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Abstract

Given an action of a Lie group G on a manifold M, and a cover N of M (such as the universal cover $\widetilde{M}$) the natural question arises of whether the action lifts to a cover of M. In these notes, we address this question and determine whether the G-action itself lifts or whether it is necessary to pass to a cover of G (there is always a lift of the action of the universal cover $\widetilde{G}$). The results are presumably well-known to experts, but do not seem to be available in print.

These notes started life as the first section of [5], though it was felt that they were too detailed for a paper primarily on symplectic reduction. We are therefore making this available as a MIMS preprint.

Introduction

Suppose a connected Lie group G acts on a connected manifold M, and suppose N is a covering of M. Then it may not be possible to lift the action of G, but there is a natural lift to universal covers giving an action of $\widetilde{G}$ on $\widetilde{M}$. This can then be used to define an action of $\widetilde{G}$ on the given cover N. This general construction must be well-known, but we were unable to find it in the literature, and consequently we establish the main results about these lifted actions. For example, since N can be written as a quotient of $\widetilde{M}$ by a subgroup of the group of deck transformations, we use this to determine exactly which subgroup of $\widetilde{G}$ acts trivially on N. We also determine the relation between isotropy subgroups of the G action on M and the lifted action on N, and we show that the action on M is proper, then so is the lifted action on N.

1 The category of covering spaces

We begin by recalling a few facts about covering spaces. Many of the details can be found in any introductory book on Algebraic Topology, for example Hatcher [3]. Let (M,z_0) be a connected manifold with a chosen base point z_0, and let $q_M:(\widetilde{M},\tilde{z}_0)\to(M,z_0)$ be the universal covering. We realize the universal cover as the set of homotopy classes of paths in M with base point z_0. For definiteness, we take the base point in $\widetilde{M}$ to be the homotopy class $\tilde{z}_0$ of the trivial loop at z_0. Throughout, 'homotopic paths' will mean homotopy with fixed end-points, and all paths will be parametrized by $t\in[0,1]$, and composition of paths $a*b$ is defined by

$$(a*b)(t)=\begin{cases} a(2t) & \text{if } t\in[0,1/2]\\ b(2t-1) & \text{if } t\in[1/2,1].\end{cases}$$

(Of course, it is assumed that $a(1)=b(0)$.)

Any cover $p_N:(N,y_0)\to(M,z_0)$ has the same universal cover $(\widetilde{M},\tilde{z}_0)$ as (M,z_0), and the covering map $q_N:(\widetilde{M},\tilde{z}_0)\to(N,y_0)$ can be constructed as follows: Let $\tilde{z}\in\widetilde{M}$ and let $z(t)$ be a representative path in M, so $z(0)=z_0$. By the path lifting property of the covering map p_N, $z(t)$ can be lifted uniquely to a path $y(t)$ in (N,y_0). Then $q_N(\tilde{z})=y(1)$.

Let $\mathfrak{C}$ be the category of all covers of (M,z_0). The morphisms are the covering maps. Since any element $(N,y_0)\in\mathfrak{C}$ also shares $\widetilde{M}$ as universal cover, it sits in a diagram,

$$(\widetilde{M},\tilde{z}_0)\xrightarrow{q_N}(N,y_0)\xrightarrow{p_N}(M,z_0).$$

Note that with this notation for the covering maps, the map $\widetilde{M} \to M$ can be written both as q_M and as $p_{\widetilde{M}}$.

It is well-known that this category is isomorphic to the category of subgroups of the fundamental group $\pi_1(M,z_0)$ of M, where the morphisms are the inclusion homomorphisms of subgroups. The isomorphism is defined as follows. Let $p_N : (N,y_0) \to (M,z_0)$ be a cover. Then $\Gamma_N := p_{N*}(\pi_1(N,y_0))$ is the required subgroup of $\Gamma := \pi_1(M,z_0)$. Γ_N consists of the homotopy classes of closed paths in (M,z_0) whose lift to (N,y_0) is also closed, and the number of sheets of the covering p_N is equal to the index $\Gamma : \Gamma_N$. Note that since $\widetilde{M}$ is simply connected, $\Gamma_{\widetilde{M}}$ is trivial.

The inverse of this isomorphism can be defined using deck transformations. Let $\Gamma = \pi_1(M,z_0)$. Then Γ is the fibre of q_M over z_0, and it acts on $\widetilde{M}$ by deck transformations defined via the homotopy product: if $\gamma \in \Gamma$ and $\tilde{z} \in \widetilde{M}$ then $\gamma * \tilde{z}$ gives the action of γ on $\tilde{z}$. Then given $\Gamma_1 < \Gamma$, define $N = \widetilde{M}/\Gamma_1$, and put $y_0 = \Gamma_1\tilde{z}_0$. Then from the long exact sequence of homotopy, it follows that $\pi_1(N,y_0) \simeq \Gamma_1$. Furthermore, if $\Gamma_1 < \Gamma_2 < \Gamma$ then there is a well-defined morphism (covering map) $p : N_1 \to N_2$, where $N_j = \widetilde{M}/\Gamma_j$, obtained from noting that any Γ_1-orbit is contained in a unique Γ_2-orbit, so we put $p(\Gamma_1\tilde{z}) = \Gamma_2\tilde{z}$.

