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Border collision bifurcations, snap-back repellers and chaos

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The normal form for codimension one border collision bifurcations of fixed points of discrete time piecewise smooth dynamical systems is considered in the unstable case. We show that in appropriate parameter regions there is a snap-back repeller immediately after the bifurcation, and hence that the bifurcation creates chaos. Although the chaotic solutions are repellers they may explain observations, and this is illustrated through an example.

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Many applications of computer software involve the modelling and control of systems which depend on both discrete and continuous variables, and so a good understanding of the interaction between these components of the system is important. These systems are often called hybrid or embedded systems, and there is a growing literature of both applications and theory [1–4]. From the perspective of dynamical systems a description of the simple bifurcations which can occur in these systems is important, and many examples are now well understood.

The systems considered here are piecewise smooth and have discrete time and continuous variables. There is a switching surface $\Sigma$ dividing the regions in which the dynamics is determined by smooth maps, and the equations are continuous across $\Sigma$. Thus the left and right sides of $\Sigma$ could be labelled by $L$ and $R$ respectively, and a discrete variable defined to take values in $\{L, R\}$ according to which side of $\Sigma$ the continuous variables are at time $n$. This discrete variable then determines which dynamical system is applied at the next time step.

Perhaps the most obvious codimension one bifurcation of such systems occurs if a fixed point (or periodic orbit) of the system is on (or has a point on) the switching boundary $\Sigma$. The two-dimensional normal form for this bifurcation was derived in [1, 5], and if the switching surface is transformed to be the $y$-axis ($x = 0$) then the local evolution with $\mathbf{x} = (x, y)^T$ is

$$
\mathbf{x}_{n+1} = \begin{cases} 
A_L \mathbf{x} + \mathbf{m} & \text{if } x \leq 0 \\
A_R \mathbf{x} + \mathbf{m} & \text{if } x \geq 0
\end{cases}
$$

(1)

where the matrices $A_L$ and $A_R$, and the vector $\mathbf{m}$ are defined as

$$
A_a = \begin{pmatrix} T_a & 1 \\
-D_a & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} \mu \\
0 \end{pmatrix}
$$

for $a = L, R$. The constants $T_a$ and $D_a$ are the trace and determinant of the Jacobian of the defining equations on the left and right of $\Sigma$ whilst $\mu$ is the bifurcation parameter. If $\mu = 0$ then the origin is a fixed point, and this is clearly in $\Sigma$. The question for bifurcation theory is what happens close to the origin when $|\mu|$ is small.

If $|D_R|$ and $|D_L|$ are less than one then the local dynamics which can occur has been discussed in a number of papers [1, 3, 6–9]. In this case bifurcations analogous to the standard saddlenode are possible, as is a border crossing in which the fixed point simply moves across the boundary. Depending on the values of the other constants, more complicated possibilities occur, with the creation of other periodic orbits and even chaos. Since the determinant of the Jacobian matrix of a map shows how areas are increased or decreased by iteration, the determinant less than one cases can be expected to give information about the stable dynamics which can be observed. It might be imagined that the case of a determinant with modulus greater than one is either uninteresting or could be obtained from the modulus less than one case by reversing time. However, neither of these is the case – since the map is not necessarily invertible we cannot simply reverse time, and the dynamics described below is certainly interesting and relevant to some examples.

The demonstration that there is complicated (chaotic) dynamics in the case of a determinant with modulus greater than one follows the ideas used in the study of snap-back repellers [10–13]. Since the systems considered here are not differentiable across the boundary, we give a brief description of how this theory works below. Next we prove the existence of a snap-back repeller and hence chaos in (1) for appropriate values of the constants defining $A_L$ and $A_R$ and then show how this bifurcation can be observed in an example of a blowout bifurcation.

