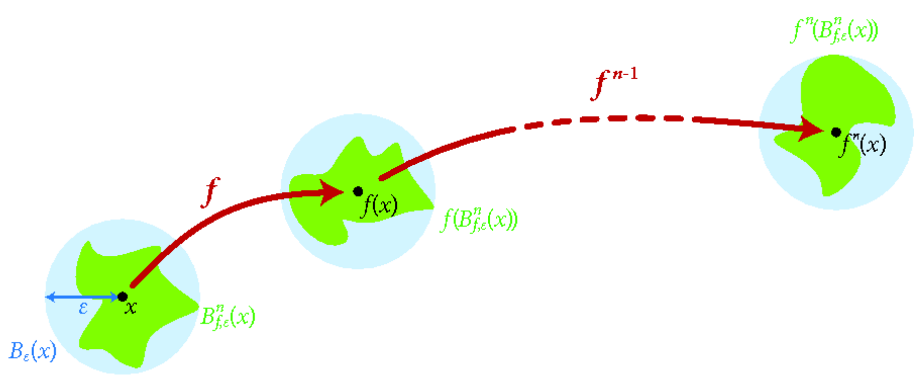


Figure 4: Definition of the (f,n,ε)-balls $B^f_{n,\varepsilon}(x)$.

Remark 3.4. *Note that the minimal sets M_ρ of Theorem 2.2 have to project down to a minimal set for the underlying transformation $r : \Theta \to \Theta$. Since we assume r to be uniquely ergodic, the only such minimal set is the topological support* $\mathrm{supp}(\mu)$ *of the unique r-invariant probability measure μ. Thus for any $\rho_1, \rho_2 \in \rho_{\mathrm{fib}}(F)$ we have $\pi_1(M_{\rho_1}) \cap \pi_1(M_{\rho_2}) =$* $\mathrm{supp}(\mu) \neq \emptyset$. *In particular, if r is an irrational rotation, $\pi_1(M_\rho) = \mathbb{T}^1$ for any $\rho \in \rho_{\mathrm{fib}}(F)$.*

Remark 3.5. *Suppose $\rho \in \rho_{\mathrm{fib}}(F)$ and $t = t(\rho)$ and F_t are chosen as in the above proof. Denote by f_t the UEF monotone circle map induced by F_t. Then the minimal sets M_ρ defined in the above proof have the property that they do not intersect the set of plateaus of f_t, that is M_ρ is disjoint from $\pi(\mathcal{V}(F_t))$ where*

$$\mathcal{V}(F_t) := \bigcup_{\theta \in \Theta} \{\theta\} \times \mathcal{U}(F_{t,\theta}) \ .$$

By invariance, M_ρ is also disjoint from all the preimages $f^{-n}(\pi(\mathcal{V}(F)))$, $n \in \mathbb{N}$. This will become important in the proof of Theorem 2.6 .

4 Topological Entropy: Proof of Theorem 2.3

First, we briefly review the definition of topological entropy, following [27] (see also [26]). Suppose (X,d) is a compact metric space and $f : X \to X$ is a continuous map. Then a sequence of metrics on X is given by $d^f_n(y,z) := \max_{i=0}^n d(f^i(y), f^i(z))$. For a given $\varepsilon > 0$, ε-balls with respect to d^f_n are called (f,n,ε)-*balls* and denoted by $B^f_{n,\varepsilon}(x)$. We let

$$R(f,n,\varepsilon) \ := \ \min\left\{ k \in \mathbb{N} \mid \exists y_1,\ldots,y_k \in X : X \subseteq \bigcup_{i=1}^k B^f_{n,\varepsilon}(y_i) \right\}$$

and

$$S(f,n,\varepsilon) \ := \ \max\left\{ k \in \mathbb{N} \mid \exists y_1,\ldots,y_k \in X : d^f_n(y_i,y_j) \geq \varepsilon \text{ if } i \neq j \right\} \ .$$

We say a finite set $S \subseteq X$ is (f,n,ε)-*separated* if $d^f_n(y,z) \geq \varepsilon \ \forall y,z \in S : y \neq z$. Then $S(f,n,\varepsilon)$ is the maximal cardinality of a (f,n,ε)-separated set in X. Similarly, $R(f,n,\varepsilon)$ is the minimal cardinality of a cover of X by (f,n,ε)-balls. It is easy to see that these quantities are non-increasing in ε and satisfy $S(f,n,2\varepsilon) \leq R(f,n,\varepsilon) \leq S(f,n,\varepsilon)$. Next, we define

$$h_\varepsilon(f) := \limsup_{n\to\infty} \frac{1}{n} \log R(f,n,\varepsilon)$$

and

$$\tilde{h}_\varepsilon(f) := \limsup_{n\to\infty} \frac{1}{n} \log S(f,n,\varepsilon) \ .$$

Again, these two quantities are non-increasing in ε, and the inequalities $\tilde{h}_{2\varepsilon}(f) \leq h_\varepsilon(f) \leq \tilde{h}_\varepsilon(f)$ hold. The topological entropy of f is defined as

$$h_{\text{top}}(f) \ := \ \lim_{\varepsilon \to 0} h_\varepsilon(f) \ = \ \sup_{\varepsilon > 0} h_\varepsilon(f) \ ,$$

and from the preceding discussion it follows that we also have

$$h_{\text{top}}(f) \ = \ \lim_{\varepsilon \to 0} \tilde{h}_\varepsilon(f) \ = \ \sup_{\varepsilon > 0} \tilde{h}_\varepsilon(f) \ .$$

We remark that replacing the metric d by another metric d' which is equivalent (meaning that d and d' induce the same topology) does not change the topological entropy. In particular, there is no need to specify below which metric we choose on the product space $\Theta \times \mathbb{T}^1$, any metric compatible with the product topology will do. However, for simplicity we will assume the metric on $\Theta \times \mathbb{T}^1$ is chosen such that $d((\theta_1, x_1), (\theta_2, x_2)) \geq \max\{d(\theta_1, \theta_2), d(x_1, x_2)\}$.

For the proof of Theorem 2.3, it will be convenient to work with a lift of the original map to a finite covering space of $\Theta \times \mathbb{T}^1$. Hence, we would like to know that this does not alter the topological entropy. We denote the k-fold cover of the circle by $\mathbb{T}^1_k = \mathbb{R}/k\mathbb{Z}$ and write $\hat{\pi}_k$ for the covering map $\hat{\pi}_k : \Theta \times \mathbb{T}^1_k \to \Theta \times \mathbb{T}^1$.