Let (N_1,y_1) be a cover of (M,z_0) with group Γ_1, and let $\Gamma_2 = \gamma\Gamma_1\gamma^{-1}$ be a subgroup conjugate to Γ_1 (where $\gamma \in \Gamma$). Then $N_2 = \widetilde{M}/\Gamma_2$ is diffeomorphic to N_1, but the base point is now $y_2 = \Gamma_2\tilde{z}_0$. The diffeomorphism is simply induced from the diffeomorphism $\tilde{z} \mapsto \gamma \cdot \tilde{z}$ of $\widetilde{M}$, which does not in general map y_1 to y_2.

If $\Gamma_1 \lhd \Gamma$ (normal subgroup), then the cover (N,y_1) is said to be a *normal cover*, In this case the Γ-action (by deck transformations) on $\widetilde{M}$ descends to an action on N (with kernel Γ_1), and Γ/Γ_1 is the group of deck transformations of the covering $N \to M$. For a general covering, the group of deck transformations is isomorphic to $N_\Gamma(\Gamma_1)/\Gamma_1$, where $N_\Gamma(\Gamma_1)$ is the normalizer of Γ_1 in Γ. Only for normal covers does the group of deck transformations act transitively on the sheets of the covering. See [3] for examples.

Let us emphasize here that we view $\Gamma = \pi_1(M,z_0)$ both as a group acting on $\widetilde{M}$ by deck transformations, and as a discrete subset of $\widetilde{M}$—the fibre over z_0. In particular, for $\gamma \in \Gamma$,

$$\gamma * \tilde{z}_0 = \gamma \tag{1.1}$$

In other words, $\tilde{z}_0$ is the identity element in Γ.

2 Lifting the group action

Now let G be a connected Lie group acting on the connected manifold M, and let $p_N : (N,y_0) \to (M,z_0)$ be a covering. To define the lifted action on N, we first describe the lift to $\widetilde{M}$ and then show it induces an action on N, using the covering $q_N : \widetilde{M} \to N$.

The action of G on M does not in general lift to an action of G on $\widetilde{M}$ but of the universal cover $\widetilde{G}$, which is also defined using homotopy classes of paths, with base point the identity element e. The covering map is denoted $q_G : \widetilde{G} \to G$. So if $\widetilde{g}$ is represented by a path $g(t)$ then $q_G(\widetilde{g}) = g(1)$. The product structure in $\widetilde{G}$ is given by pointwise multiplication of paths: if $\widetilde{g}_1$ is represented by a path $g_1(t)$ and $\widetilde{g}_2$ by $g_2(t)$, then $\widetilde{g}_1\widetilde{g}_2$ is represented by the path $t \mapsto g_1(t)g_2(t)$.

Definition 2.1 Let $\widetilde{g} \in \widetilde{G}$ be represented by a path $g(t)$ (with $g(0) = e$), and $\tilde{z} \in \widetilde{M}$ be represented by a path $z(t)$ (with $z(0) = z_0$). Then we define $\widetilde{g} \cdot \tilde{z}$ to be $\tilde{y} \in \widetilde{M}$, where $\tilde{y}$ is the homotopy class represented by the path $t \mapsto g(t) \cdot z(t)$. It is readily checked that the homotopy class of this path depends only on the homotopy classes $\widetilde{g}$ and $\tilde{z}$.

With this definition for the action of $\widetilde{G}$ on $\widetilde{M}$, it is clear that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{G} \times \widetilde{M} & \longrightarrow & \widetilde{M} \\ \downarrow & & \downarrow \\ G \times M & \longrightarrow & M \end{array} \tag{2.1}$$

where the vertical arrows are $q_G \times q_M$ and q_M respectively, and the horizontal arrows are the group actions. In particular,

$$\tilde{y} = \widetilde{g} \cdot \tilde{z} \quad \Longrightarrow \quad y = g \cdot z \tag{2.2}$$

where for $\tilde{z} \in \widetilde{M}$ we denote its projection to M by z, and similarly with elements of $\widetilde{G}$. Note for future reference that it follows immediately from (2.2) that the isotropy subgroups satisfy

$$\widetilde{g} \in \widetilde{G}_{\tilde{z}} \quad \Longrightarrow \quad g \in G_z. \tag{2.3}$$

Remark 2.2 A second approach to defining the action of $\widetilde{G}$ on $\widetilde{M}$ is as follows. The action of G gives rise to an 'action' of the Lie algebra $\mathfrak{g}$. That is, to each $\xi \in \mathfrak{g}$ there is associated a vector field ξ_M on M; these are the so-called generating vector fields of the G-action. Let $N \to M$ be any covering. The covering map is a local diffeomorphism, so the vector fields ξ_M can be lifted to vector fields ξ_N on N. Because this covering map is a local diffeomorphism, this gives rise to an 'action' of $\mathfrak{g}$ on N. Now $\mathfrak{g}$ is the Lie algebra of a unique simply connected Lie group $\widetilde{G}$. To see that the vector fields on N are complete, so defining an action of $\widetilde{G}$, one needs to compare the local actions on M and N. It is not hard to see that the two definitions of actions of $\widetilde{G}$ are equivalent.

Lemma 2.3 *Let $g(t)$ be a path in G with $g(0) = e$, and $z(t)$ a path in M with $z(0) = z_0$ and $z(1) = z_1$. Then the following three homotopy classes coincide:*

$$g(t) \cdot z(t), \quad [g(t) \cdot z_0] * [g(1) \cdot z(t)], \quad z(t) * [g(t) \cdot z_1],$$

where $*$ is the homotopy product of paths.