Let $G_L$ denote the half-plane with $x \leq 0$ and $G_R$ denote the half-plane with $x \geq 0$. For a map such as (1), we will use the notation $F_L$ and $F_R$ to denote the map in $G_L$ and $G_R$ respectively. We say that the map has a simple snap-back repeller if, possibly after the transfor-
FIG. 1: The geometry of a simple snap-back repellor.

nation \( x \to -x \) (which exchanges the roles of \( L \) and \( R \)), there exists a fixed point \( x_0^L \) in \( x > 0 \) and

1. the eigenvalues \( s_{\pm} \) of \( A_R \) satisfy \( |s_+| \geq |s_-| > 1 \);
2. there is a point \( x_0^R \) in \( x < 0 \) such that \( F_L(x_0^R) = x_0^R \); and
3. there exists a sequence \( x_i^R \) in \( x > 0 \) which tends to \( x_0^R \) as \( i \to \infty \) such that \( F_L(x_{i+1}^R) = x_i^R, i = 1, 2, 3, \ldots \) and \( F_R(x_i^R) = x_{i+1}^R \).

This is shown in Figure 1. Note that the first condition implies that \( x_0^R \) is a source with a two dimensional local unstable manifold and so there can be no conventional homoclinic orbits such as those of the Lozi map.

Of course, more complicated connections are possible, with several passages across the boundary, or more than one path, but this is sufficient for our needs here.

We need to define a set of neighbourhoods of the points which define the snap-back repellor and on which the chaotic dynamics can be defined. We start by choosing a closed ball of radius \( r \) centred on \( x_0^L, B(1, r) \), and take \( r \) small enough so that \( B(1, r) \subset G_R, F_R(B(1, r)) = B(0, r) \subset G_L \) and none of the other points \( x_i^R, i = 2, 3, \ldots \), are contained in \( B(1, r) \).

Now define \( B(2, r) \) to be those points in \( G_R \) which map to \( B(1, r) \) under one iteration of \( F_R \), and again, possibly reducing the size of \( r \) this is a closed set in \( G_R \) which does not contain any points in \( \Sigma \). Note that if \( s_{\pm} \) are real and distinct then \( B(2, r) \) is an ellipse with major and minor axes of length \( |s_{\pm}|^{-1}r \). Define \( B(n, r) \) inductively so that \( F_R(B(n, r)) = B(n-1, r) \) and \( B(n, r) \) does not intersect \( \Sigma \). Note that after a finite number of steps these sets, which are ellipses with major and minor axes of length \( |s_{\pm}|^{-n+1}r \) in the case of real eigenvalues, will be sufficiently close to \( x_0^R \) and sufficiently small so that no reduction of \( r \) will be necessary.

Let \( N(r) = F_L(B(0, r)) \), which is a set in \( G_R \) (possibly having reduced \( r \) to ensure no intersection with \( \Sigma \) containing \( x_0^R \) in its interior. By definition, the sets \( B(n, r) \) converge to \( x_0^R \) and their maximal diameters tend to zero, so there exists \( K > 0 \) such that \( B(k, r) \subset N(r) \) for all \( k > K \). By construction \( F_L \circ F_R(B(k, r)) = N(r) \) and \( F_L \circ F_R \) restricted to \( B(k, r) \) is a homeomorphism (in fact, affine). Hence for every \( k > K \) there exists a closed connected set \( B(k, k_1, r) \subset B(k, r) \) such that \( F_L \circ F_R(B(k, k_1, r)) = B(k_1, r) \). The standard induction argument for dynamical systems (using the convergence of nested close sets) implies that for any \( M > 0 \) and any sequence \( k_0, k_1, k_2, \ldots \) with \( K < k_i < K + M \) there exists a non-empty set \( B(k_0, k_1, k_2, \ldots, r) \subset B(k_0, r) \) such that

\[ F_L \circ F_R^M(B(k_0, k_1, k_2, \ldots, r)) = B(k_1, k_2, \ldots, r) \]

and hence that there are infinitely many periodic points and countably many aperiodic points in the dynamics of the map close to the simple snap-back repellor.

Because the argument is so simple in this restricted case it has been worth reworking how the simple snap-back repellor implies chaos, as there has been some controversy about the original idea [10, 11], and the system (1) does not formally satisfy all the conditions usually imposed, although [13] does apply here.