Lemma 4.1. *Suppose $f : \Theta \times \mathbb{T}^1 \to \Theta \times \mathbb{T}^1$ is continuous and homotopic to the identity, let $X = \Theta \times \mathbb{T}^1_k$ and assume $\hat{f} : X \to X$ is a lift of f. Then $h_{\text{top}}(f) = h_{\text{top}}(\hat{f})$.*

Proof. A covering of X with $(\hat{f}, n, \varepsilon)$-balls projects to a covering of $\Theta \times \mathbb{T}^1$ with (f, n, ε)-balls. Hence $R(f, n, \varepsilon) \leq R(\hat{f}, n, \varepsilon)$ and thus $h_{\text{top}}(f) \leq h_{\text{top}}(\hat{f})$.

In order to prove the converse inequality, let $\varepsilon_0 \in (0, \frac{1}{4})$ be such that $d(y, z) < \varepsilon_0$ implies $d(\hat{f}(y), \hat{f}(z)) < \frac{1}{2}$ for all $y, z \in X$. We will show that for any $0 < \varepsilon \leq \varepsilon_0$ we have $R(\hat{f}, n, \varepsilon) \leq kR(f, n, \varepsilon)$, which immediately implies $h_{\text{top}}(\hat{f}) \leq h_{\text{top}}(f)$. Fix $0 < \varepsilon \leq \varepsilon_0$, let $R := R(f, n, \varepsilon)$ and choose $y_1, \ldots, y_R \in \Theta \times \mathbb{T}^1$ such that $\Theta \times \mathbb{T}^1 \subseteq \bigcup_{i=1}^R B^f_{n,\varepsilon}(y_i)$. For any $i \in \{1, \ldots, R\}$, the point y_i has exactly k lifts z^j_i (Figure 5), with $d(z^j_i, z^l_i) \geq 1$ whenever $j \neq l$. We claim that $X \subseteq \bigcup_{i=1}^R \bigcup_{j=1}^k B^{\hat{f}}_{n,\varepsilon}(z^j_i)$, so $R(\hat{f}, n, \varepsilon) \leq kR$ as required. In order to see this, note that for any $z \in X$ we must have $\hat{\pi}_k(z) \in B^f_{n,\varepsilon}(y_i)$ for some $i \in \{1, \ldots, R\}$. In particular $\hat{\pi}_k(z) \in B_\varepsilon(y_i)$, and therefore $z \in B_\varepsilon(z^j_i)$ for some $j \in \{1, \ldots, k\}$ (Figure 5). Now $\hat{\pi}_k(z) \in B^f_{n,\varepsilon}(y_i)$ implies that $\hat{f}(z)$ is contained in one of the k ε-balls that make up $(\hat{\pi}_k)^{-1} B_\varepsilon(f(y_i))$. All of these are pairwise disjoint and have distance $\geq \frac{1}{2}$ to each other, since $\varepsilon \leq \varepsilon_0 < \frac{1}{4}$. Due to the choice of ε_0, we must have $\hat{f}(z) \in B_\varepsilon(\hat{f}(z^j_i))$. By induction on m, we thus obtain $\hat{f}^m(z) \in B_\varepsilon(\hat{f}^m(z^j_i))$ for all $m = 0, \ldots, n$. Hence $z \in B^{\hat{f}}_{n,\varepsilon}(z^j_i)$. As $z \in X$ was arbitrary, this completes the proof. □

Proof of Theorem 2.3 . Suppose f is a UEF circle endomorphism, homotopic to the identity. Further, assume F is a lift of f and the rotation interval $\rho_{\text{fib}}(F)$ has non-empty interior. We will work with a lift $\hat{f} : X \to X$ to the finite covering space $X = \Theta \times \mathbb{T}^1_4$ and show that the numbers $S(\hat{f}, n, 1)$ grow exponentially.

For any $\rho \in \rho_{\text{fib}}(F)$, let M_ρ be the minimal set provided by Theorem 2.2. Choose $\rho_1, \rho_2 \in \rho_{\text{fib}}(F)$ with $\rho_2 > \rho_1$ and let $\epsilon = \frac{1}{4}(\rho_2 - \rho_1)$. Recall that $\pi_1(M_{\rho_1}) \cap \pi_1(M_{\rho_2}) \neq \emptyset$ (Remark 3.4). By the uniform convergence of the quantities $\frac{1}{n}(F^n_\theta(x) - x)$ on M_{ρ_1} and M_{ρ_2} there exists $N \in \mathbb{N}$ such that for any $\theta \in \pi_1(M_{\rho_1}) \cap \pi_1(M_{\rho_2})$ we have for all $n \geq N$

$$|F^n_\theta(x_1) - x_1 - n\rho_1| < n\epsilon$$
$$|F^n_\theta(x_2) - x_2 - n\rho_2| < n\epsilon$$

for any x_1, x_2 such that $(\theta, x_1) \in \pi^{-1}(M_{\rho_1})$ and $(\theta, x_2) \in \pi^{-1}(M_{\rho_2})$. Thus

$$F^n_\theta(x_1) - x_1 > n\rho_1 - n\epsilon$$
$$F^n_\theta(x_2) - x_2 < n\rho_2 + n\epsilon$$