PROOF: Denote the three curves by $a(t), b(t)$ and $c(t)$ respectively. So for example,

$$c(t) = \begin{cases} z(2t) & \text{if } t \in [0, \frac{1}{2}] \\ g(2t-1) \cdot z_1 & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$$

A homotopy between a and b can be given by

$$A(t,s) = \begin{cases} g((1+s)t) \cdot z((1-s^2)t) & \text{if } t \leq \frac{1}{1+s} \\ g(1) \cdot z((1+s)t - s) & \text{if } t \geq \frac{1}{1+s} \end{cases}.$$

Then, $A(t,0) = a(t)$ and $A(t,1) = b(t)$. It is readily checked that $A(t,s)$ is continuous. A similar homotopy can be defined between a and c. ❒

Recall that $\Gamma := \pi_1(M, z_0)$ acts on $\widetilde{M}$ by deck transformations; that is, given $\gamma \in \Gamma$ and $\tilde{z} \in \widetilde{M}$ then $\gamma \cdot \tilde{z} := \gamma * \tilde{z}$. This action is transitive on fibres of the covering map q_M. Furthermore, the fibre $q_M^{-1}(z_0)$ is the Γ-orbit of the constant loop $\tilde{z}_0$ which we identify with Γ, see equation (1.1).

Proposition 2.4 *The action of $\widetilde{G}$ on $\widetilde{M}$ commutes with the deck transformations. Furthermore, for each $\widetilde{g} \in \pi_1(G, e)$ the homotopy class $g(t) \cdot z_0$ lies in the centre of $\pi_1(M, z_0)$.*

PROOF: Let $\widetilde{g} \in \widetilde{G}$, $\delta \in \Gamma$ and $\tilde{z} \in \widetilde{M}$ with $q_M(\tilde{z}) = y \in M$. We want to show that $\widetilde{g} \cdot (\delta \cdot \tilde{z}) = \delta \cdot (\widetilde{g} \cdot \tilde{z})$. By Lemma 2.3 (applied with $\gamma = \delta * \tilde{z}$), we have

$$\widetilde{g} \cdot (\delta \cdot \tilde{z}) = [\delta * \tilde{z}] * [\widetilde{g} \cdot y],$$

while again by Lemma 2.3 (now with $\gamma = \tilde{z}$),

$$\delta \cdot (\widetilde{g} \cdot \tilde{z}) = \delta * [\tilde{z} * (\widetilde{g} \cdot y)].$$

The result follows from the associativity of the homotopy product.

Now let $\widetilde{g} \in \pi_1(G,e)$ and $\delta \in \Gamma$. We want to show that $[\widetilde{g} \cdot \tilde{z}_0] * \delta = \delta * [\widetilde{g} \cdot \tilde{z}_0]$, where $\tilde{z}_0$ is the constant loop at x. By Lemma 2.3, $\delta * [\widetilde{g} \cdot \tilde{z}_0] = \widetilde{g} \cdot \delta = [\widetilde{g} \cdot \tilde{z}_0] * \delta$ (since $g(1) = e$), as required. ❐

As a particular example, this leads to the following well-known result

Corollary 2.5 $\pi_1(G,e)$ *lies in the centre of* $\widetilde{G}$*. Consequently the following is a central extension:*

$$1 \to \pi_1(G,e) \to \widetilde{G} \xrightarrow{q_G} G \to 1. \tag{2.4}$$

PROOF: This follows by applying the proposition to the left action of $\widetilde{G}$ on itself. ❐

Now we are in a position to define the action of $\widetilde{G}$ on an arbitrary cover (N,y_0) of (M,z_0). As in §1, let $\Gamma_N = p_{N*}(\pi_1(N,y_0)) < \Gamma$. So, $N \simeq \widetilde{M}/\Gamma_N$. That is, a point in N is a Γ_N-orbit of points in $\widetilde{M}$.

Definition 2.6 The $\widetilde{G}$-action on N is defined simply by

$$\widetilde{g} \cdot \Gamma_N \tilde{z} := \Gamma_N(\widetilde{g} \cdot \tilde{z}).$$

This is well-defined as the actions of $\widetilde{G}$ and Γ commute, by Proposition 2.4. It is clear too that the analogues of (2.1), (2.2), and (2.3) hold with N in place of $\widetilde{M}$.

Proposition 2.7 *Let* $p_N : (N,y_0) \to (M,z_0)$ *be a covering map. The* $\widetilde{G}$*-orbits on* N *are the connected components of the inverse images under* p_N *of the orbits on* M*. More precisely, if* $y \in p_N^{-1}(z) \subset N$ *then* $\widetilde{G} \cdot y$ *is the connected component of* $p_N^{-1}(G \cdot z)$ *containing* y*. In particular if the* G*-orbits in* M *are closed, so too are the* $\widetilde{G}$*-orbits in* N*.*

PROOF: Let $Z \subset M$ be any submanifold. Then $Z' := p_N^{-1}(Z)$ is a submanifold of N and the projection $p_N|_{Z'} : Z' \to Z$ is a covering, and if Z is closed so too is Z'. Moreover, if Z is G-invariant (hence $\widetilde{G}$-invariant), then by the equivariance of p_N so is Z', and if Z is a single orbit, then Z' is a discrete union of orbits: discrete because p_N is a covering. Since $\widetilde{G}$ is connected, the orbits are the connected components of Z'. ❐

3 The kernel of the lifted action

The natural action of $\widetilde{G}$ on $\widetilde{M}$ described above need not be effective, even if the action of G on M is, and the kernel is a subgroup of $\pi_1(G,e)$ (the kernel of a_{z_0} described below). But first let us recall some work of Daniel Gottlieb [2].