Now return to the normal form (1). Suppose that \( T_n \) and \( D_n, \alpha = L, R \), are given. The fixed points of the maps are given by

\[ x^*_n = \frac{\mu}{1 - T_n + D_n}, \quad y^*_n = -D_n x^*_n, \quad \alpha = L, R \]

and \( x^*_R \) exists provided \( x^*_R > 0 \), with a similar inequality for the existence of \( x^*_L \). Given \( T_n \) and \( D_n \), these inequalities define the sign of \( \mu \) for which these fixed points exist. The fixed points in \( G_R \) and \( G_L \) coincide at the origin (on \( \Sigma \)) if \( \mu = 0 \).

The geometry of the map near the boundary controls much of what can be observed. The image of the boundary \( \Sigma \) is the \( x \)-axis, and since \( y_{n+1} = -D_n x_n, G_R \) (with \( x > 0 \)) is mapped to the upper (respectively lower) half plane if \( D_R < 0 \) (resp. \( D_R > 0 \)) and \( G_L \) is mapped to the upper (respectively lower) half plane if \( D_L > 0 \) (resp. \( D_L < 0 \)). In other words, the images of \( G_L \) and \( G_R \) overlap if \( D_L \) and \( D_L \) have opposite signs (one positive and one negative), and do not overlap if they have the same signs. This observation goes a long way towards explaining why boundary crossing occurs if the determinants have the same sign, and more complicated bifurcations can occur otherwise.

To fix ideas we will consider the case

\[ D_R > 1, \quad D_L < 0 \]

and aim to show the existence of a snap-back repellor to \( x^*_R \). For geometric simplicity we will make the further assumption that \( x_0^R \) is an unstable node, so the eigenvalues of \( A_R \) are real and distinct and greater than one. This corresponds to the additional condition

\[ T_R > 2, \quad T_R^2 > 4D_R, \quad 1 - T_R + D_R > 0 \]

which implies that the fixed point in \( G_R \) given by (4) exists if \( \mu > 0 \). Then (5) implies that the images of \( G_R \) and \( G_L \) lie in the lower half plane, and hence that there is a preimage of \( x_0^R \) in \( G_R \) if \( y^*_R < 0 \), which is automatically satisfied from (4) as \( D_R > 0 \).
A short calculation using (1) shows that this point, $x_0^L = (x_0, y_0)$ in the notation of the previous section is given by

$$x_0 = \frac{D_R}{D_L} x_1^R, \quad y_0 = \frac{1}{D_L} \left( T_R D_L - T_L D_R - D_L D_R \right) x_1^R \quad (7)$$

By definition of Figure 1 is a preimage of $x_0^R$ in $x > 0$ and for this preimage to exist we must have $y_0 < 0$ since the images of both $G_R$ and $G_L$ are in the lower half plane. Since $D_L < 0$ and $x^R > 0$ this implies that the condition

$$T_R D_L - T_L D_R - D_L D_R > 0 \quad (8)$$

must hold. In this case $x_0^R = (x_1, y_1)$ exists and by definition $x_0^R$ is mapped to it by the map in $x > 0$ so

$$x_0 = T_R x_1 + y_1 + \mu$$

$$y_0 = -D_R x_1 \quad (9)$$

and hence, using (4) and (7)

$$x_1 = -\frac{1}{D_R D_L} (T_R D_L - T_L D_R - D_L D_R) x_1^R$$

$$y_1 = \frac{1}{D_R D_L} (D_R D_L - D_L D_R + T_R (T_R D_L - T_L D_R)) x_1^R \quad (10)$$

Note that (8) ensures that this point does exist in $G_R$.

Looking back to the definition of a snap-back repeller it remains to show that $x_0^R$ lies in the two dimensional unstable manifold of $x_0^R$, i.e. if $x_0^R$ is iterated in backwards time using the map in $x > 0$ then this orbit remains in $x > 0$ and converges to $x_0^R$.

By (6) the eigenvalues and eigenvectors of the linear part of the map in $x > 0$ are

$$s_\pm = \frac{1}{2} (T_R \pm \sqrt{T_R^2 - 4 D_R}), \quad e_\pm = \left( \begin{array}{c} s_\pm \\ -D_R \end{array} \right) \quad (11)$$

with $s_\pm > 1$ and hence the eigenvectors both have negative slopes. Except for solutions on $e_+$, orbits of the linear map therefor converge in backwards time to $x_0^R$ on generalized parabolas which are tangential to $e_-$ (the eigenvalue with smaller modulus) at the fixed point, and this is the eigenvector with the steeper slope. Thus solutions in backwards time lie on curves as sketched in Figure 2, and clearly all solutions in $y < 0$ which start to the left of $e_+$ tend to the fixed point along solution curves which lie in $x > 0$ and $y < 0$ for all time and so there will be a simple snap-back repeller.