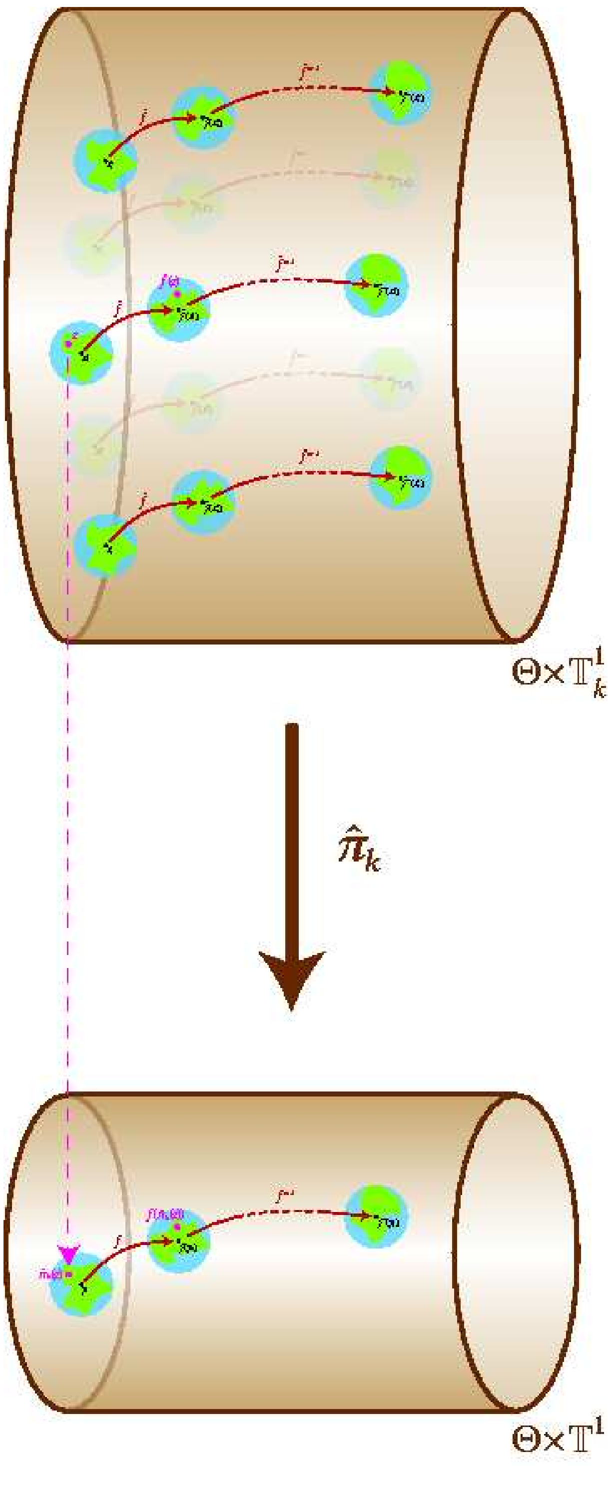


Figure 5: Illustration of the proof of Lemma 4.1, with $z \in z_1^2$

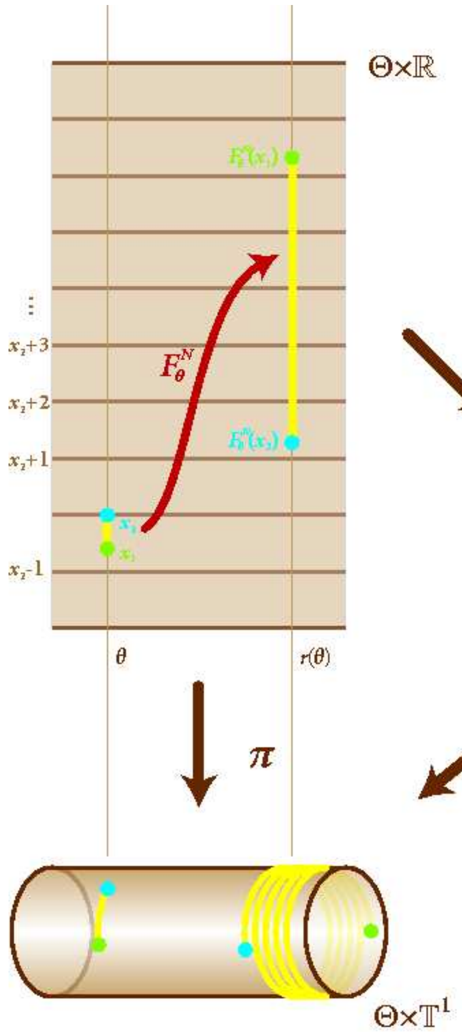


Figure 6: Illustration of the proof of Theorem 2.3

so that

$$F_\theta^n(x_1) - x_1 + n\rho_2 + n\epsilon > F_\theta^n(x_2) - x_2 + n\rho_1 - n\epsilon$$

and hence (recall that $\rho_2 - \rho_1 = 4\epsilon$):

$$\begin{aligned} F_\theta^n(x_1) - F_\theta^n(x_2) &> x_1 - x_2 + n(\rho_1 - \rho_2 - 2\epsilon) \\ &> x_1 - x_2 + 2n\epsilon \end{aligned}$$

By 3.1 we can take $x_2 - 1 < x_1 < x_2$ without loss of generality. We then choose N sufficiently large such that $2N\epsilon > 5$ and hence

$$F_\theta^N(x_1) - F_\theta^N(x_2) > 4$$

for any x_1, x_2 such that $(\theta, x_1) \in \pi^{-1}(M_{\rho_1})$ and $(\theta, x_2) \in \pi^{-1}(M_{\rho_2})$ and $x_2 - 1 < x_1 < x_2$. This implies that for any $\theta \in \pi_1(M_{\rho_1}) \cap \pi_1(M_{\rho_2})$ the map $\hat{f}_\theta^N$ sends each of the intervals $I_i := [i-1, i] \subseteq \mathbb{T}^1_4,\ i = 1, \ldots, 4$ surjectively onto $\mathbb{T}^1_4$ (Figure 6). For any such θ and any finite sequence $\sigma \in \{1,3\}^{n+1}, n \in \mathbb{N}$ of the symbols 1 and 3 define the set (Figure 7)

$$I_\sigma^n := \cap_{i=0}^n (\hat{f}_\theta^{iN})^{-1}(I_{\sigma_i}) \tag{4.1}$$

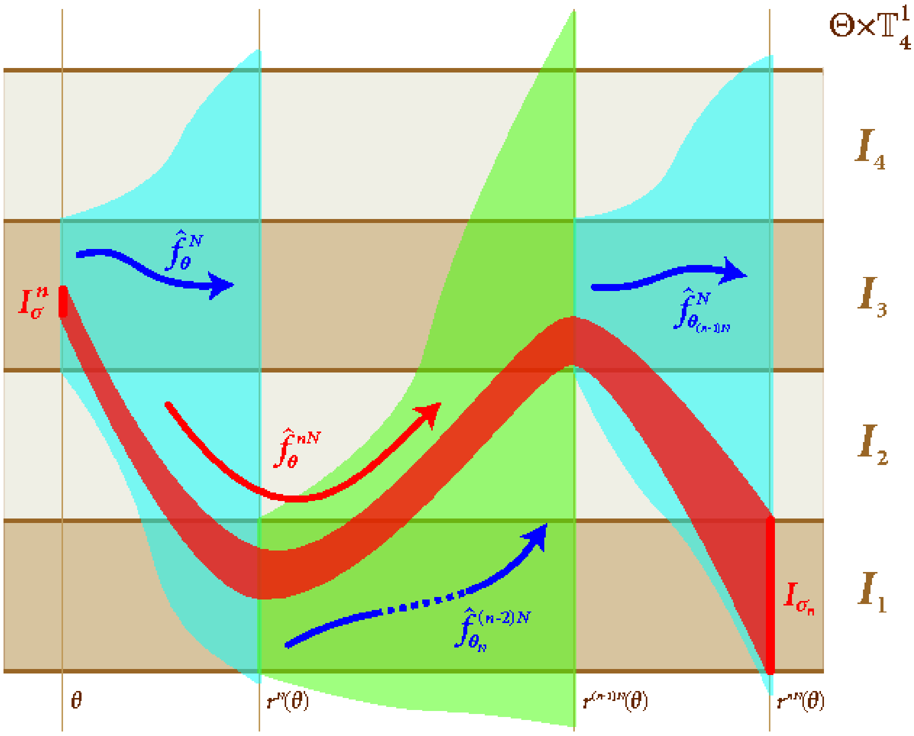


Figure 7: Construction of the Set I^n_σ defined by (4.1). The map $\hat{f}^N_\theta$ maps any of the intervals $I_1, \ldots, I_4$ at least once around the whole of $\mathbb{T}^1_4$.

