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ON LP-MODELS OF ARITHMETIC

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Abstract. We answer some problems set by Priest in [11] and [12], in particular refuting Priest's Conjecture that all LP-models of Th($\mathbb{N}$) essentially arise via congruence relations on classical models of $Th(\mathbb{N})$. We also show that the analogue of Priest's Conjecture for $I\Delta_0 + Exp$ implies the existence of truth definitions for intervals $[0, a] \subset_e M \models I\Delta_0 + \mathrm{Exp}$ in any cut $[0, a] \subset_e K \subseteq_e M$ closed under successor and multiplication.

§1. Introduction. In [11] Graham Priest, continuing a theme introduced by R. Meyer and C. Mortensen [4, 6, 5], investigated finite models of the complete theory of $\mathbb{N}$ based on the paraconsistent logic LP,[1] aiming at a characterization of all such models. Although he did not fully achieve this aim, a combination of his results and those of J. Paris and N. Pathmanathan [8] led to a complete characterization of all finite LP models of arithmetic. In a second paper, [12], Priest considered also infinite LP models of arithmetic, for which the picture is not as clear as for the finite case. Our aim in this paper is to consider some of the problems and a conjecture that Priest posed in [11] and [12].

In the rest of this section, we will give some definitions (from [11]). In section 2 we will give a negative answer to the second problem in [12], which concerns the structure of infinite LP-models.[2] Section 3 is dedicated to answering the first problem in [11], which concerns the number of LP-models of cardinality n, for $n \in \mathbb{N}$. In the final section of this paper we will give a negative answer to Priest's Conjecture.

Following the background of [8] and [11], we define an LP-structure for a (first order) language $\mathscr{L}$ to be a pair $\langle D, I\rangle$, with D being the domain and I an assignment to the non logical symbols of the language such that:

- $I(c) \in D$, for every constant symbol c.
- $I(f)$ is an n-ary function on D, for every n-ary function symbol f.
- $I(P)$ is the pair $\langle I^+(P), I^-(P)\rangle$, with $I^+(P), I^-(P)$ being the extension and anti-extension of P, satisfying $I^+(P) \cup I^-(P) = D^n$, for every n-ary predicate symbol P.
- $I^+(=) = \{\, \langle x, x\rangle : x \in D \,\}$, $I^-(=) \supseteq \{\, \langle x, y\rangle : x, y \in D, x \neq y \,\}$.

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[1] LP stands for 'logic of paradox', see [10, 11].

[2] Given this flirting with paraconsistency we should perhaps reassure the reader that throughout the meta-theory we work in is intended to be classical and consistent, for example ZFC suffices.

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Let $M = \langle D, I\rangle$ be an LP-structure for $\mathscr{L}$. We define satisfiability in M as follows:

For a term $t(\vec{x})$, a formula $\theta(\vec{x})$ of $\mathscr{L}$ and an assignment v from the free variables of the language into D we define $t^{M,v}(\vec{x})$, $M, v \vDash \theta(\vec{x})$ and $M, v \vDash \neg\theta(\vec{x})$ inductively as follows:

- If $t(\vec{x}) = c$ then $t^{M,v}(\vec{x}) = I(c)$, if $t(\vec{x}) = x$ then $t^{M,v}(\vec{x}) = v(x)$.
- If $t(\vec{x}) = f(t_1(\vec{x}), \ldots, t_m(\vec{x}))$ then $t^{M,v}(\vec{x}) = I(f)(t_1^{M,v}(\vec{x}), \ldots, t_m^{M,v}(\vec{x}))$.
- For an n-ary predicate symbol P,

$$\begin{aligned} M, v \vDash P(t_1(\vec{x}), \ldots, t_n(\vec{x})) &\iff \langle t_1^{M,v}(\vec{x}), \ldots, t_n^{M,v}(\vec{x})\rangle \in I^+(P), \\ M, v \vDash \neg P(t_1(\vec{x}), \ldots, t_n(\vec{x})) &\iff \langle t_1^{M,v}(\vec{x}), \ldots, t_n^{M,v}(\vec{x})\rangle \in I^-(P). \end{aligned}$$

- For formulae $\theta_1(\vec{x}), \theta_2(\vec{x})$ of $\mathscr{L}$,

$$\begin{aligned} M, v \vDash \neg\neg\theta_1(\vec{x}) &\iff M, v \vDash \theta_1(\vec{x}), \\ M, v \vDash \theta_1(\vec{x}) \wedge \theta_2(\vec{x}) &\iff M, v \vDash \theta_1(\vec{x}) \text{ and } M, v \vDash \theta_2(\vec{x}), \\ M, v \vDash \neg(\theta_1(\vec{x}) \wedge \theta_2(\vec{x})) &\iff M, v \vDash \neg\theta_1(\vec{x}) \text{ or } M, v \vDash \neg\theta_2(\vec{x}), \\ M, v \vDash \theta_1(\vec{x}) \vee \theta_2(\vec{x}) &\iff M, v \vDash \theta_1(\vec{x}) \text{ or } M, v \vDash \theta_2(\vec{x}), \\ M, v \vDash \neg(\theta_1(\vec{x}) \vee \theta_2(\vec{x})) &\iff M, v \vDash \neg\theta_1(\vec{x}) \text{ and } M, v \vDash \neg\theta_2(\vec{x}), \\ M, v \vDash \theta_1(\vec{x}) \to \theta_2(\vec{x}) &\iff M, v \vDash \neg\theta_1(\vec{x}) \text{ or } M, v \vDash \theta_2(\vec{x}), \\ M, v \vDash \neg(\theta_1(\vec{x}) \to \theta_2(\vec{x})) &\iff M, v \vDash \theta_1(\vec{x}) \text{ and } M, v \vDash \neg\theta_2(\vec{x}). \end{aligned}$$

- For a formula $\theta(y, \vec{x})$,

$$\begin{aligned} M, v \vDash \exists y\ \theta(y, \vec{x}) &\iff \text{for some } a \in D,\ M, v' \vDash \theta(y, \vec{x}), \\ M, v \vDash \neg\exists y\ \theta(y, \vec{x}) &\iff \text{for all } a \in D,\ M, v' \vDash \neg\theta(y, \vec{x}), \\ M, v \vDash \forall y\ \theta(y, \vec{x}) &\iff \text{for all } a \in D,\ M, v' \vDash \theta(y, \vec{x}), \\ M, v \vDash \neg\forall y\ \theta(y, \vec{x}) &\iff \text{for some } a \in D,\ M, v' \vDash \neg\theta(y, \vec{x}), \end{aligned}$$

  where v' agrees with v on all variables except possibly y when $v'(y) = a$.

As usual, we shall often write $M \vDash \theta(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in D$, in place of $M, v \vDash \theta(x_1, \ldots, x_n)$, where v is some (equivalently any) assignment such that $v(x_i) = a_i$ for $i = 1, \ldots, n$.

