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2008

MIMS EPrint: **2008.86**

ISSN 1749-9097

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Contents

1. Introduction

This note is a complement to the paper [Tr2], where super real closed rings are introduced and studied. A super real closed ring A is a commutative unital ring A together with an operation $F_A : A^n \longrightarrow A$ for every continuous map $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, so that all term equalities between the F's remain valid for the F_A's. For example if $C(X)$ is the ring of real valued continuous functions on a topological space X, then $C(X)$ carries a natural super real closed ring structure, where $F_{C(X)}$ is composition with F. Super real closed rings provide a natural framework for the algebra and model theory of rings of continuous functions.

A *bounded* super real closed ring A is a commutative unital ring A together with an operation $F_A : A^n \longrightarrow A$ for every *bounded* continuous map $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, so that all term equalities between the F's remain valid for the F_A's (cf. (2.7) below).

In particular every super real closed ring is a bounded super real closed ring by forgetting the operation of the unbounded functions. An example of a bounded super real closed ring, which is not a super real closed ring, is the ring $C^{\mathrm{pol}}(\mathbb{R}^n)$ of all polynomially bounded continuous functions $\mathbb{R}^n \longrightarrow \mathbb{R}$.

We show that

- bounded super real closed rings are precisely the classical localizations of super real closed rings (cf. (3.6)).
- bounded super real closed rings are precisely the convex subrings of super real closed rings (cf. (4.6)).
- there is an idempotent mono-reflector $A \mapsto \hat{A}$ from the category of bounded super real closed rings to the category of super real closed rings (cf. (5.12)). This means that every bounded super real closed ring A has a super real closed hull $\hat{A}$, $\hat{A}$ is minimal and uniquely determined up to a unique A-isomorphism. For example $\widehat{C^{\mathrm{pol}}(\mathbb{R}^n)} = C(\mathbb{R}^n)$
- Inside every bounded super real closed ring A there is a largest super real closed ring A^{Υ} (cf. (6.2)). For example $(C^{\mathrm{pol}}(\mathbb{R}^n))^{\Upsilon} = C^*(\mathbb{R}^n)$ (=the ring of bounded continuous functions $\mathbb{R}^n \longrightarrow \mathbb{R}$).

2000 Mathematics Subject Classification: Primary 03C60; Secondary 46E25, 54C05, 03E15
Keywords and Phrases: real closed rings, rings of continuous functions, model theory.

The motivation for writing this note is as follows: If A is a ring of continuous functions say $A = C(\mathbb{R}^n)$, then the Zariski sheaf $\operatorname{Spec} A$ is in general not a sheaf of rings of continuous functions. It is indeed **not** a super real closed ring in a natural way. On the other hand by the localization theorem (3.6) below, $\operatorname{Spec} A$ is always a sheaf of bounded super real closed rings and this is also true if we start with a bounded super real closed ring A.

Hence it is desirable to extend the commutative algebra of super real closed rings to the bounded case. In order not to repeat arguments, we develop tools which allow the use of the reflector $A \mapsto \hat{A}$ to explain what's going on in A. For example, forming reside rings and classical localizations behave well with respect to the reflector, cf. (5.11) and (5.13). The close relation of the ideals of A and $\hat{A}$ is worked out in (5.8) and in (5.9).

The results mentioned above can be used to transfer most of the commutative algebra, developed in [Tr2] sections 11, 12 and 13 (with the appropriate adaptions) to bounded super real closed rings. It would be tedious to elaborate this here, instead we present instruments which allow such a transfer easily, whenever it is needed in subsequent work.

The results in section 6 are not of this instrumental style. As stated above, we prove the existence of a largest super real closed ring *inside* every bounded super real closed ring and we state two explicit descriptions of this ring.

2. DEFINITION OF BOUNDED SUPER REAL CLOSED RINGS

We shall make use of the theory of **real closed ring** introduced by N. Schwartz (cf. [Schw]). However we will use it in the way explained in [Tr2], section 2. We recall this briefly. Let $\mathcal{C}_n$ be the set of all continuous maps $\mathbb{R}^n \longrightarrow \mathbb{R}$ which are 0-definable in the field $\mathbb{R}$; in other words whose graph is a boolean combination of subsets of $\mathbb{R}^n \times \mathbb{R}$ defined by polynomial inequalities $P(\bar{x}, y) \geq 0$ with $P(\bar{x}, y) \in \mathbb{Z}[\bar{x}, y]$. A ring A is real closed if there is a collection of functions $(f_A : A^n \longrightarrow A \mid n \in \mathbb{N}, f \in \mathcal{C}_n)$, such that

1. If $f \in \mathcal{C}_n$ is constant 0 or constant 1, then f_A is constant 0 or constant 1; if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the identity, then $f_A : A \longrightarrow A$ is the identity; if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is addition or multiplication in $\mathbb{R}$, respectively, then $f_A : A^2 \longrightarrow A$ is addition or multiplication in A, respectively.
2. If $f \in \mathcal{C}_n$, $k \in \mathbb{N}$ and $f_i \in \mathcal{C}_k$ $(1 \leq i \leq n)$, then
$$[f \circ (f_1, ..., f_n)]_A = f_A \circ (f_{1,A}, ..., f_{n,A}).$$

(2.1) FACT. *Every real closed ring is reduced ([Tr2], (2.2)) and*

(i) For every real closed ring there is a unique collection of functions as in the definition above. This is [Tr2], (2.13), where the functions f_A are explicitly constructed from the pure ring A.