By definition, $I^0_\sigma = I_{\sigma_0}$ and by the above $I_{\sigma_1} \subset \hat{f}^N_\theta(I_{\sigma_0})$. Hence $\hat{f}^N_\theta(I^1_\sigma) = I_{\sigma_1}$. Similarly $I_{\sigma_2} \subset \hat{f}^N_\theta(I_{\sigma_1}) = \hat{f}^N_\theta(\hat{f}^N_\theta(I^1_\sigma)) = \hat{f}^{2N}_\theta(I^1_\sigma)$ and so $\hat{f}^{2N}_\theta(I^2_\sigma) = I_{\sigma_2}$. Continuing by induction we see that

$$\hat{f}^{iN}_\theta(I^n_\sigma) \; = \; I_{\sigma_i} \; .$$

and in particular, I^n_σ is non-empty for any $n \in \mathbb{N}$. Clearly, for any $x \in I^n_\sigma$ and $x' \in I^n_{\sigma'}$ with $\sigma \neq \sigma'$, the points (θ, x) and (θ, x') are $(\hat{f}^N, n, 1)$-separated. Thus $S(\hat{f}^N, n, 1) \geq 2^{n+1}$. But $S(\hat{f}, nN, 1) \geq S(\hat{f}^N, n, 1)$ and therefore

$$\begin{aligned} \tilde{h}_1(\hat{f}) &= \limsup_{n\to\infty} \frac{1}{n} \log S(\hat{f}, n, 1) \\ &\geq \limsup_{n\to\infty} \frac{1}{nN} \log 2^{n+1} \;=\; \frac{\log 2}{N} \;>\; 0 \; . \end{aligned}$$

Since by definition $h_{\text{top}}(\hat{f}) \geq = h_1(\hat{f})$ and $h_{\text{top}}(f) = h_{\text{top}}(\hat{f})$ by Lemma 4.1, this completes the proof.

□

The above proof shows that the positive entropy of f is even realised on single fibres, meaning that for suitable $\theta \in \Theta$ we can find an exponentially growing number of (f, n, ε)-separated points contained in $\{\theta\} \times \mathbb{T}^1$. However, this is by no means surprising. In fact, when $h_{\text{top}}(r) = 0$, as in the quasiperiodically forced case, it is the only way to obtain positive topological entropy for the skew-product transformation. This follows from a well-known result by Bowen. In order to state it, we have to introduce the topological entropy of a subset $K \subseteq X$, where as at the beginning of this section we assume that X is a compact metric space. We let

$$R(f, K, n, \varepsilon) \;:=\; \min \left\{ k \in \mathbb{N} \mid \exists y_1, \ldots, y_k \in X : K \subseteq \bigcup_{i=1}^k B^f_{n,\varepsilon}(y_i) \right\}$$

and then define $h_\varepsilon(f,K) := \lim_{n\to\infty} \frac{1}{n}\log R(f,K,n,\varepsilon)$ and $h_{\rm top}(f,K) := \lim_{\varepsilon\to 0} h_\varepsilon(f,K)$. The numbers $S(f,K,n,\varepsilon)$ and $\tilde{h}_\varepsilon(f,K)$ are defined similarly, as above.

Theorem 4.2 (Bowen [27]). *Suppose X,Z are compact metric spaces, $r : X \to X$, $f : Z \to Z$ and $p : Z \to X$ are continuous maps, with p surjective and $p\circ f = r\circ p$. Then*

$$h_{\rm top}(f) \ \leq \ h_{\rm top}(r) + \sup_{y\in X} h_{\rm top}(f,p^{-1}\{y\}) \ .$$

Hence, if f is a UEF circle endomorphism and $h_{\rm top}(r) = 0$, then $h_{\rm top}(f) > 0$ implies that there exists some $\theta\in\Theta$ with $h_{\rm top}(f,\{\theta\}\times\mathbb{T}^1) > 0$. Conversely, if all fibre maps are monotone then the above theorem easily entails the following

Corollary 4.3. *Suppose f is a UEF monotone circle map. Then $h_{\rm top}(f) = h_{\rm top}(r)$. In particular, if f is a QPF monotone circle map, then $h_{\rm top}(f) = 0$. Note that f need not necessarily be homotopic to the identity.*

Proof. For any $\theta\in\Theta$, let $\mathbb{T}_\theta := \{\theta\}\times\mathbb{T}^1$. In view of Theorem 4.2, we only have to prove that

$$h_{\rm top}(f,\mathbb{T}_\theta) = 0 \quad \forall\theta\in\Theta \ .$$

In order to do so, we will show that the numbers $R(f,\mathbb{T}_\theta,n,\varepsilon)$ can grow at most linearly with n. To that end, fix $\theta_0\in\Theta$ and $\varepsilon > n$. By compactness, there exists a finite cover of $\Theta\times\mathbb{T}^1$ by boxes $A^i_j = A_i\times I_j$, $(i,j = 1,\ldots,N)$, with the following properties:

(i) each set A^i_j has diameter less then ε;

(ii) there exist $a_0 < a_1 <,\ldots,< a_N = a_0 \in\mathbb{T}^1$, such that $I_j = [a_{j-1},a_j]$;

(iii) for all $m\in\mathbb{N}$ and $j = 1,\ldots,N$, the point a_j has a unique preimage under the map $f^m_{\theta_0}$;

Concerning (iii) note that, due to monotonicity, for each fibre map f_θ the set of points on which f_θ is not injective is an at most countable union of intervals, and each of these intervals is mapped to a single point. Consequently, for any m there is an at most countable set E_m of exceptional points, whose preimage under $f^m_{\theta_0}$ is not unique. It suffices to choose the a_j outside the resulting countable union $\bigcup_{m\in\mathbb{N}} E_m$.