We say that M is an LP-model[3] of a set of sentences T if for all $\theta \in T$, $M \vDash \theta$ (as usual, the choice of v does not matter here).

In [11] Priest gives a method for making (in particular) LP-models of arithmetic. Namely let M be a classical, non-standard, model of arithmetic and $\sim$ a congruence relation[4] on M. Now define $D_\sim$ to be the set of equivalence classes, say $[a]$ is the equivalence class containing $a \in M$, and define $I_\sim(0) = [0], I_\sim(')([a]) = [a'], I_\sim(+)([a],[b]) = [a+b], I_\sim(\times)([a],[b]) = [ab]$ and

[3] Because of the close connection of this paper to Priest's work we shall adopt his notation here rather than harken back to Meyer and Mortensen's earlier notion from [4, 6, 5] of an *RM*3-*assignment*.

[4] I.e., $\sim$ is an equivalence relation and satisfies that if $a_1 \sim a_2, b_1 \sim b_2$ then $a_1' \sim a_2', a_1 + b_1 \sim a_2 + b_2$ and $a_1 b_1 \sim a_2 b_2$.

$$I^+_\sim(=) = \{\, \langle [a], [b] \rangle : a \sim b \,\},$$
$$I^-_\sim(=) = \{\, \langle [a], [b] \rangle \;:\; a \neq b \,\}.$$

Then $M/\sim\, = \langle D_\sim, I_\sim \rangle$ is an LP-model of the theory of M. In such a case we shall refer to $M/\sim$ as a *collapse* of M.

Notice here that for $a \in M$, $\langle [a], [a] \rangle \in I^-(=)$ if and only if $[a]$ has at least 2 elements. By the Extension Lemma (see [11], alternatively the Extendability Lemma of [6], p. 513) we also obtain an LP-model of T if we take any enlargement of this anti-extension of $=$ whilst keeping everything else the same. We refer to this as an *extension* of the original LP-model.

Henceforth we shall restrict ourselves to the case where $\mathscr{L}$ is the language of arithmetic and unless otherwise indicated *LP-model* will be short for LP-model of some *fixed complete, consistent (classical) extension* T of Peano's Axioms PA.[5] Priest actually concentrated on the case when T was the theory of true arithmetic, i.e., $\mathrm{Th}(\mathbb{N})$, though in fact the results (up to now) do not depend on the particular choice of T.

§2. Properties of infinite LP-models.

Let $i \in M$. Then the set

$$N(i) := \{\, x \in M : M \models i \leq x \leq i \,\}$$

is called *nucleus* of i, where, as usual, $x \leq y$ is defined to be $\exists z\ (x + z = y)$. If there is p such that $i + p = i$ (in M of course) then we say that p is a *period* of i. Thus i may have more than one period. Observe that for any $j \in N(i)$ we have that if p is a period of i, then p is also a period of j. Also observe that $N(i) = N(j)$, for any $j \in N(i)$. It follows that we may omit i, and just write N instead of $N(i)$ and say that p is a *period* of nucleus N, if there is $i \in N$ such that $i + p = i$. When a nucleus has a period greater than zero we shall say that it is *proper*.[6]

As we indicated in introduction, a nucleus may have more than one period. Priest in [4] proposed the following:

Problem 1. *Can a nucleus have an infinitely descending sequence of periods?*

The answer is positive and the construction of such a nucleus is given in the next proposition.

Proposition 1. *There is an LP-model with a nucleus that has an infinitely descending sequence of periods.*

Proof. Suppose M is a (classical) nonstandard model of $\mathrm{Th}(\mathbb{N})$, $v \in M$, nonstandard, $u = v!$ and $I := u^{\mathbb{N}} = \{\, a \in M : \exists n \in \mathbb{N},\ a \leq u^n \,\}$. So I is a proper initial segment, or cut, in M, denoted $I \subset_e M$, $u \in I$ and I is closed under addition and multiplication. Since $u \in I$ has infinitely many divisors there is a strictly decreasing sequence $(p_i)_{i \in \mathbb{N}}$, such that $p_{i+1} | p_i$, with $p_0 = u$. Define

$$a \equiv b \iff \begin{cases} a, b \in I \text{ and } a = b \text{ or} \\ a, b > I \text{ and } a = b \bmod p_i, \text{ for some } i \geq 1, \end{cases}$$

where $a > I$, etc., means that $a > x$ for all $x \in I$.

[5] PA alone is not enough because Modus Ponens is no longer sound with respect to LP-structures.

[6] I.e., having more than one element, see for details [11].

First we check that $\equiv$ is an equivalence relation:

(i) $a \equiv a$ obviously holds;
(ii) If $a \equiv b$, then obviously $b \equiv a$;
(iii) Suppose $a \equiv b$ and $b \equiv c$. We will show that $a \equiv c$. Indeed, if $a \in I$, then $b, c \in I$, and obviously $a \equiv c$ holds. If $a > I$, then $b, c > I$. Now suppose $a = b \bmod p_i$ and $b = c \bmod p_j$. Setting $p_k = \min\{p_i, p_j\}$, since either $p_i | p_j$ or $p_j | p_i$, we get that $a = b \bmod p_k$, $b = c \bmod p_k$, with $k \geq 1$. So $a \equiv c$.
Now we will show that it is also a congruence relation.
(iv) Suppose that $a \equiv b$. If $a \in I$, then $b \in I$. But I is closed under successor, so $a', b' \in I$ and $a' \equiv b'$ holds. If $a > I$, then $b > I$. Now suppose $a = b \bmod p_i$; then $a', b' > I$ and $a' = b' \bmod p_i$, so again $a' \equiv b'$.
(v) Suppose $a_1 \equiv a_2$, $b_1 \equiv b_2$. If $a_1, b_1 \in I$, then $a_2, b_2 \in I$. So, since I is closed under addition, $a_1 + b_1, a_2 + b_2 \in I$ and $a_1 + b_1 \equiv a_2 + b_2$ hold. If $a_1 > I$ or $b_1 > I$, then $a_1 + b_1 > I$. Now suppose $a_1 = b_1 \bmod p_i$ and $a_2 = b_2 \bmod p_j$; then set $p_k = \min\{p_i, p_j\}$, where i or j can take value 0. Since $a_1 + b_1, a_2 + b_2 > I$ and $k > 0$, we have that $a_1 + b_1 \equiv a_2 + b_2$.
(vi) Exactly as above, if $a_1 \equiv a_2$, $b_1 \equiv b_2$ then $a_1 b_1 \equiv a_2 b_2$, because I is closed under multiplication.