Given a manifold M (or more generally a CW complex) with z_0 as base point. Let $I = [0,1]$ and let $H : M \times I \to M$ be a cyclic homotopy, which is a homotopy satisfying

$$H(z,0) = H(z,1) = z, \quad \forall z \in M.$$

The *trace* of a cyclic homotopy is defined to be the curve $H(z_0,t)$, $t \in I$, which is a closed curve and so defines an element of $\pi_1(M,z_0)$. The set of all such elements forms a subgroup of $\pi_1(M,z_0)$ that Gottlieb denotes $G(M,z_0)$, and his paper [2] is dedicated to determining properties of this subgroup; here we quote two particular results.

Theorem 3.1 (Gottlieb [2]) *Let* $G(M,z_0)$ *be as defined above. Then* (i). *$G(M,z_0)$ is a subgroup of $P(M,z_0)$ (defined below);*

(ii). *if M has the homotopy type of a compact polyhedron and $\chi(M) \neq 0$ (Euler characteristic) then $G(M,z_0)$ is trivial.*

The subgroup $P(M,z_0) < \pi_1(M,z_0)$ is defined as follows. For each $k > 0$ there is a natural action of $\pi_1(M,z_0)$ on $\pi_k(M,z_0)$ and $P(M,z_0)$ is the common kernel of these actions; that is, it is the subgroup of the fundamental group that acts trivially on all the homotopy groups π_k for $k \geq 1$. In particular, it is a subgroup of the centre $\mathcal{Z}(\pi_1(M,z_0))$.

Part (i) of Gottlieb's theorem refines Proposition 2.4 above, and its proof is similar. The proof of part (ii) relies on ideas from Nielsen-Wecken fixed point theory.

Now return to the lifted group action. Let $\widetilde{g} \in \pi_1(G,e)$ be represented by a path $g(t)$, with $g(1) = e$. The path $g(t)$ determines a cyclic homotopy, whose trace $g(t) \cdot z_0$ determines an element of Gottlieb's group $G(M,z_0) < \pi_1(M,z_0)$. Moreover, homotopic loops in G give rise to homotopic loops in M, so this induces a well-defined homomorphism

$$a_{z_0} : \pi_1(G,e) \to \pi_1(M,z_0), \tag{3.1}$$

whose image lies in $G(M)$.

Proposition 3.2 **(i)** *The kernel $K < \pi_1(G,e)$ of a_{z_0} is independent of z_0 and acts trivially on $\widetilde{M}$ and hence on every cover of M.*

(ii) *If (N,y_0) is a cover of (M,z_0), with associated subgroup Γ_N of $\pi_1(M,z_0)$, then $K_N := a_{z_0}^{-1}(\Gamma_N)$ is independent of the choice of base point y_0 in N, and acts trivially on N.*

(iii) *If G acts effectively on M then $G_N := \widetilde{G}/K_N$ acts effectively on N.*

Note that since the domain of a_{z_0} is $\pi_1(G,e)$ which is in the centre of $\widetilde{G}$, it follows that K_N is a normal subgroup of $\widetilde{G}$. And with the notation of the proposition, $K = K_{\widetilde{M}}$ since $\Gamma_{\widetilde{M}}$ is trivial. We will write

$$G' := \widetilde{G}/K \tag{3.2}$$

for the group acting on $\widetilde{M}$.

In particular, if a_{z_0} is trivial then $K = \pi_1(G,e)$ and the G-action on M lifts to an action of G on $\widetilde{M}$. That is, a_{z_0} is the obstruction to lifting the G-action. A particular case is where the action of G on M has a fixed point. If z_0 is such a fixed point then $a_{z_0} = 0$ and so the action on M lifts to an action of G on $\widetilde{M}$, and hence on any other cover N. More generally this is true if any (and hence every) G-orbit in M is contractible in M, since in that case too a_{z_0} is trivial.

PROOF: **(i)** Let $z_0, z_1 \in M$ and let η be any path from z_0 to z_1 (recall we are assuming M is a connected manifold), and let $\widetilde{g} \in \pi_1(G,e)$ with a representative path $g(t)$. For $T \in [0,1]$ define $g^T(t) = g(Tt)$ (for $t \in [0,1]$), so $g^T \in \widetilde{G}$. Then varying T defines a homotopy from η to $(g^T \cdot \tilde{z}_0) * (g(T)(\eta)) * ((g^T)^{-1}\tilde{z}_0')$. In particular, putting $T = 1$ shows that η is homotopic to $a_{z_0}(\widetilde{g}) * \eta * a_{z_1}(\widetilde{g}^{-1})$, or equivalently that

$$\eta * a_{z_1}(\widetilde{g}^{-1}) * \overline{\eta} = a_{z_0}(\widetilde{g}^{-1}),$$

where $\overline{\eta}$ is the reverse of the path η. This composition of paths defines the standard isomorphism $\eta_* : \pi_1(M,z_1) \to \pi_1(M,z_0)$. We have shown therefore that $a_{z_0} = \eta_* \circ a_{z_1}$, and so both have the same kernel. That K acts trivially on $\widetilde{M}$ follows from the definition of a_{z_0}: let $\tilde{z} \in \widetilde{M}$ and $\widetilde{g} \in K$, then $\widetilde{g} \cdot \tilde{z} = \widetilde{g} \cdot (\tilde{z}_0 * \tilde{z}) = a_{z_0}(\widetilde{g}) * \tilde{z} = \tilde{z}$ (using Lemma 2.3).