We denote by $\mathcal{A}^n$ the n-th refinement of the cover $\mathcal{A} = \{A^i_j \mid i,j\in\{1,\ldots,N\}\}$, that is

$$\mathcal{A}^n \ := \ \left\{\alpha\subseteq\Theta\times\mathbb{T}^1 \ \middle| \ \alpha = \bigcap_{k=0}^{n} f^{-k}\left(A^{i_k}_{j_k}\right), \ i_k,j_k\in\{1,\ldots,N\} \ \forall k = 0,\ldots,n\right\} \ .$$

By $\mathcal{A}^n_{\theta_0}$, we denote the restriction of $\mathcal{A}^n$ to the θ_0-fibre, that is

$$\mathcal{A}^n_{\theta_0} \ := \ \{\alpha\cap\mathbb{T}_{\theta_0} \mid \alpha\in\mathcal{A}^n\}\setminus\{\emptyset\} \ .$$

Choose points $x_\beta\in\Theta\times\mathbb{T}^1$, such that $x_\beta\in\beta \ \forall\beta\in\mathcal{A}^n_{\theta_0}$. Since the sets A^i_j all have diameter less than ε, we obtain

$$\mathbb{T}_{\theta_0} \ \subseteq \ \bigcup_{\beta\in\mathcal{A}^n_{\theta_0}} B^f_{n,\varepsilon}(x_\beta) \ .$$

Consequently, $R(f,\mathbb{T}_{\theta_0},n,\varepsilon) \leq \#\mathcal{A}^n_{\theta_0}$ where $\#\mathcal{A}^n_{\theta_0}$ denotes the number of elements in the partition $\mathcal{A}^n_{\theta_0}$. However, if $k\in\{0,\ldots,n\}$ is fixed, then due to montonicity and properties (ii) and (iii) above the preimages of the intervals I_j under the map $f^k_{\theta_0}$ are all intervals with pairwise disjoint interiors. It follows that $\mathcal{A}^n_\theta$ is a partition of the circle $\mathbb{T}_{\theta_0}$, given by the points $(\theta_0,(f^k_{\theta_0})^{-1}(a_j))$, $j = 1,\ldots,N, k = 0,\ldots,n$. This implies that $\#\mathcal{A}^n_\theta \leq (n+1)N$, and consequently $R(f,\mathbb{T}_{\theta_0},n,\varepsilon) \leq (n+1)N$. Since $\theta_0\in\Theta$ was arbitrary, this completes the proof.

□

Proof of Theorem 2.2 (ii) . This follows immediately from Corollary 4.3. Recall from the proof of part (i) that the orbit under F of any point in M_ρ coincides with the orbit under the UEF monotone map $F_{t(\rho)}$. Hence $h_{\rm top}(f|_{M_\rho}) \leq h_{\rm top}(f_{t(\rho)})$, where $F_{t(\rho)}$ is a lift of $f_{t(\rho)}$. But, since $F_{t(\rho)}$ is monotone we have $h_{\rm top}(f_{t(\rho)}) = 0$, as required.

□

Figure 8: Essentially bounded sets on the torus. Green sets are essentially bounded, red sets are not.

5 Strangely Dispersed Minimal Sets: Proof of Theorem 2.6

Let $q : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the quotient map. We call a subset $E \subseteq \mathbb{T}^2$ *essentially bounded*, if all connected components of $q^{-1}(E)$ are bounded, Figure 8. The following proposition will be the key ingredient in the proof of Theorem 2.6 .

Proposition 5.1. *Suppose f is a QPF circle endomorphism, homotopic to the identity. Further, assume $E \subseteq \mathbb{T}^2$ is open and essentially bounded and $M \subseteq E$ is a minimal set. Then M is strangely dispersed.*

Proof. Since M is minimal by assumption, it remains to show that it has properties (ii) and (iii) in Definition 2.4 .

In order to see that connected components of M are contained in single fibres, suppose that $\hat{E}_0$ is a connected component of $\hat{E} := q^{-1}(E)$ such that $\hat{M}_0 := q^{-1}(M) \cap \hat{E}_0 \neq \emptyset$. Since E is essentially bounded, $\hat{E}_0$ is bounded and hence $\hat{M}_0$ is compact. Thus the first coordinate of points in $\hat{M}_0$ attains a minimal value, at say $(\hat{\theta}_0, \hat{x}_0) \in \hat{M}_0$ (more precisely $\hat{\theta}_0 = \inf\{\theta \in \mathbb{R} \mid \exists x \in \mathbb{R} : (\theta, x) \in \hat{M}_0\}$). Let $(\theta_0, x_0) := q(\hat{\theta}_0, \hat{x}_0)$ and $E_0 := q(\hat{E}_0)$. Observe that $E_0 \subseteq E$ is a connected component of E and in particular, E_0 is open. Also $(\theta_0, x_0) \in M$ and $(\theta_0, x_0) \in E_0$

Now, assume that $C \subseteq M$ is a connected component of M which is not contained in a single fibre. Then $\pi_1(C)$ is connected and hence an interval of positive length, say $\pi_1(C) = [a, b]$ with $\delta = d(a, b) > 0$. We assume for simplicity of exposition that $d(a, b) < \frac{1}{2}$. Choose $(\theta, x) \in C$ with $d(\theta, a) = d(\theta, b) = \delta/2$. Observe that for any $n \in \mathbb{N}$, we have $\pi_1(f^n(C)) = [r^n(a), r^n(b)]$ which also has length δ, and $d(\theta_n, a) = d(\theta_n, b) = \delta/2$ where $(\theta_n, x_n) = f^n(\theta, x)$.

Since M is minimal, the orbit of (θ, x) is dense in M. By the above, E_0 is open and contains $(\theta_0, x_0) \in M$, so that there exists some $n \in \mathbb{N}$, such that $f^n(\theta, x) \in E_0 \cap B_{\delta/4}(\theta_0, x_0)$. The set $f^n(C)$ is connected and $f^n(C) \subseteq M \subseteq E$ for all $n \in \mathbb{N}$. Hence $f^n(C)$ is contained in a connected component of E. Since $f^n(C)$ contains $f^n(\theta, x) \in E_0$ this connected component must by E_0, that is $f^n(C) \subseteq E_0$.

Define $\hat{D}_0$ as the unique connected component of $q^{-1}(f^n(C))$ that contains the unique point $(\hat{\theta}^*, \hat{x}^*)$ in $q^{-1}\{f^n(\theta, x)\} \cap B_{\delta/4}(\hat{\theta}_0, \hat{x}_0)$. Then $(\hat{\theta}^*, \hat{x}^*) \in \hat{E}_0$, and by connectedness $\hat{D}_0 \subseteq \hat{E}_0$ and hence $\hat{D}_0 \subseteq \hat{M}_0$. Since $q(\hat{D}_0) = f^n(C)$, $q(\hat{\theta}^*, \hat{x}^*) = f^n(\theta, x) = (\theta_n, x_n)$ and $\pi_1(f^n(C)) = [\theta_n - \delta/2, \theta_n + \delta/2]$ we have $\pi_1(\hat{D}_0) = [\hat{\theta}^* - \delta/2, \hat{\theta}^* + \delta/2]$.