Clearly for $a \in I$ the nucleus of $[a]$, $N([a])$, is just $\{[a]\}$ itself. Fix $r > I$. Thus when $x > I$, we have that $[r] \leq [x] \leq [r]$. So the distinct nuclei in $M/\equiv$ are the $N([a])$ for $a \in I$ and $N([r])$. Furthermore for each i $r, r + p_i > I$ so $[r] = [r + p_i]$ and in $M/\equiv$ the $[p_i]$ (being in I) are strictly decreasing. Thus $N([r])$ has an infinitely descending sequence of periods. ⊣

In [12] Priest proved that, in the finite case, proper nuclei are always closed under addition and multiplication and posed the following problem.

Problem 2. *Must proper nuclei always be closed under addition and multiplication*?

We give a negative answer in our next result.

Proposition 2. *There is an infinite LP-model with a proper nucleus that is not closed under addition* (*and thus neither under multiplication*).

Proof. Suppose $M \models PA$, (classical) nonstandard. Take again $q \in M$, with infinitely many divisors and an infinite strictly decreasing sequence $(p_i)_{i \in \mathbb{N}}$, such that $p_{i+1} | p_i$, with $p_0 = q$ (thus all of them are nonstandard). Define

$$a \equiv b \iff \begin{cases} a, b \in \mathbb{N} \text{ and } a = b \text{ or} \\ a, b > \mathbb{N},\ a = b \bmod p_i, \text{ for some } i, \text{ and} \\ a - \frac{a}{\lambda} \leq b \leq a + \frac{a}{\lambda}, \text{ for some } \lambda > \mathbb{N}. \end{cases}$$

So we get an infinite $M/\equiv$ with an infinite number of nuclei. When the first case holds, i.e., we are in $\mathbb{N}$, then it is obvious that $\equiv$ is a congruence relation. Now we should check that $\equiv$ is equivalence relation, assuming that we are in the second case.
(i) $a \equiv a$ obviously holds.
(ii) Suppose $a \equiv b$, so $a = b \bmod p_i$, for some i, and

$$a - \frac{a}{\lambda} \leq b \leq a + \frac{a}{\lambda}, \tag{1}$$

for some $\lambda > \mathbb{N}$. Thus

$$\frac{a}{\lambda-1} - \frac{a}{\lambda(\lambda-1)} \leq \frac{b}{\lambda-1} \leq \frac{a}{\lambda-1} + \frac{a}{\lambda(\lambda-1)}. \tag{2}$$

So by adding (1) and (2) we deduce that

$$a = a - \frac{a}{\lambda} + \frac{a}{\lambda-1} - \frac{a}{\lambda(\lambda-1)} \leq b + \frac{b}{\lambda-1}.$$

Similarly, by subtracting (2) from (1), we obtain

$$b - \frac{b}{\lambda-1} \leq a.$$

Thus $b \equiv a$.

(iii) Suppose $a \equiv b$ and $b \equiv c$, i.e., $a - \frac{a}{\lambda_1} \leq b \leq a + \frac{a}{\lambda_1}$ and $b - \frac{b}{\lambda_2} \leq c \leq b + \frac{b}{\lambda_2}$, for some $\lambda_1, \lambda_2 > \mathbb{N}$.
Set $\lambda = \min\{\lambda_1, \lambda_2\}$. Then $a - \frac{a}{\lambda} \leq b$ gives $a - \frac{a}{\lambda} - \frac{b}{\lambda} \leq b - \frac{b}{\lambda} \leq c$.
But $\frac{b}{\lambda} \leq \frac{a}{\lambda} + \frac{a}{\lambda^2}$, thus $a - \frac{a}{\lambda} - \frac{a}{\lambda} - \frac{a}{\lambda^2} \leq c$. So $a - \frac{a}{(\lambda/3)} \leq c$.
Similarly $c \leq a + \frac{a}{(\lambda/3)}$. It follows that $a \equiv c$.

Now we will show that $\equiv$ is also a congruence relation.

(iv) Suppose $a \equiv b$, i.e., $a - \frac{a}{\lambda} \leq b \leq a + \frac{a}{\lambda}$, for some $\lambda > \mathbb{N}$. Adding 1 to all sides, we obtain $a+1-\frac{a}{\lambda} \leq b+1 \leq a+1+\frac{a}{\lambda}$, for some $\lambda > \mathbb{N}$, so $a' - \frac{a'}{\lambda} \leq b' \leq a' + \frac{a'}{\lambda}$. Thus $a' \equiv b'$.

(v) Suppose $a_1 \equiv a_2$, $b_1 \equiv b_2$, i.e., $a_1 - \frac{a_1}{\lambda_1} \leq b_1 \leq a_1 + \frac{a_1}{\lambda_1}$ and $a_2 - \frac{a_2}{\lambda_2} \leq b_2 \leq a_2 + \frac{a_2}{\lambda_2}$. for some $\lambda_1, \lambda_2 > \mathbb{N}$.

Again set $\lambda = \min\{\lambda_1, \lambda_2\}$. So the previous two inequalities hold for λ, and by adding them we get $a_1 + a_2 - \frac{a_1+a_2}{\lambda} \leq b_1 + b_2 \leq a_1 + a_2 + \frac{a_1+a_2}{\lambda}$, thus we have $a_1 + b_1 \equiv a_2 + b_2$.

(vi) Suppose $a_1 \equiv a_2$, $b_1 \equiv b_2$, i.e.,

$$a_1 - \frac{a_1}{\lambda_1} \leq b_1 \leq a_1 + \frac{a_1}{\lambda_1} \tag{3}$$

and

$$a_2 - \frac{a_2}{\lambda_2} \leq b_2 \leq a_2 + \frac{a_2}{\lambda_2}, \tag{4}$$

for some $\lambda_1, \lambda_2 > \mathbb{N}$. Again set $\lambda = \min\{\lambda_1, \lambda_2\}$.

So (3), (4) hold for λ, and by multiplying them we obtain

$$a_1a_2 - \frac{2a_1a_2}{\lambda} + \frac{a_1a_2}{\lambda^2} \leq b_1b_2 \leq a_1a_2 + \frac{2a_1a_2}{\lambda} + \frac{a_1a_2}{\lambda^2}.$$

Observe now that

$$a_1a_2 - \frac{a_1a_2}{\left(\frac{\lambda^2}{2\lambda-1}\right)} \leq b_1b_2 \leq a_1a_2 + \frac{a_1a_2}{\left(\frac{\lambda^2}{2\lambda-1}\right)}.$$

Noting that $\frac{\lambda^2}{2\lambda-1} > \mathbb{N}$, we have $a_1b_1 \equiv a_2b_2$.

Finally observe that for any $a \in M - \mathbb{N}$, a and $2a$ are not in the same (proper) nuclei. So the answer is negative to Priest's second problem. ⊣

§3. **The number of finite LP-models.** In his paper [11] Priest described the structure of the finite LP-models of (true) arithmetic and set the following two problems relative to it.

PROBLEM 3. *What is the number of LP-models* $(\mathit{of}\ T)$ *of finite cardinality* n?

PROBLEM 4. *Is there a characterization of the finite LP-models* $(\mathit{of}\ T)$?