(ii) Let $y_0, y_1 \in N$, let $z_j = p_N(y_j) \in M$ and let ζ be any path from y_0 to y_1, with η its projection to M. The result follows from the fact that the following diagram commutes:

$$\begin{tikzcd}
 & \pi_1(G,e) \arrow[dl, "a_{z_0}"'] \arrow[dr, "a_{z_1}"] & \\
\pi_1(M,z_0) \arrow[rr, "\eta_*"] & & \pi_1(M,z_1) \\
\pi_1(N,y_0) \arrow[u, "p_{(N,y_0)_*}"] \arrow[rr, "\zeta_*"] & & \pi_1(N,y_1) \arrow[u, "p_{(N,y_1)_*}"']
\end{tikzcd}$$

Writing $N = \widetilde{M}/\Gamma_N$, if $\widetilde{g} \in a_{z_0}^{-1}(\Gamma_N)$ then $\widetilde{g} \in K\Gamma_N$ and, $\widetilde{g}\Gamma_N\tilde{z} \subset \Gamma_N K\tilde{z} = \Gamma_N\tilde{z}$ so $\widetilde{g}$ acts trivially (using Proposition 2.4 and part (i)).

(iii) Suppose $\widetilde{g} \in \widetilde{G}$ acts trivially on N, so for all $y \in N$, $\widetilde{g} \cdot y = y$. Projecting to M, this implies that $g(1) \cdot z = z$ (for all $z \in M$) so $g(1) \in \cap_{z\in M} G_z = \{e\}$. Thus $\widetilde{g} \in \pi_1(G,e)$.

To prove the statement, we first consider the case $N = \widetilde{M}$. If $\widetilde{g} \notin K$ then $a_{z_0}(\widetilde{g}) \neq \tilde{z}_0 \in \pi_1(M,z_0)$. Since $\pi_1(M,z_0)$ acts effectively (by deck transformations) on the fibre $q_M^{-1}(z_0) \simeq \pi_1(M,z_0) \subset \widetilde{M}$ it follows that $a_{z_0}(\widetilde{g})$ acts non-trivially, which is in contradiction with the assumption that $\widetilde{g}$ acts trivially.

Now suppose $\widetilde{g} \in \widetilde{G}$ acts trivially on N. We have $\widetilde{g}\Gamma_N\tilde{z}_0 = \Gamma_N\tilde{z}_0$, so that $\widetilde{g} \in \Gamma_N K = a_{z_0}^{-1}(\Gamma_N)$ as required. ❒

In conclusion we have shown that a_{z_0} is the obstruction to lifting the G-action, and the following result therefore follows from Gottlieb's theorem above.

Corollary 3.3 *If M has the homotopy type of a compact polyhedron, and $\chi(M) \neq 0$ then the G-action on M lifts to a G-action on any cover of M.*

4 Isotropy subgroups

In this section we consider the isotropy subgroups for the lifted action of G_N on N and relate them to the isotropy subgroups for the original G-action on M.

Fix $y_0 \in N$ and let $\widetilde{g} \in \widetilde{G}_{y_0}$, the isotropy subgroup at y_0 for the $\widetilde{G}$ action on N. It follows that $q_G(\widetilde{g}) \in G_{z_0}$, where $z_0 = p_N(y_0)$, since $\widetilde{g} \cdot y = y \Rightarrow g \cdot z = z$. Consequently, $\widetilde{G}_{y_0}$ is a subgroup of $\Lambda_{z_0} := q_G^{-1}(G_{z_0})$. Restricting the exact sequence (2.4), we have

$$1 \to \pi_1(G,e) \to \Lambda_{z_0} \xrightarrow{q_G} G_{z_0} \to 1. \tag{4.1}$$

The group Λ_{z_0} consists of those homotopy classes of paths $g(t)$ with $g(0) = e$ and $g(1) \in G_{z_0}$. It follows that $g(t) \cdot z_0$ is a closed loop, and so determines a well-defined element of $\pi(M,z_0)$. That is, the homomorphism a_{z_0} described above extends naturally to a homomorphism

$$\bar{a}_{z_0} : \Lambda_{z_0} \to \pi_1(M,z_0).$$

In contrast to a_{z_0}, this homomorphism *does* depend on z_0. Let L_{z_0} be the kernel of this homomorphism (which obviously contains K), and $L_{(N,y_0)} := \bar{a}_{z_0}^{-1}(\Gamma_N)$ (which contains K_N). Recall that $G_N := \widetilde{G}/K_N$ from Proposition 3.2.

Remark 4.1 While the image of a_{z_0} lies in the centre of the fundamental group of M, the same cannot be said of the image of $\bar{a}_{z_0}$. The most one can say in general is that it lies in the (image of) $\pi_1(M_{(H)},z_0)$, and centralizes $\pi_1(M^H, z_0)$, where $H = G_{z_0}$ and M^H the set of H-fixed points, and $M_{(H)} = G \cdot M^H$ is the set of points of isotropy type H. (The last statement follows like Proposition 2.4, but with $\delta \in \pi_1(M^H, z_0)$.)