But recall that $\hat{\theta}_0 = \inf\{\theta \in \mathbb{R} \mid \exists x \in \mathbb{R} : (\theta, x) \in \hat{M}_0\}$ and since $\hat{D}_0 \subseteq \hat{M}_0$ we must have $\hat{\theta}_0 \leq \hat{\theta}^* - \delta/2$. On the other hand $(\hat{\theta}^*, \hat{x}^*) \in B_{\delta/4}(\hat{\theta}_0, \hat{x}_0)$, so that $\hat{\theta}^* \leq \hat{\theta}_0 + \delta/4$ which implies that $\hat{\theta}^* - \delta/2 \leq \hat{\theta}_0 - \delta/4$. Combining these two inequalities yields the contradiction

$$\hat{\theta}_0 \;\leq\; \hat{\theta}^* - \delta/2 \;\leq\; \hat{\theta}_0 - \delta/4 \; .$$

Hence any connected component of M must be contained in a single fibre, proving property (ii).

It remains to prove that for any $(\theta, x) \in M$ and any open neighbourhood U of (θ, x), the set $\pi_1(U \cap M)$ contains a non-empty open interval, i.e. property (iii). First observe that if this property holds for some $(\theta, x) \in M$ then it holds for $f(\theta, x)$ (and hence $f^n(\theta, x)$ for any $n \in \mathbb{N}$). To see this, let U be an open neighbourhood of (θ, x). Then $f^{-1}(U)$ is an open neighbourhood of (θ, x), and hence $\pi_1(f^{-1}(U) \cap M)$ contains a non-empty open interval (a, b). Hence $\pi_1(U \cap M)$ contains $(r(a), r(b))$ which has the same length as (a, b) and hence is a non-empty open interval. Also, property (iii) is closed, that is if it holds for a convergent sequence of points $(\theta_i, x_i) \in M$ with $(\theta_i, x_i) \to (\theta, x)$ then it holds for the limit point (θ, x). This is because if U is an open neighbourhood U of (θ, x) then it is an open neighbourhood of (θ_i, x_i) for all sufficiently large i.

Thus, property (iii) is both closed and invariant and hence, it either holds for all or for no point in M since by minimality the only closed invariant subsets of M are the empty set and M itself. Arguing by contradiction, let us assume that every $z \in M$ has a neighbourhood $U(z)$, such that $\pi_1(U(z) \cap M)$ contains no open interval and hence is nowhere dense. By compactness, M is covered by a finite number $U(z_1), \ldots, U(z_N)$ of such neighbourhoods. However, this would imply that $\pi_1(M)$ is the union of a finite number of nowhere dense sets and hence is itself nowhere dense. This is clearly a contradiction, since $\pi_1(M)$ must be the whole circle, because this is the only closed invariant set of the underlying irrational rotation.

□

Proof of Theorem 2.6. Suppose f is given by (1.1), and consequently has a lift F with fibre maps

$$F_\theta(x) \; = \; x + \tau + \frac{\alpha}{2\pi}\sin(2\pi x) + \beta \sin(2\pi\theta) \; .$$

Part (a). Recall that the QPF plateau maps F_t in the proof of Theorem 2.2 were given by $F_{t,\theta} := (F_\theta)_t$, with the mapping $[0,1] \times \mathcal{E},\ (t, G) \mapsto G_t$ provided by Proposition 3.1 . Any F_t induces a QPF monotone circle map, which we will denote by f_t. If we let

$$G(x) \; := \; x + \tau + \frac{\alpha}{2\pi}\sin(2\pi x) \; ,$$

then $F_{t,\theta}(x) = G_t(x) + \beta\sin(2\pi\theta)$. In particular, the plateaus of $F_{t,\theta}$ do not depend on θ. The fact that f in (1.1) is bimodal further implies that these plateaus are unique modulo addition of integers, that is $\mathcal{U}(G_t) = \bigcup_{n\in\mathbb{Z}} I_t + n$ for some interval $I_t \subseteq \mathbb{R}$. Hence, recalling Remark 3.5, we have (Figure 9a))

$$\mathcal{V}(F_t) \; = \; \bigcup_{\theta\in\mathbb{T}^1} \{\theta\} \times \mathcal{U}(F_{t,\theta}) \; = \; \bigcup_{k\in\mathbb{Z}} \mathbb{T}^1 \times (I_t + k) \; .$$

Now suppose that $\rho \in \rho_{\text{fib}}(F)$ and denote by M_ρ the minimal set obtained in the proof of Theorem 2.2. Let $t = t(\rho) \in [0,1]$ be the corresponding parameter such that $\rho(F_t) = \rho$ and M_ρ is a F_t-minimal set. As mentioned in Remark 3.5, M_ρ is disjoint from the set $\bigcup_{n\in\mathbb{N}} f^{-n}(\pi(\mathcal{V}(F_t))$. Now if $I' = [a,b] \subseteq I_t$ is a closed interval let

$$E \; := \; \mathbb{T}^2 \setminus \left(\mathbb{T}^1 \times \pi(I') \cup f_t^{-1}(\mathbb{T}^1 \times \pi(I'))\right) \; ,$$

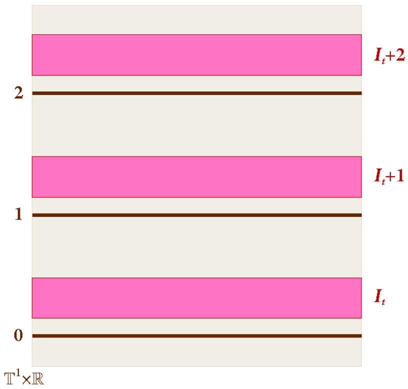
as indicated in Figure 9(b,c). Then $M_\rho \subseteq E$, and in view of Proposition 5.1 we only have to show that E is essentially bounded. Since the complement $q^{-1}(E)^c$ of $q^{-1}(E)$ contains the horizontal line $\mathbb{R} \times \{a\}$ and all its integer translates, it is obvious that all connected components of $q^{-1}(E)$ are bounded in the vertical direction.