In relation to Problem 4, we recall a result obtained by J. Paris and N. Pathmanathan, see [8],[7] that completed the characterization given by Priest in [11]. In what follows all models should be considered classical, unless we specify them as LP-models.

Suppose that M is a nonstandard model of T, $p_0, p_1, \dots, p_m \in \mathbb{N}$, with $m, p_1 \geq 1$, $p_{i+1} | p_i$, for all $1 \leq i < m$ and either $m = 1$ or $p_0 > 0$. Take $C_1, \dots, C_m$ to be a sequence of strictly increasing proper cuts[8] in M, with $C_m = M$. Define for $a, b \in M$

$$a \equiv b \iff \begin{cases} a = b < p_0 \text{ or } p_0 \leq a, b \in C_i - C_{i-1} \\ \text{for some } i \text{ (take } C_0 = \emptyset) \text{ and } a = b \bmod p_i. \end{cases}$$

Then the relation $\equiv$ is a congruence relation on M and consequently $M/\equiv$ is a finite LP-model of T. Notice that the particular choice of M and $C_1, C_2, \dots, C_{m-1}$ is actually irrelevant here, up to isomorphism $M/\equiv$ depends only on the natural numbers $p_0, p_1, \dots, p_m$. [Indeed, as pointed out in [8], this would still be true if we started with M being a model of just a suitable fragment of T.] The anti-extension $I^-(=)$ of $=$ in $M/\equiv$ contains all pairs of elements except $\langle 0^{(i)}, 0^{(i)} \rangle$ ($0^{(i)}$ is the ith successor of 0) for $i = 0, 1, \dots, p_0 - 1$. As pointed out earlier, by the Extension Lemma (see [11]) we will continue to have a finite LP-model of T, of the same size, if we replace this anti-extension $I^-(=)$ in $M/\equiv$ by any of its 2^{p_0} possible extensions. Following the notation in [8] we call such an LP-model a *Linear Plus* LP-model of T.

THEOREM 3. [8] *The finite LP-models of T are exactly the Linear Plus LP-models of T.*

Having now a picture of the finite LP-models of T, we can answer Problem 3 concerning the number of LP-models of T of cardinality n.

Let $p_0, p_1, \dots, p_m$ be as above for an LP-model of cardinality n. Notice that since $p_1 \geq 1$ we must have $p_0 < n$. If $p_0 = 0$ then $m = 1$ and there is only one possible LP-model of cardinality n. Otherwise any such LP-model is determined by p_0, its $I^-(=)$ (from 2^{p_0} possibilities) and some $r_1, r_2, \dots, r_m \geq 1$ such that $m \geq 1$

$$n - p_0 = (r_1 r_2 \dots r_m) + (r_2 r_3 \dots r_m) + \dots + (r_{m-1} r_m) + r_m,$$

(i.e., $r_i r_{i-1} \dots r_m = p_i$). Let $\beta(n - p_0)$ be the number of choices of $r_1, r_2, \dots, r_m$ where m is also allowed to vary. Then since this final r_m must be a divisor of $n - p_0$,

[7] Unfortunately the condition that either $m = 1$ or $p_0 > 0$ was omitted from the version stated in [8] (see [9]). The necessity of this follows because if $p_0 = 0$ and $m > 0$ then, in the notation of that paper, $b_1 = 0$ so

$$b_1 = 0 = b_2 b_1 = b_2 b_1^{(p_1)} = b_2 b_1 + p_1 b_2 = p_1 b_2 = b_2,$$

contradicting the non-equivalence of b_1, b_2.

[8] I.e., closed under successor, addition and multiplication.

the function β can be defined recursively by $\beta(0) = 1$ and for $k > 0$,

$$\beta(k) = \sum_{d|k} \beta(kd^{-1} - 1).$$

Altogether then the number of LP-models of T of cardinality n is

$$1 + \sum_{p=1}^{n-1} 2^p \beta(n-p)$$

(where p is to be thought of as p_0). ⊣

§4. The characterization problem for infinite LP-models. In the finite case, we saw that every LP-model of T is derived from some nonstandard model of T (or any theory extending some small fragment of PA in fact, see [8]) by collapsing via a congruence relation and then possibly taking an extension. Priest in [12] made the following conjecture in the particular case when $T = \mathrm{Th}(\mathbb{N})$:

PRIEST'S CONJECTURE (PC). Every infinite LP-model of T is obtained by collapsing a classical model of T and, possibly, extending the collapse.

Note that extensions (of collapsed models) can be derived only by extending the anti-extension of $=$, since this is the unique predicate symbol of the language of arithmetic $\mathscr{L}$.

Richard Benham, [1], has shown that every LP-model of T *is* obtained by collapsing a *substructure* of a classical model of T and, possibly, extending the collapse. However as the following construction and lemmas show Priest's Conjecture is false even in the original form when $T = \mathrm{Th}(\mathbb{N})$.

Let M be a nonstandard model of T, $\mathbb{N} < K \subset_e M$ with K closed under successor and multiplication. Define the congruence relation $\sim$ on M by

$$c \sim d \iff \begin{cases} c = d \in K \text{ or} \\ K < c, d. \end{cases}$$

So $M/{\sim}$ looks like K with one new element, ∞ say, stuck on top. By Priest's results it is an LP-model of T. It has infinitely many nuclei and they are all improper singletons (in the sense of not being closed under successor) except the last one, which is a proper singleton.

Furthermore in K itself we can interpret the functions $+, \times, '$ of $M/{\sim}$, as $\dot{+}, \dot{\times}, \dot{'}$ say, together with the extensions of $=$ and $\neq$. To this end for $c = [b] \in M/{\sim}$ set the code $\dot{c}$ of c in K to be $\langle 0, b\rangle$ if $b \in K$ and $\langle 1, 0\rangle$ if $b \notin K$, equivalently $c = \infty$. Then:

LEMMA 4. *For any formula $\theta(\vec{x})$ (of arithmetic) there are formulae $\theta^+(\vec{x}), \theta^-(\vec{x})$ such that for any $c_1, \dots, c_n \in M/{\sim}$,*

$$M/{\sim} \models \theta(c_1, \dots, c_n) \iff K \models \theta^+(\dot{c}_1, \dots, \dot{c}_n),$$
$$M/{\sim} \models \neg\theta(c_1, \dots, c_n) \iff K \models \theta^-(\dot{c}_1, \dots, \dot{c}_n).$$

PROOF. By induction on formulae.