The remaining (sufficient but not necessary) condition for the snap-back repeller to exist if $T_R > 0$ is that $x_0^R$ lies to the left of $e_+$ (the second iterate has the same eigenvector as the first iterate). The line of the eigenvector through $x_0^R$ is $y = -\frac{D_R}{s_+} (x - (s_+ - 1) x_1^R)$ and so the geometric condition which guarantees the existence of the snap-back repeller is

$$s_+ y_1 \leq -D_R (x_1 + (s_+ - 1) x_1^R) \quad (12)$$

which, after some manipulation gives

$$s_+ D_R (D_R - D_L) + (s_+ T_R - D_R) (T_R D_L - T_L D_R) \geq 0 \quad (13)$$

In the arguments above we have accumulated a number of conditions: (5), (6), (8), and (13) with $T_R > 0$. We now need to show that there are some values of the parameters which satisfy all these conditions simultaneously. We start by setting

$$D_R = 10, \quad T_R = 7 \quad (14)$$

in which case $s_+ = 5$ and since $1 - T_R + D_R = 4$ the fixed point exists if $\mu > 0$. The final two constraints give

$$-3 D_L - 10 T_L > 0, \quad 10 + 3 D_L - 5 T_L > 0 \quad (15)$$

To show how the existence of the snap-back repeller is to some extent independent of the linear type of the flow in $G_L$ we consider briefly several possibilities which satisfy these constraints.

First suppose that

$$T_L = 0, \quad 0 < -D_L < 1 \quad (16)$$

Then both conditions in (15) are satisfied so the snap-back repeller exists if $\mu > 0$. The fixed point in $G_L$ is stable and since $1 - T_L + D_L > 0$ it exists if $\mu < 0$. Hence, as $\mu$ increases through zero a stable fixed point is destroyed and an unstable fixed point with a strange invariant set from the snap-back repeller is created (possibly with other recurrent dynamics; we have not made an exhaustive study here). If $T_L = -1$ and $D_L = -\frac{1}{2}$ there is a similar bifurcation but in this case the stable fixed point in $\mu < 0$ is replaced by a saddle.

Another interesting transformation occurs if

$$D_L = -2, \quad T_L < \frac{3}{5} \quad (17)$$
where parameters in the problem. Over an unbounded set of the values of the other > 0. If \( e > \frac{1}{2a} \) non-synchronized orbits can be created from the two boundaries \( x = a^{-1} \) and \( y = -a^{-1} \). One of the simplest border collisions in this example is for orbits of period three. If \( a = 1.8 \) and \( e = 0.18 \), then there are two period 3 orbits, one with a point close to \( x = a^{-1} \approx 0.555 \) at \( (0.553, 0.737) \) and the other with a point at \( (0.559, 0.736) \). As \( e \) increases these tend to the boundary, one from the left and the other from the right, and there is a border collision at \( e \approx 0.1845 \). For the third iterate of the map these are repelling fixed points, and (again for the third iterate) \( T_L \approx -9.83, D_L \approx 21.77, T_R \approx -1.44 \) and \( D_R \approx -27.21 \). Condition (6) is satisfied, and since this is the negative trace case we have checked numerically that the backward orbit of the preimage of the periodic point does indeed tend to the periodic point in backwards time without crossing the border, which shows that the left hand period 3 point \((0.553, 0.737)\) is a snap-back repeller. Figure 3 shows some of the bifurcating orbits.

In this note we have shown that snap-back repellors exist in the normal form for unstable border collision bifurcations, which makes it possible to predict the existence of chaotic solutions. These solutions are repelling, but can help explain the existence of periodic and aperiodic orbits which play an important role in the dynamics of the system as described in [15]. A fuller description of the cases, including \( D_R < -1 \), will be given elsewhere.

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