We also claim that $q^{-1}(E)^c$ contains a continuous curve joining $\mathbb{R}\times\{a\}$ and $\mathbb{R}\times\{a+1\}$, which implies immediately that it is bounded horizontally. We in fact show that $V := \pi^{-1}(E)$ contains a continuous curve joining $\mathbb{T}^1\times\{a+k\}$ and $\mathbb{T}^1\times\{a+k+1\}$ for some $k \in \mathbb{Z}$. Observe that $F_t^{-1}(\mathbb{T}^1 \times I')$ contains a curve Γ that is the graph $\Gamma := \{(\theta, \gamma(\theta) \mid \theta \in \mathbb{T}^1\}$ of a continuous function $\gamma : \mathbb{T}^1 \to \mathbb{R}$ (Figure 9d). Since Γ is mapped into $\mathbb{T}^1 \times I'$ we have

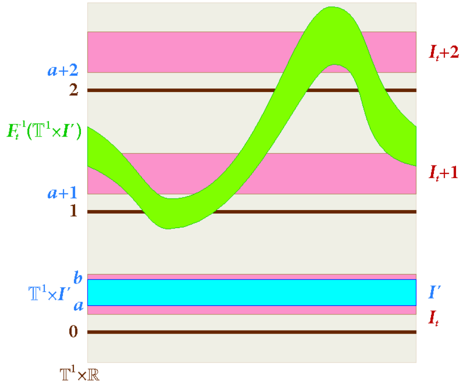
$$F_{t,\theta}(\gamma(\theta)) \; = \; G_t(\gamma(\theta)) + \beta\sin(2\pi\theta) \; \in I' \quad \forall \theta \in \mathbb{T}^1 \; .$$

Since we assume that $\beta \geq \frac{3}{2}$ we have

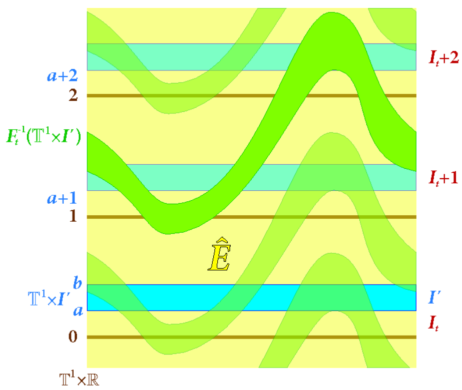
$$F_{t,\theta}(\gamma(1/4)) = G_t(\gamma(1/4))) + \beta \geq G_t(\gamma(1/4)) + \frac{3}{2}$$
$$F_{t,\theta}(\gamma(3/4)) = G_t(\gamma(3/4)) - \beta \leq G_t(\gamma(3/4)) - \frac{3}{2}.$$



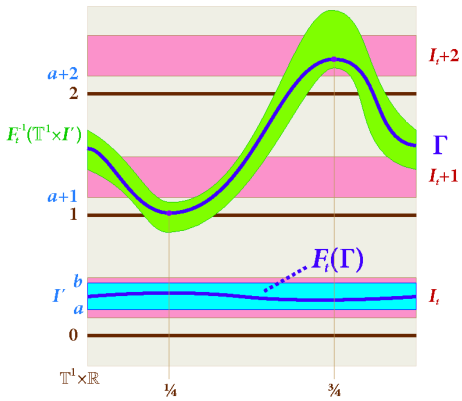
(a) The set $\mathcal{V}(F_t)$



(b) The sets $\mathbb{T}^1 \times I'$ and $F_t^{-1}(\mathbb{T}^1 \times I')$.



(c) The set $\hat{E}$, coloured yellow, which is the lift of $E = \mathbb{T}^2 \setminus \left(\mathbb{T}^1 \times \pi(I') \cup f_t^{-1}(\mathbb{T}^1 \times \pi(I'))\right)$



(d) The curve Γ in $F_t^{-1}(\mathbb{T}^1 \times I')$ and its image under F_t in $\mathbb{T}^1 \times I'$.

Figure 9: Proof of Theorem 2.6. Construction of the set E. This is given by the projection to $\mathbb{T}^2$of the complement, coloured yellow, of $\mathbb{T}^1 \times I'$ (blue) and $F_t^{-1}(\mathbb{T}^1 \times I')$ (green) in b).

Since the length of I' is less than 1 this implies

$$1 > G_t(\gamma(1/4)) + \frac{3}{2} - G_t(\gamma(3/4)) + \frac{3}{2}$$

and hence

$$G_t(\gamma(3/4)) - G_t(\gamma(1/4)) > 2.$$

Since G_t is monotone and $G_t(x+n) = G_t(x) + n \ \forall n \in \mathbb{Z}$, if $\gamma(3/4) \leq \gamma(1/4) + 2$ then $G_t(\gamma(3/4)) \leq G_t(\gamma(1/4)+2) = G_t(\gamma(1/4))+2$, so that $G_t(\gamma(3/4)) - G_t(\gamma(1/4)) \leq 2$. Hence we must have $\gamma(3/4) > \gamma(1/4) + 2$, or in other words

$$\gamma(1/4) - \gamma(3/4) \geq 2.$$

Hence there exists $k \in \mathbb{Z}$, such that Γ intersects both $\mathbb{T}^1 \times \{a+k\}$ and $\mathbb{T}^1 \times \{a+k+1\}$. This proves our claim.

Part (b) Recall from the proof of Theorem 2.2 that $\rho_{\text{fib}}(F) = [\rho_1, \rho_2]$, where $\rho_1 = \rho(F^-)$ and $\rho(F_2) = \rho(F^+)$. For any $x \in \mathbb{R}$, let $x^- := \inf\{y \in \mathbb{Z} + \frac{3}{4} \mid y \geq x\}$ and $x^+ := \sup\{y \in \mathbb{Z} + \frac{1}{4} \mid y \leq x\}$ (Figure 10). Then $x^- = x^+ + \frac{1}{2}$ or $x^- = x^+ + \frac{3}{2}$. Note that

$$F_\theta(x^+) = x^+ + \tau + \frac{\alpha}{2\pi} + \beta \sin(2\pi\theta)$$
$$F_\theta(x^-) = x^- + \tau - \frac{\alpha}{2\pi} + \beta \sin(2\pi\theta)$$

and hence

$$F_\theta(x^+) - F_\theta(x^-) = x^+ - x^- + \frac{\alpha}{\pi}.$$

Recall from the defintion of plateau maps that if $x' \leq x$ then $F_\theta^-(x') \leq F(x)$ and $F_\theta^+(x) \geq F(x')$. Since $x^+ \leq x \leq x^-$ we have

$$F_\theta^+(x) \geq F_\theta(x^+)$$
$$F_\theta(x^-) \geq \ F_\theta^-(x)$$

for all $(\theta, x) \in \mathbb{T}^1 \times \mathbb{R}$. Thus if $\alpha \geq \frac{5}{2}\pi$, it follows that

$$F_\theta^+(x) \ \geq F_\theta(x^+) \ \geq \ F_\theta(x^-) + \frac{5}{2} - (x^- - x^+) \ \geq \ F_\theta^-(x) + 1 \ .$$

Hence we have $F_\theta^+(x) \geq F_\theta^-(x) + 1$ for all $(\theta, x) \in \mathbb{T}^1 \times \mathbb{R}$, which implies $\rho_2 \geq \rho_1 + 1$. Hence $\rho_{\text{fib}}(F) = [\rho_1, \rho_2]$ has positive length, as required.

□

References

[1] Grebogi, C., E. Ott, S. Pelikan and J.A. Yorke Strange attractors which are nor chaotic, *Physica D* 13:261–268, 1984.