- Atomic: $\theta(\vec{x})$ is $t_1(\vec{x}) = t_2(\vec{x})$, t_1, t_2 terms. Then set $\theta^+(\vec{x})$ to be $\dot{t}_1(\vec{x}) = \dot{t}_2(\vec{x})$ and $\theta^-(\vec{x})$ to be $(\dot{t}_1(\vec{x}) \neq \dot{t}_2(\vec{x})) \vee (\dot{t}_1(\vec{x}) = \langle 1,0\rangle) \vee (\dot{t}_2(\vec{x}) = \langle 1,0\rangle)$. Thus, by the definition of $M/{\sim}$, we get that for all $\vec{c} \in M/{\sim}$

$$\begin{aligned} M/{\sim} \models t_1(\vec{c}) = t_2(\vec{c}) &\iff t_1(\vec{c}) = [b] = t_2(\vec{c}) \text{ for some } b \in K \\ &\qquad \text{or } t_1(\vec{c}) = \infty = t_2(\vec{c}) \\ &\iff K \models \dot{t}_1(\vec{\dot{c}}) = \dot{t}_2(\vec{\dot{c}}) \\ &\iff K \models \theta^+(\vec{\dot{c}}) \end{aligned}$$

and

$$\begin{aligned} M/{\sim} \models t_1(\vec{c}) \neq t_2(\vec{c}) &\iff t_1(\vec{c}) = [b_1],\ t_2(\vec{c}) = [b_2] \text{ and either } b_1, b_2 \in K \\ &\qquad \text{and } b_1 \neq b_2 \text{ or } b_1 \notin K \text{ or } b_2 \notin K \\ &\iff \dot{t}_1(\vec{\dot{c}}) \neq \dot{t}_2(\vec{\dot{c}}) \text{ or } \dot{t}_1(\vec{\dot{c}}) = \infty \text{ or } \dot{t}_2(\vec{\dot{c}}) = \infty \\ &\iff K \models \theta^-(\vec{\dot{c}}). \end{aligned}$$

- $\theta(\vec{x})$ is $\neg\phi(\vec{x})$. This case, and those for the other connectives follows as in the proof of Lemma 8.
- $\theta(\vec{x})$ is $\forall y\ \phi(y, \vec{x})$, with the induction hypothesis holding for $\phi(y, \vec{x})$. Clearly the set of codes is definable in K, say by the formula $\eta(x)$. Setting $\theta^+(\vec{x})$ to be $\forall y\ (\eta(y) \to \phi^+(y, \vec{x}))$ and $\theta^-(\vec{x})$ to be $\exists y\ (\eta(y) \wedge \phi^-(y, \vec{x})$ now gives the required equivalences.

$\dashv$

LEMMA 5. *Assume Priest's conjecture for T. Let $K \subseteq_e M \models T$, with K closed under successor and multiplication. Then if $H \equiv K$ there exist G such that $H \subseteq_e G \models T$.*

PROOF. The interpretation of the LP-model $\langle K, \infty\rangle$ in K described in Lemma 4 gives in H an interpretation of a logically equivalent (in the obvious sense) LP-model $\langle H, \infty\rangle$ of T. By Priest's Conjecture $\langle H, \infty\rangle$ is of the form $G/{\sim}$ with G a model of T (extending the $I^-(=)$ is not necessary in this case) and $\sim$ a congruence relation on G. Indeed H must be an initial segment in G since otherwise we would have that $G/{\sim} \models c \neq c$ for some $c \in H$. $\dashv$

COROLLARY 6. *Priest's Conjecture is false for any complete consistent extension T of PA.*

PROOF. Starting with $PA + \Pi_1(T) + \{a > \underline{n} : n \in \mathbb{N}\}$, where $\Pi_1(T)$ is the set of Π_1 sentences in T, a is a new constant and $\underline{n}$ is the numeral of n, we can make a model H of this theory which omits the type

$$\{\ulcorner \underline{\theta} \urcorner \in z : \theta \in \Pi_2(T)\} \cup \{\ulcorner \underline{\theta} \urcorner \notin z : \theta \notin \Pi_2(T)\}.$$

The reason being that if not there would, by the Omitting Types Theorem (see for example [2]), be a formula $\phi(z)$ such that

$$PA + \Pi_1(T) + \{a > \underline{n} : n \in \mathbb{N}\} + \exists z\ \phi(z) \text{ is consistent,}$$

and for each $\theta \in \Pi_2(T)$

$$PA + \Pi_1(T) + \{a > \underline{n} : n \in \mathbb{N}\} \vdash \forall z\ [\phi(z) \to \ulcorner \underline{\theta} \urcorner \in z],$$

whilst for each $\theta \notin \Pi_2(T)$

$$PA + \Pi_1(T) + \{\, a > \underline{n} : n \in \mathbb{N} \,\} \vdash \forall z\, [\phi(z) \to \ulcorner \underline{\theta} \urcorner \notin z].$$

But in this case $\Pi_2(T)$ would be Σ_2 definable in T, which it is not.

Hence H is a nonstandard model of $PA + \Pi_1(T)$ in which $\Pi_2(T)$ is not coded. Since $H \models \Pi_1(T)$ it has an extension to a model M of T and by a theorem of Gaifman [3], $K \equiv H$ where K is the initial segment of M in which H is cofinal.

However if we assume Priest's Conjecture for T then by Lemma 5 there is a $G \models T$ such that $H \subseteq_e G$, so $\Pi_2(T)$ *must* be coded in H since H is nonstandard and it is coded in G, contradiction! $\dashv$

We now show that Priest's Conjecture also fails if we replace the complete theory T by simply PA.

THEOREM 7. *There is an LP-model of PA that is not an extension of a collapse of a nonstandard model of PA.*

In order to prove this theorem, we first prove some lemmas.

LEMMA 8. *Let K be a nonstandard model of some fragment of PA, and let K^- be the LP-model*[9] *obtained by extending the anti-extension of $=$ in K, i.e., $\{\, \langle a, b\rangle \in K^2 : a \neq b \,\}$, to be all possible pairs of elements of K. Then for each $\theta(\vec{x})$, there are $\theta^+(\vec{x})$ and $\theta^-(\vec{x})$ such that for all $\vec{a} \in K^-$*

$$K^- \models \theta(\vec{a}) \iff K \models \theta^+(\vec{a}),$$
$$K^- \models \neg\theta(\vec{a}) \iff K \models \theta^-(\vec{a}),$$

where the left hand side refers to LP and the right hand side to classical logic.

PROOF. By induction on the formula θ.