Proposition 4.2 *The isotropy subgroups for the lifted actions are as follows:*

(i) *at $\tilde{z}_0$ for the $\widetilde{G}$-action on $\widetilde{M}$ it is $\widetilde{G}_{\tilde{z}_0} = L_{z_0}$ and for G' it is $G'_{\tilde{z}_0} = L_{z_0}/K$*

(ii) *at $y_0 \in N$ for the $\widetilde{G}$-action on N it is $\widetilde{G}_{y_0} \simeq L_{(N,y_0)}$ and consequently, $(G_N)_{y_0} \simeq L_{(N,y_0)}/K_N$.*

PROOF: We just prove (ii) as (i) is a special case. Let $\widetilde{g} \in \widetilde{G}$ be represented by a path $g(t)$. Then $\widetilde{g} \cdot y_0 = y_0$ implies $g(1) \in G_{z_0}$; that is, $\widetilde{g} \in \Lambda_{z_0}$. Using $y_0 = \Gamma_N \tilde{z}_0$, we have $\widetilde{g} \cdot \Gamma_N \tilde{z}_0 = \Gamma_N \tilde{z}_0$ and this is equivalent to $\widetilde{g} \cdot \tilde{z}_0 \in \Gamma_N \tilde{z}_0 = \Gamma_N$ (as in (1.1)); that is, $\bar{a}_{z_0}(\widetilde{g}) \in \Gamma$, so we are done. ❐

Corollary 4.3 *If the G-action on M is free, then so is the G_N-action on N.*

PROOF: Since G_{z_0} is trivial, we have $\Lambda_{z_0} = \pi_1(G, e)$ and $\bar{a}_{z_0} = a_{z_0}$ and thus $L_{(N,y_0)} = K_N$, so $(G_N)_{y_0}$ is trivial. ❐

To identify the isotropy subgroups L_{z_0}/K or $L_{(N,y_0)}/K_N$ with subgroups of the isotropy subgroup G_{z_0} we define a homomorphism

$$\begin{array}{cccll} \psi_{z_0} & : & G_{z_0} & \longrightarrow & \operatorname{coker}(a_{z_0}) \\ & & g & \longmapsto & \widetilde{g} \cdot z_0 \mod \operatorname{image}(a_{z_0}) \end{array} \tag{4.2}$$

where $\widetilde{g} \in \Lambda_{z_0}$ is any lift of g. We take *right* cosets, so $g \bmod H = Hg$.

The homomorphism ψ_{z_0} is well defined, for given any two lifts $\widetilde{g}_1$ and $\widetilde{g}_2$ of $g \in G_z$, define $\widetilde{g}_0 \in \pi_1(G, e)$ to be the homotopy product of the path $g_1(t)$ and the reverse path of $g_2(t)$ (which goes from g to e):

$$g_0(t) = \begin{cases} g_1(2t) & \text{for } t \in [0, \frac{1}{2}] \\ g_2(2-2t) & \text{for } t \in [\frac{1}{2}, 1] \end{cases} .$$

Then $\widetilde{g}_1 \cdot \tilde{z}_0 = (\widetilde{g}_0 \cdot \tilde{z}_0) * (\widetilde{g}_2 \cdot \tilde{z}_0) \in \operatorname{image}(a_{z_0}).(\widetilde{g}_2 \cdot \tilde{z}_0)$, as required.

The homomorphism $\bar{a}_{z_0}$ induces a morphism between two short exact sequences, the lower two rows of the following commutative diagram:

$$\begin{tikzcd} K \arrow[d] & L_{z_0} \arrow[d] & G_{z_0} \arrow[d] \\ \pi_1(G,e) \arrow[r] \arrow[d, "a_{z_0}"] & \Lambda_{z_0} \arrow[r, "p_G"] \arrow[d, "\bar{a}_{z_0}"] & G_{z_0} \arrow[d] \\ \pi_1(M,z_0) \arrow[r, "="] & \pi_1(M,z_0) \arrow[r] & 1 \end{tikzcd}$$

where the first row consists of the kernels of the vertical homomorphisms.

Proposition 4.4 (i). *There is an exact sequence*

$$0 \to K \to L_{z_0} \to G_{z_0} \xrightarrow{\psi_{z_0}} \operatorname{coker}(a_{z_0}) \to \operatorname{coker}(\bar{a}_{z_0}) \to 0 \tag{4.3}$$

where the homomorphism $\psi_{z_0} : G_{z_0} \to \operatorname{coker}(a_{z_0})$ is defined above (4.2). Consequently, (ii). $G'_{\tilde{z}_0}$ *is isomorphic to* $\ker \psi_{z_0}$, *which is a subgroup of* G_{z_0}

(iii). $(G_N)_{y_0}$ *is isomorphic to* $\psi_{z_0}^{-1}(\Gamma_N \bmod \operatorname{image}(a_{z_0}))$.

Since $\operatorname{image}(\bar{a}_{z_0})$ is not in general normal in $\pi_1(M,z_0)$, $\operatorname{coker}(\bar{a}_{z_0})$ here is just the set of right cosets of $\operatorname{image}(\bar{a}_{z_0})$ in $\pi_1(M,z_0)$; and exactness at $\operatorname{coker}(\bar{a}_{z_0})$ means only that the map $\operatorname{coker}(a_{z_0}) \to \operatorname{coker}(\bar{a}_{z_0})$ is surjective (which is obvious as $\bar{a}_{z_0}$ is an extension of a_{z_0}). The first part of the proposition would be an instance of the snake lemma, but for the fact that the groups here are not all abelian.