[2] Ding, M., C. Grebogi and E. Ott Evolution of attractors in quasiperiodically forced systems - from quasiperiodic to strange nonchaotic to chaotic. *Phys. Rev. A* 39:2593–2598, 1989.

[3] Romeiras, F.J., A. Bondeson, E. Ott, T.M. Antonsen Jr. and C. Grebogi Quasiperiodically forced dynamical systems with strange nonchaotic attractors. *Physica D* 26:277–294, 1987

[4] Romeiras, F.J., A. Bondeson, E. Ott, T.M. Antonsen Jr. and C. Grebogi Quasiperiodic forcing and the observability of strange nonchaotic attractors *Phys. Scripta* 40:442–444, 1989.

[5] Romeiras, F.J. and E. Ott Strange nonchaotic attractors of the damped pendulum with quasiperiodic forcing. *Phys. Rev. A* 35:4404–4413, 1987.

[6] Feudel, U., J. Kurths and A.S. Pikovsky Strange nonchaotic attractor in a quasiperiodically forced circle map *Physica D* 88:176–186, 1995.

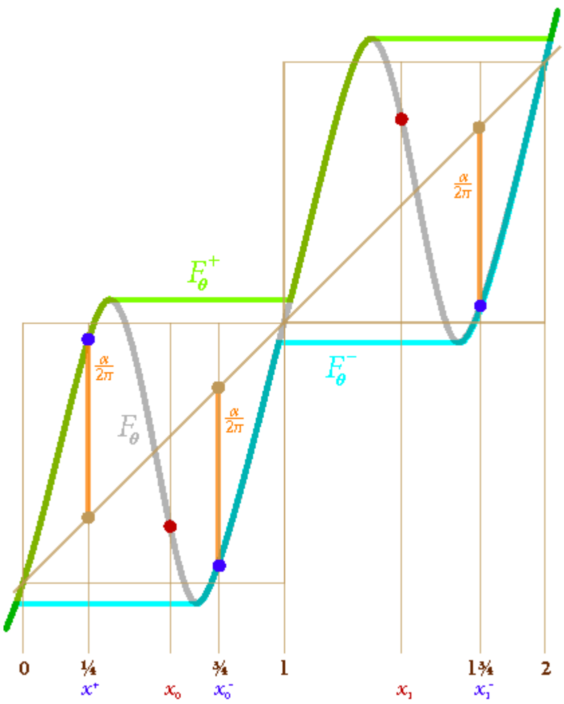


Figure 10: Proof of the fact that $F_\theta^+(x) \geq F_\theta^-(x) + \alpha/\pi - (x^- - x^+)$ We consider two cases: one where $1/4 \leq x \leq 3/4$ (shown as x_0) and the other where $3/4 \leq x \leq 7/4$ (shown as x_1). For both of these $x^+ = 1/4$, whereas $x^- = 3/4$ and $7/4$ respectively, indicated as x_0^- and x_1^- on the figure.

[7] Pikovsky, A.S. and U. Feudel Characterizing strange nonchaotic attractors. *Chaos* 5:253-260, 1995.

[8] Chastell, P. R., Glendinning, P. A., and Stark, J. Determining the locations of bifurcations in quasiperiodic systems. *Phys. Lett. A* 200: 17–26. 1995.

[9] Glendinning, P., Feudel, U,, Pikovsky, A. S., and Stark, J. The structure of mode-locked regions in quasi-periodically forced circle maps. *Physica D* 140: 227–243, 2000.

[10] Michael R. Herman. Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2. *Commentarii Mathematici Helvetici*, 58:453–502, 1983.

[11] Keller, G. A note on strange nonchaotic attractors. *Fundamenta Mathematicae* 151: 139–148, 1996.

[12] T. Jäger. On the structure of strange non-chaotic attractors in pinched skew products *Ergodic Theory and Dynamical Systems*, 27:493–510, 2007.

[13] Glendinning, P. A., T. Jäger and Keller, G. How chaotic are strange non-chaotic attractors? *Nonlinearity* 19:2005–2022, 2006.

[14] Stark, J., Invariant graphs for forced systems. *Physica D* 109: 163–179, 1997.

[15] Sturman, R., and Stark, J. Semi-uniform ergodic theorems and applications to forced systems. *Nonlinearity* 13:113–143, 2000.

[16] Kristian Bjerklöv. Positive Lyapunov exponent and minimality for a class of one-dimensional quasi-periodic Schrödinger equations. *Ergodic Theory and Dynamical Systems*, 25:1015–1045, 2005.

[17] T. Jäger. The creation of strange non-chaotic attractors in non-smooth saddle-node bifurcations. Preprint 2006, to appear in *Memoirs of the AMS*.

[18] T. Jäger. Strange non-chaotic attractors in quasiperiodically forced circle maps. Preprint 2007.

[19] Stark, J., Feudel, U., Glendinning, P. A., and Pikovsky, A. Rotation numbers for quasi-periodically forced monotone circle maps. *Dynamical Systems* 17: 1–28, 2002.

[20] Stark, J. Transitive sets for quasi-periodically forced monotone maps *Dynamical Systems* 18: 351–364, 2003.

[21] T. Jäger and G. Keller. The Denjoy type-of argument for quasiperiodically forced circle diffeomorphisms. *Ergodic Theory and Dynamical Systems*, 26(2):447–465, 2006.

[22] T. Jäger and J. Stark Towards a classification for quasiperiodically forced circle homeomorphisms. *J. London Math. Soc.*, 73:727–744, 2006.

[23] F. Beguin, S. Crovisier, T. Jäger, and F. LeRoux. Denjoy constructions for fibered homeomorphisms of the two-torus. Preprint 2007.

[24] W. Huang and Y. Yi. Almost periodically forced circle flows. Preprint 2007.

[25] MacKay, R.S. and Tresser, C Transition to Topological Chaos for Circle Maps. *Physica D* 19:206–237, 1986.

[26] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1997.

[27] Rufus Bowen. Entropy for group endomorphisms and homogeneous spaces. *Transactions of the AMS*, 153:401–413, 1971.

[28] M. Misiurewicz and K. Ziemian. Rotation sets for maps of tori. *Journal of the London Mathematical Society*, 40:490–506, 1989.

[29] M. Misiurewicz (2006) Rotation Theory. Online Proceedings of the RIMS Workshop "Dynamical Systems and Applications: Recent Progress" http://www.math.kyoto-u.ac.jp/ kokubu/RIMS2006/RIMS_Online_Proceedings.html

[30] P.L. Boyland. Bifurcations of circle maps: Arnold tongues, bistability and rotation intervals. *Communications in Mathematical Physics*, 106:353–381, 1986.