- Atomic: $\theta(\vec{x})$ is $t_1(\vec{x}) = t_2(\vec{x})$, t_1, t_2 terms. Then set $\theta^+(\vec{x})$ to be $t_1(\vec{x}) = t_2(\vec{x})$ and $\theta^-(\vec{x})$ to be $t_1(\vec{x}) = t_1(\vec{x})$. Thus, by the definition of K^-, we get that for all $\vec{a} \in K^-$

$$K^- \models t_1(\vec{a}) = t_2(\vec{a}) \iff K \models t_1(\vec{a}) = t_2(\vec{a}),$$
$$K^- \models t_1(\vec{a}) \neq t_2(\vec{a}) \iff K \models t_1(\vec{a}) = t_1(\vec{a}).$$

- $\theta(\vec{x})$ is $\neg\phi(\vec{x})$, with the induction hypothesis holding for $\phi(\vec{x})$. Then there are $\phi^+(\vec{x})$ and $\phi^-(\vec{x})$ such that for all $\vec{a} \in K^-$

$$K^- \models \neg\phi(\vec{a}) \iff K \models \phi^-(\vec{a}),$$
$$K^- \models \neg\neg\phi(\vec{a}) \iff K \models \phi^+(\vec{a}).$$

  Set $\theta^+(\vec{x})$ to be $\phi^-(\vec{x})$ and $\theta^-(\vec{x})$ to be $\phi^+(\vec{x})$. Then for all $\vec{a} \in K^-$

$$K^- \models \theta(\vec{a}) \iff K \models \theta^+(\vec{a}),$$
$$K^- \models \neg\theta(\vec{a}) \iff K \models \theta^-(\vec{a}).$$

[9] By the Extension Lemma (see [11]) it is an LP-model of $\mathrm{Th}(K)$.

- $\theta(\vec{x})$ is $\phi(\vec{x}) \wedge \psi(\vec{x})$, with the induction hypothesis holding for $\phi(\vec{x})$ and $\psi(\vec{x})$. Then set $\theta^+(\vec{x})$ to be $\phi^+(\vec{x}) \wedge \psi^+(\vec{x})$ and set $\theta^-(\vec{x})$ to be $\phi^-(\vec{x}) \vee \psi^-(\vec{x})$. Thus for all $\vec{a} \in K^-$

$$K^- \models \theta(\vec{a}) \iff K^- \models \phi(\vec{a}) \wedge \psi(\vec{a}) \iff K \models \theta^+(\vec{a}),$$
$$K^- \models \neg\theta(\vec{a}) \iff K^- \models \neg\phi(\vec{a}) \vee \neg\psi(\vec{a}) \iff K \models \theta^-(\vec{a}).$$

- $\theta(\vec{x})$ is $(\phi(\vec{x}) \to \psi(\vec{x}))$, with the induction hypothesis holding for $\phi(\vec{x})$ and $\psi(\vec{x})$. Setting $\theta^+(\vec{x})$ to be $\phi^-(\vec{x}) \vee \psi^+(\vec{x})$ and $\theta^-(\vec{x})$ to be $\phi^+(\vec{x}) \wedge \psi^-(\vec{x})$ gives the required result.
- $\theta(\vec{x})$ is $\forall y\ \phi(y, \vec{x})$, with the induction hypothesis holding for $\phi(y, \vec{x})$. Then setting $\theta^+(\vec{x})$ to be $\forall y\ \phi^+(y, \vec{x})$ and $\theta^-(\vec{x})$ to be $\exists y\ \phi^-(y, \vec{x})$ gives the required result.

⊣

LEMMA 9. *Let K be a model of some fragment of PA and let K^- be defined as in Lemma 8. Then, for each $\theta(\vec{x})$,*

$$\textit{either } K \models \forall\vec{x}\ \theta^+(\vec{x}) \textit{ or } K \models \forall\vec{x}\ \theta^-(\vec{x}).$$

PROOF. By induction on formulae, making repeated use of the proof of Lemma 8.

- Atomic: $\theta(\vec{x})$ is $t_1(\vec{x}) = t_2(\vec{x})$. Then

$$\begin{aligned} K^- \models \forall\vec{x}\ t_1(\vec{x}) \neq t_2(\vec{x}) &\iff K \models (\forall\vec{x}\ (t_1(\vec{x}) \neq t_2(\vec{x})))^+ \\ &\iff K \models \forall\vec{x}\ (t_1(\vec{x}) = t_2(\vec{x}))^- \\ &\iff K \models \forall\vec{x}\ t_1(\vec{x}) = t_1(\vec{x}). \end{aligned}$$

- $\theta(\vec{x})$ is $\neg\phi(\vec{x})$, with the induction hypothesis holding for $\phi(\vec{x})$. Thus

$$K \models \forall\vec{x}\ \phi^+(\vec{x}) \quad \text{or} \quad K \models \forall\vec{x}\ \phi^-(\vec{x}).$$

So

$$K \models \forall\vec{x}\ (\neg\phi(\vec{x}))^- \quad \text{or} \quad K \models \forall\vec{x}\ (\neg\phi(\vec{x}))^+.$$

- $\theta(\vec{x})$ is $\phi(\vec{x}) \wedge \psi(\vec{x})$, with the induction hypothesis holding for $\phi(\vec{x})$ and $\psi(\vec{x})$. Thus

$$K \models \forall\vec{x}\ \phi^+(\vec{x}) \quad \text{or} \quad K \models \forall\vec{x}\ \phi^-(\vec{x})$$

and

$$K \models \forall\vec{x}\ \psi^+(\vec{x}) \quad \text{or} \quad K \models \forall\vec{x}\ \psi^-(\vec{x}).$$

If we have

$$K \models \forall\vec{x}\ \phi^+(\vec{x}) \quad \text{and} \quad K \models \forall\vec{x}\psi^+(\vec{x}),$$

then it implies that

$$K \models \forall\vec{x}\ \phi^+(\vec{x}) \wedge \forall\vec{x}\ \psi^+(\vec{x}),$$

so

$$K \models \forall\vec{x}\ (\phi^+(\vec{x}) \wedge \psi^+(\vec{x})).$$

Otherwise

$$K \models \forall\vec{x}\ \phi^-(\vec{x}) \vee \forall\vec{x}\ \psi^-(\vec{x})$$

and this implies

$$K \models \forall \vec{x}\, (\phi^-(\vec{x}) \vee \psi^-(\vec{x})).$$

So we obtain

$$K \models \forall \vec{x}\, (\phi(\vec{x}) \wedge \psi(\vec{x}))^-.$$

- $\theta(\vec{x})$ is $(\phi(\vec{x}) \to \psi(\vec{x}))$, with the induction hypothesis holding for $\phi(\vec{x})$ and $\psi(\vec{x})$. So again we have

$$K \models \forall \vec{x}\, \phi^+(\vec{x}) \quad \text{or} \quad K \models \forall \vec{x}\, \phi^-(\vec{x})$$

and

$$K \models \forall \vec{x}\, \psi^+(\vec{x}) \quad \text{or} \quad K \models \forall \vec{x} \psi^-(\vec{x}).$$

If we have

$$K \models \forall \vec{x}\, \phi^-(\vec{x}) \quad \text{or} \quad K \models \forall \vec{x}\, \psi^+(\vec{x})$$

then it implies that

$$K \models \forall \vec{x}\, \phi^-(\vec{x}) \vee \forall \vec{x}\, \psi^+(\vec{x}).$$

So we obtain

$$K \models \forall \vec{x}\, (\phi(\vec{x}) \to \psi(\vec{x}))^+.$$

Otherwise we get

$$K \models \forall \vec{x}\, (\phi(\vec{x}) \to \psi(\vec{x}))^-.$$

- $\theta(\vec{x})$ is $\forall y\ \phi(y, \vec{x})$, with the induction hypothesis holding for $\phi(y, \vec{x})$. So

$$K \models \forall \vec{x}\, \forall y\ \phi^+(y, \vec{x}) \quad \text{or} \quad K \models \forall \vec{x}\, \forall y\ \phi^-(y, \vec{x}).$$

In the first case we get

$$K \models \forall \vec{x}\, (\forall y\ \phi(y, \vec{x}))^+.$$

The second case gives us

$$K \models \forall \vec{x}\, \exists y\ \phi^-(y, \vec{x})$$

and this implies

$$K \models \forall \vec{x}\, (\forall y\ \phi(y, \vec{x}))^-.$$

$\dashv$

We recall the following principle, as it will be used in the next proof.