PROOF: (i) Although not all the groups involved are abelian, the proof follows the usual diagram chasing proof of the snake lemma, so the details are omitted. Let us just make explicit the argument at $\operatorname{coker}(a_{z_0})$. Write $j : \operatorname{coker}(a_{z_0}) \to \operatorname{coker}(\bar{a}_{z_0})$, and let $\gamma \in \ker(j) \subset \pi_1(M,z_0)$. Then $\gamma \in \operatorname{image}(\bar{a}_{z_0})$, so $\exists \tilde{g} \in \Lambda_{z_0}$ such that $\gamma = \bar{a}_{z_0}(\tilde{g})$. Then $p_G(\tilde{g}) = g \in G_{z_0}$, and $\psi_{z_0}(g) = \gamma$ as required.

(ii) By (i), $\ker \psi_{z_0} = \operatorname{image}[L_{z_0} \to G_{z_0}] \simeq L_{z_0}/K$ which is $(G')_{\tilde{z}_0}$ by Proposition 4.2.

(iii) If we replace $\pi_1(M,z_0)$ by $\Delta := \pi_1(M,z_0)/\Gamma_N$ in the bottom row of the diagram above, then $a'_{z_0} : \pi_1(G,e) \to \Delta$ has kernel equal to $K_N = a_{z_0}^{-1}(\Gamma_N)$ and $\bar{a}'_{z_0} : \Lambda_{z_0} \to \Delta$ has kernel equal to $L_{(N,y_0)}$. The proof follows now in the same way as the proof of (ii). ❒

Notice firstly that the connected component of the identity $G^o_{z_0}$ of G_{z_0} is contained in $\ker \psi_{z_0}$. To see this it is enough to take $\tilde{g}$ to be a path contained entirely in $G^o_{z_0}$.

On the other hand,

$$\operatorname{image}(\psi_{z_0}) \simeq \frac{\operatorname{image}(\bar{a}_{z_0})}{\operatorname{image}(a_{z_0})}$$

so that for a given isotropy subgroup G_{z_0}, the larger the difference between the images of a_{z_0} and $\bar{a}_{z_0}$, the smaller the isotropy subgroup $G'_{\tilde{z}}$.

Example 4.5 Let M be the open Mobius band: $M = \mathbb{R} \times S^1/\sim$, where $(x,\theta) \sim (-x,\theta+\pi)$, and $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We will denote points of M as $[x,\theta]$. The fundamental group $\pi_1(M,z_0)$ is isomorphic to $\mathbb{Z}$. Consider the usual action of $G = S^1$ on M given by $\phi \cdot [x,\theta] = [x,\theta+\phi]$. Then for any $z_0 \in M$, $\operatorname{image}(a_{z_0}) = 2\mathbb{Z} < \mathbb{Z}$, so that $\psi_{z_0} : G_{z_0} \to \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$. On the other hand, for z_0 on the 'equator', $G_{z_0} \simeq \mathbb{Z}_2$ and $\operatorname{image}(\bar{a}_{z_0}) = \mathbb{Z}$. Consequently for such z_0, $\psi_{z_0} : \mathbb{Z}_2 \to \mathbb{Z}_2$ is an isomorphism, and the action of S^1 on the universal cover is then free.

Theorem 4.6 *Let N be a cover of M, and suppose the G-action on M is effective and proper. Then the G_N-action on N is also proper.*

PROOF: Since G acts properly on M there is a G-invariant Riemannian metric on M. This metric can be lifted by the covering map to one on N. Since the covering map is equivariant, it follows that the lifted metric is also G_N-invariant.

To show that action is proper, we need to show that the action map $\Phi_N : G_N \times N \to N \times N$ is closed and has compact fibres. The fibre $\Phi_N^{-1}(x,y) = \{(g,y) \in G_N \times N \mid g \cdot x = y\}$. If this is non-empty, and $h \cdot x = y$ then $\Phi_N^{-1}(x,y) \simeq h(G_N)_x$, which is compact since the G-action is proper, using Proposition 4.4.

To see that the action map is closed, consider a sequence (g_i,x_i) in $G_N \times N$ for which $(g_i \cdot x_i, x_i)$ converges to (y,z). Then of course $x_i \to z$. We claim that $g_i \cdot z \to y$. This is because,

$$d(g_i \cdot z, y) \leq d(g_i \cdot z, g_i \cdot x_i) + d(g_i \cdot x_i, y) = d(z,x_i) + d(g_i \cdot x_i, y),$$

where d is the G_N-invariant metric on N defined above. Both terms on the right tend to 0 so that $d(g_i \cdot z, y) \to 0$ as required.

Now, by Proposition 2.7 the G_N-orbits in N are closed and hence there is an $g \in G_N$ with $y = g \cdot z$. That is, $g_i \cdot z \to g \cdot z$. Consequently, $g_i(G_N)_z \to g(G_N)_z$ in $G_N/(G_N)_z$. By taking a slice to the proper $(G_N)_z$-action on G, this can be rewritten as $g_i h_i \to g$ in G_N, for some sequence $h_i \in (G_N)_z$. Since $(G_N)_z$ is compact, (h_i) has a convergent subsequence, $h_{i_k} \to h$. Then $g_{i_k} \to gh^{-1}$. It follows therefore that $(g_{i_k}, x_{i_k}) \to (gh^{-1}, z)$ and $\Phi_N(gh^{-1}, z) = (y, z)$. ❒

Remark 4.7 There is an alternative argument for proving this theorem as follows. Any invariant (Riemannian) metric on M lifts to an invariant metric on N. By a standard result, the group $I(N)$ of isometries of N acts properly on N (see [6, problem 26, p.31] and [1, p.106], although neither give a detailed proof). Since the action of G_N is by isometries, it follows from the monomorphism $A : G_N \to I(N)$ that the action of G_N is proper. The argument we give is more direct, using the covering structure of the action.