Overspill Principle. Let M be a nonstandard model of PA, $\phi(x)$ be an $\mathscr{L}$-formula and $I \subset_e M$ closed under successor. If

$$M \models \phi(a), \quad \text{for all } a \in I,$$

then there is $b \in M - I$, such that $M \models \forall x \leq b\ \phi(x)$.

PROOF OF THEOREM 7. Let J be a nonstandard model of PA and let $\mathbb{N} < a \in J$. Set K to be the substructure of J with universe the set of non-negative $p(a)$, for $p(x)$ a polynomial over the integers $\mathbb{Z}$. Then the formula $x^2 \leq a$ defines the set of natural numbers $\mathbb{N}$ in K, because every $n \in \mathbb{N}$ satisfies $n^2 < a$ but there is no polynomial $p(x)$ over $\mathbb{Z}$ with $K \models p(a)^2 \leq a$ and $p(a) > \mathbb{N}$. So $\mathbb{N}$ is $\exists_1$-definable in K. But note that

$$PA \vdash \forall z\, [0 < z \rightarrow \exists y\, (y^2 < z \leq (y+1)^2)].$$

Thus K is not a model of PA, though it does satisfy all the axioms except induction (this is easy to see).

Assume now that K^- is defined as above. By the Extension Lemma, K^- certainly satisfies the non-induction axioms of PA. In fact K^- also satisfies the induction schema. To see this, notice that for any formula $\phi(y, \vec{z})$, the instance of induction

$$\phi(0, \vec{z}) \wedge \forall y\, (\phi(y, \vec{z}) \rightarrow \phi(y', \vec{z})) \rightarrow \forall y\, \phi(y, \vec{z})$$

holds in K^- just if

$$[\phi(0, \vec{z}) \wedge \forall y\, (\phi(y, \vec{z}) \rightarrow \phi(y', \vec{z})) \rightarrow \forall y\, \phi(y, \vec{z})]^+$$

holds in K. But this is equivalent to

$$\phi^-(0, \vec{z}) \vee \forall y\, [\phi^+(y, \vec{z}) \vee \phi^-(y', \vec{z})] \vee \forall y\, \phi^+(y, \vec{z})$$

holding in K, which is certainly the case, since by Lemma 9 at least one of $\phi^-(0, \vec{z})$ and $\forall y\, \phi^+(y, \vec{z})$ hold in K. Thus K^- is an LP-model of PA.

However K^- is not of the form of an $I^-(=)$ extension of some $M/\equiv$, where $M \models PA$ and $\equiv$ is a congruence relation on M. Indeed, if it were, say $[\alpha] = a$ where $\alpha \in M$, then M would have to be nonstandard and so, by overspill, contain a nonstandard element β satisfying $\beta^2 \leq \alpha$ and in $M/\equiv [\beta]$ would still be nonstandard and satisfy $[\beta]^2 \leq [\alpha] = a$, contradiction. $\dashv$

Notice that the LP-model constructed in the proof above cannot (apparently) be used as a counterexample to Priest's Conjecture, because we proved only that it is an LP-model of PA, do not know whether it is also an LP-model of some complete theory extending PA.

Theorem 7 shows that the variant of Priest's Conjecture where we replace T by PA is also false. However Benham's construction in [1] shows that the conjecture is true if instead we replace T by, say $\Pi_1(PA)$, the Π_1 consequences of PA. This suggests then that we might consider for a theory T_0 in the language of arithmetic:

PRIEST'S CONJECTURE FOR T_0, (PC(T_0)). Every LP-model of T_0 is obtained by collapsing a classical model of T_0 and, possibly, extending the collapse.

Notice that provided T_0 is reasonably expressive Lemma 5 still holds under the assumption of PC(T_0) and provides a powerful tool, always assuming of course that it is consistent! The following result, using essentially this lemma with $T_0 = I\Delta_0 + \mathrm{Exp}$, indicates that some instances of this conjecture may have interesting consequences for bounded arithmetics.

THEOREM 10 (PC($I\Delta_0 + \mathrm{Exp}$)). *Given $m \in \mathbb{N}$ there is a finite set of formulae $\chi_1, \chi_2, \ldots, \chi_n$ such that for any $e_1, e_2, \ldots, e_m < a \in M \models I\Delta_0 + \mathrm{Exp}$ and $[0, a] \subset_e$*

$K \subseteq_e M$, with K closed under successor and multiplication, there is $1 \leq i \leq n$ such that for all formulae $\theta(z, x_1, x_2, \dots, x_m)$ in the language of $[0, a]$

$$[0, a] \models \theta(a, e_1, e_2, \dots, e_m) \iff K \models \chi_i(\ulcorner \theta \urcorner, a, e_1, e_2, \dots, e_m).$$

PROOF. The result is clear if $K = \mathbb{N}$ so we may take this not to be the case. Assume $\mathrm{PC}(I\Delta_0 + \mathrm{Exp})$ and let M be a nonstandard model of $I\Delta_0 + \mathrm{Exp}$, $K \subseteq_e M$ closed under successor and multiplication, and $e_1, e_2, \dots, e_m < a \in K$. Consider the following type Σ:

$$\{ \ulcorner \theta(x, \vec{y}) \urcorner \in z \leftrightarrow [0, a] \models \theta(a, \vec{e}) : \theta(x, \vec{y}) \text{ a standard formula} \}.$$

This type is realizable in $\langle M, a, \vec{e} \rangle$ and, as the code can be taken arbitrarily small nonstandard, also realizable in $\langle K, a, \vec{e} \rangle$ because $K \subseteq_e M$.

Let $\langle H, a, \vec{e} \rangle$ be any model of $\mathrm{Th}(K, a, \vec{e})$. Thus $\langle K, a, \vec{e} \rangle \equiv \langle H, a, \vec{e} \rangle$ and the interpretation of the LP-model $\langle K, a, \vec{e}, \infty \rangle$ as in Lemma 4 gives in $\langle H, a, \vec{e} \rangle$ an elementarily equivalent (in the obvious sense) LP-model $\langle H, a, \vec{e}, \infty \rangle$ of $I\Delta_0 + \mathrm{Exp}$. By $\mathrm{PC}(I\Delta_0 + \mathrm{Exp})$ $\langle H, \infty \rangle$ is of the form $G/\!\sim$ with G a model of $I\Delta_0 + \mathrm{Exp}$ (extending the $I^-(=)$ is not necessary in this case) and $\sim$ a congruence relation on G. Indeed H must form a cut in G (so $a, \vec{e} \in G$) since otherwise we would have that $G/\!\sim\, \models c \neq c$ for some $c \in H$.

It follows that we can realize Σ in $\langle G, a, \vec{e} \rangle$, and hence in $\langle H, a, \vec{e} \rangle$. As a consequence of the Omitting Types Theorem then it must be that we cannot locally omit the type Σ in $\mathrm{Th}(K, a, \vec{e})$. Hence there must be some formula $\psi(x, \vec{y}, z)$ such that

$$\exists z\ \psi(a, \vec{e}, z) \in \mathrm{Th}(K, a, \vec{e})$$

and for all $\theta(x, \vec{y})$ the sentence

$$\forall z\ [\psi(a, \vec{e}, z) \to (\ulcorner \theta(x, \vec{y}) \urcorner \in z \leftrightarrow [0, a] \models \theta(a, \vec{e}))]$$

is in $\mathrm{Th}(K, a, \vec{e})$. Let $\chi_{a,\vec{e}}(w, x, \vec{y})$ be the formula

$$\exists z\ [\psi(x, \vec{y}, z) \wedge w \in z].$$

Then for all $\theta(x, \vec{y})$,

$$\chi_{a,\vec{e}}(\ulcorner \theta(x, \vec{y}) \urcorner, a, \vec{e}) \in \mathrm{Th}(K, a, \vec{e}) \iff [0, a] \models \theta(a, \vec{e}).$$

Of course $\chi_{a,\vec{e}}$ may vary with $\vec{e}$ and a. However notice that for a fixed finite length of $\vec{e} < a$ *some* finite set of $\chi_{a,\vec{e}}$ (not necessarily the $\chi_{a,\vec{e}}$ we initially chose here) will contain 'truth definition' representatives which work for all $\vec{e} < a$ of that finite length, i.e., there are $\chi_1, \dots, \chi_n$ such that for every $e_1, e_2, \dots, e_m < a \in K$ there is some $1 \leq i \leq n$ such that for all $\theta(z, x_1, x_2, \dots, x_m)$,

$$K \models \chi_i(\ulcorner \theta \urcorner, a, e_1, e_2, \dots, e_m) \iff [0, a] \models \theta(a, e_1, e_2, \dots, e_m).$$

For if that was not the case we could take an ultraproduct of structures $\langle K, a, \vec{e} \rangle$ with various $\vec{e}$ in which there would be no such $\chi_{a,\vec{e}}$ for some $a, \vec{e}$ in the ultraproduct, contradicting $\mathrm{PC}(I\Delta_0 + \mathrm{Exp})$ (for similar reasons as above). Similarly we can show that the χ_i can be chosen independent of the $\mathrm{Th}(M)$ and depend only on the axiom system $I\Delta_0 + \mathrm{Exp}$. The theorem follows. $\dashv$

It would be nice to improve this result to a single truth definition χ. Even so as it stands the result seems surprising. For how could these χ_i be deciding truth, clearly not in the standard way since K certainly need not be closed under exponentiation. Whilst this conclusion may seem somewhat bizarre we note that ostensibly stronger conjectures with a similar flavour, such as the Bounded Matijasevic Conjecture [7], have survived intact already for over two decades.

We conclude this section by pointing out that, like their finite counterparts, infinite LP-models of T can be very simple, even decidable. To see this let M be a countable nonstandard model of T and let C_j, $j = 1, 2, \ldots$, be a strictly increasing sequence of cuts in M closed under successor and multiplication and such that $M = \bigcup_j C_j$. Define the congruence relation $\equiv$ on M by

$$a \equiv b \iff \begin{cases} a = b = 0 \text{ or} \\ a, b \in C_j - C_{j-1} \text{ for some } j \text{ (take } C_0 = \{0\}) \end{cases}$$

Let $a_0 = 0$ and $a_j \in C_j - C_{j-1}$ for $j > 0$. Then the universe of $M/\equiv$ is the set of a_j, $j \in \mathbb{N}$, successor, addition and multiplication in $M/\equiv$ are given by

$$[a_j]' = \begin{cases} [a_1] & \text{if } j = 0, \\ [a_j] & \text{otherwise,} \end{cases}$$
$$[a_j] + [a_k] = [a_{\max\{j,k\}}],$$
$$[a_j] \times [a_k] = \begin{cases} [a_0] & \text{if } \min\{j,k\} = 0, \\ [a_{\max\{j,k\}}] & \text{otherwise,} \end{cases}$$

and the anti-extension of equality in $M/\equiv$ is all pairs except $\langle [a_0], [a_0] \rangle$. Clearly this LP-model can be interpreted in $\langle \mathbb{N}, {}', +, =, 0 \rangle$ (when we interpret $[a_j]$ as j). Hence for any sentence θ we can recursively find a sentence θ^* such that

$$M/\equiv \models \theta \iff \langle \mathbb{N}, {}', +, =, 0 \rangle \models \theta^*.$$

Since the theory of $\langle \mathbb{N}, {}', +, =, 0 \rangle$ is decidable it follows that the set of sentences true in the LP-model $M/\equiv$ is also decidable.

§5. **Conclusion.** In this paper we have continued previous work on the nature of both finite and infinite LP-models of (complete) theories $T \supseteq PA$. On one hand, we have found a recursive formula giving the number of such models with n elements, thus solving the first problem in [11]. On the other hand, we have studied properties of nuclei in infinite models and, by solving the second problem in [12], proved that their structure is different from that of nuclei in the finite case. This is an indication that the construction of infinite LP-models differs essentially from that of finite ones.

Our belief is strengthened by two more results, related to the conjecture stated by Priest in [12], concerning the nature of infinite LP-models of $\mathrm{Th}(\mathbb{N})$. The first result shows that Priest's Conjecture on the structure of LP-models of $\mathrm{Th}(\mathbb{N})$ is false and the second shows that it remains false even if we replace $\mathrm{Th}(\mathbb{N})$ by PA.

These results, and our final example of an infinite decidable LP-model of T, show that such structures can shed much of the complexity possessed by their classical counterparts. In one way this is interesting, just as the existence of finite LP-models of $Th(\mathbb{N})$ is interesting. On the other hand it suggests that we may need to consider somewhat more sophisticated LP-models (or alternative logics) if they are to tell us anything deep about classical arithmetic.

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