5 Orbit spaces and covers for free actions

It can be useful to compare the orbit spaces M/G and $\widetilde{M}/\widetilde{G}$ (or $\widetilde{M}/G'$ where $G' = \widetilde{G}/K$) when the G-action is free and proper, and more generally with N/G_N when N is a normal cover of M.

Let N be a normal cover of M (see the end of §1), with associated group Γ_N. Then there is an action of $G_N \times \Gamma$ on N (the action of Γ by deck transformations factors through one of Γ/Γ_1, and commutes with the G_N-action, by Proposition 2.4). We assume the G action on M is free and proper, in which case it follows from Proposition 4.3 and Theorem 4.6 that so too is the action of G_N on N.

Recall from equation (3.1) that the action of $\widetilde{G}$ defines a natural homomorphism $a_{z_0} : \pi_1(G, e) \to \pi_1(M, z_0)$, whose image lies in the centre of $\pi_1(M, z_0)$.

Proposition 5.1 *Let G act freely and properly on M. Then the natural map $q'_M : \widetilde{M}/G' \to M/G$ is a covering map, with deck transformation group equal to* $\operatorname{coker}(a_{z_0})$ *acting transitively on the fibres.*

More generally, if $p_N : N \to M$ is a normal covering then $p'_N : N/G_N \to M/G$ is a normal covering with deck transformation group $\operatorname{coker}(a_{z_0})/\Gamma_N \ := \ \Gamma/(\Gamma_N.\operatorname{image}(a_{z_0}))$.

PROOF: Since G acts freely and properly on M then G_N acts freely and properly on N, so both M/G and N/G_N are smooth manifolds. Moreover, since N is a normal cover of M, it follows that $\Delta_N := \Gamma/\Gamma_N$ acts freely and transitively on the fibres of the covering map, and so $M \simeq N/\Delta_N$ (as described in §1).

Consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\pi_{\widetilde{M}}} & \widetilde{M}/G' \\ {\scriptstyle q_N}\downarrow & & \downarrow{\scriptstyle q'_N} \\ N & \xrightarrow{\pi_N} & N/G_N \\ {\scriptstyle p_N}\downarrow & & \downarrow{\scriptstyle p'_N} \\ M & \xrightarrow{\pi_M} & M/G \end{array} \tag{5.1}$$

Since the coverings q_N and p_N are local diffeomorphisms, it follows that slices to the $\widetilde{G}$-actions can be chosen in $\widetilde{M}$, N and M in a way compatible with the coverings. Consequently the vertical maps on the right in the diagram are also coverings (the same is true if the cover N is not normal).

First consider the covering $q'_M : \widetilde{M}/G' \longrightarrow M/G$. Since the action of Γ on $\widetilde{M}$ commutes with the action of G', it descends to an action on $\widetilde{M}/G'$. Moreover, since $\widetilde{M}/\Gamma \simeq M$, so

$$(\widetilde{M}/G')/\Gamma \simeq \widetilde{M}/(G' \times \Gamma) \simeq M/G.$$

(All diffeomorphisms $\simeq$ are natural.) Furthermore, since Γ acts transitively on the fibres of $\widetilde{M} \to M$, so it does on the fibres of $\widetilde{M}/G' \to M/G$.

We claim that the isotropy subgroup of the action of Γ for any point in $\widetilde{M}/G'$ is $\Sigma = \mathrm{image}(a_{z_0})$. Indeed, for the action of $G' \times \Gamma$ on $\widetilde{M}$ the isotropy subgroup of $\tilde{x}$ is

$$H = \{(\widetilde{g},\gamma) \mid \widetilde{g}\cdot\gamma\cdot\tilde{x} = \tilde{x}\}.$$

Clearly then, $(\widetilde{g},\gamma) \in H$ implies in particular $\widetilde{g} \in \pi_1(G,e)$, and for such $\widetilde{g}$, $(\widetilde{g},\gamma)\cdot\tilde{x} = a_{z_0}(\widetilde{g}) * \gamma * \tilde{x}$ and so $(\widetilde{g},\gamma) \in H$ iff $a_{z_0}(\widetilde{g}) = \gamma^{-1}$. Thus $\gamma \in \Gamma$ acts trivially on $\widetilde{M}/G'$ if and only if $\exists \widetilde{g} \in G'$ such that $a_{z_0}(\widetilde{g}^{-1}) = \gamma$, as required for the claim. Consequently, for the covering q'_M, the deck transformation group is $\Gamma/\mathrm{image}(a_{z_0}) = \mathrm{coker}(a_{z_0})$, and this acts transitively on the fibres.

The same argument as above can be used for the more general normal covering $p_N : N \to M$, with G' replaced by G_N and Γ by Γ/Γ_N. ❒

Remark 5.2 If N is a cover of M but not a normal cover, then as pointed out in the proof N/G is a cover of M/G. Moreover, the fibre still has cardinality $\mathrm{coker}(a_{z_0})/\Gamma_N$, but the latter is not in this case a group.

Notice that if G acts freely and properly on M, then $\widetilde{M}/G'$ is connected and simply connected (the latter because G' is connected). Consequently, $\widetilde{M}/G'$ is the (a) universal cover of M/G.

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