(ii) Every ring-homomorphism $A \longrightarrow B$ between real closed rings respects the new functions f_A and f_B ([Tr2], (2.16)).

□

Because of (2.1) we may identify a real closed ring with its underlying pure ring.

(2.2) FACT. *Let A be a real closed ring. The relation $f \leq g \Leftrightarrow \exists h \in A :\ f - g = h^2$ defines a partial order on A and A together with $\leq$ is a lattice ordered ring. The supremum of f and $-f$ is denoted by $|f|$. A subring B of A is* **convex** *if $f \leq h \leq g$ and $f, g \in B$ implies $h \in B$. By [Tr2], (10.5) we have* *(i)* *There is a smallest convex subring* $\mathrm{Hol}(A)$ *of* A*, called the* **holomorphy ring***, namely*

$$\mathrm{Hol}(A) = \{f \in A \mid |f| \leq N \text{ for some } N \in \mathbb{N}\}.$$

(ii) *The convex subrings of* A *are precisely the subrings of* A *containing* $\mathrm{Hol}\, A$ *and all these subrings are real closed.*

(iii) *If* B *is a convex subring of* A*, then* $A = S^{-1}{\cdot}B$ *is the localization of* B *at* $S = A^\times \cap B$*.*

(iv) *There is a largest real closed ring* C *having* A *as a convex subring.* C *is called the* **convex closure** *of* A *(cf. [Tr2]:(11.2)). If* $C = A$*, then* A *is called* **convexly closed***. For example, real closed fields are convexly closed.*

□

For a topological space X let $C(X)$, $C^*(X)$ denote the continuous functions, the bounded continuous functions $X \longrightarrow \mathbb{R}$, respectively.

(2.3) Definition. A function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called **polynomially bounded** if there is some polynomial $P \in \mathbb{R}[X_1, ..., X_n]$ with

$$|F(x)| \leq P(x) \ (x \in \mathbb{R}^n).$$

Let $C^{\mathrm{pol}}(\mathbb{R}^n)$ be the ring of polynomially bounded continuous functions $\mathbb{R}^n \longrightarrow \mathbb{R}$.

(2.4) Observation. *Since every polynomial from* $\mathbb{R}[X_1, ..., X_n]$ *is bounded by a power of the polynomial* $2 + X_1^2 + ... + X_n^2$*, a function* $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is polynomially bounded if and only if if there is some* $p \in \mathbb{N}$ *such that*

$$\frac{|F(x)|}{(2 + x_1^2 + ... + x_n^2)^p} \leq 1 \ (x \in \mathbb{R}^n).$$

(2.5) Proposition.

(i) $C^{\mathrm{pol}}(\mathbb{R}^n)$ *is a convex subring of* $C(\mathbb{R}^n)$

(ii) $C^{\mathrm{pol}}(\mathbb{R}^n) = C^*(\mathbb{R}^n)[x_1, ..., x_n]$

(iii) $C^{\mathrm{pol}}(\mathbb{R}^n) = S^{-1}{\cdot}C^*(\mathbb{R}^n)$*, where*

$$S = \{F \in C^*(\mathbb{R}^n) \mid \text{ there is } Q \in \mathbb{R}[X_1, ..., X_n] \text{ with } F{\cdot}Q \geq 1 \text{ on } \mathbb{R}^n\}.$$

(iv) $C^{\mathrm{pol}}(\mathbb{R}^n) = C^*(\mathbb{R}^n)_P$*, where* $C^*(\mathbb{R}^n)$ *denotes the ring of bounded continuous functions* $\mathbb{R}^n \longrightarrow \mathbb{R}$ *and* P *is the polynomial* $2 + X_1^2 + ... + X_n^2$*.*

Proof. This is obvious from (2.4). □

(2.6) Definition. Recall from [Tr1], 5.1 the following notation:

$$\Upsilon := \{s : \mathbb{R} \longrightarrow \mathbb{R} \mid s \text{ is continuous and } s^{-1}(0) = \{0\}\}.$$

We define

$$\Upsilon^{\mathrm{pol}} := \Upsilon \cap C^{\mathrm{pol}}(\mathbb{R}).$$

(2.7) Definition.

(a) Let $\mathscr{L}_{\Upsilon^{\mathrm{pol}}}$ be the first order language extending the language $\{+, -, \cdot\ , 0, 1\}$ of rings, which has in addition an n-ary function symbol $\underline{F}$ for every polynomially bounded continuous function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ and every $n \in \mathbb{N}_0$.

(b) Let $T_{\Upsilon^{\mathrm{pol}}}$ be the $\mathscr{L}_{\Upsilon^{\mathrm{pol}}}$-theory which extends the theory of real closed rings and which has the following additional axioms:

1. The axioms of a commutative unital ring (with 1) in the language $\{+, -, \cdot, 0, 1\}$.

2. The axiom $\forall xy\ (\underline{+}(x,y) = x+y \wedge \underline{\cdot}(x,y) = x\cdot y \wedge \underline{\mathrm{id}}(x) = x \wedge \underline{-}(x) = -x \wedge \underline{1}(x) = 1 \wedge \underline{0}(x) = 1)$. Hence the symbols from the language of rings have the same meaning as the corresponding symbols when reintroduced in $\mathscr{L}_{\Upsilon^{\mathrm{pol}}}$ as symbols, naming continuous functions.

3. All the sentences

$$\forall \bar{x}\ \underline{F}(\underline{f}_1(\bar{x}), ..., \underline{f}_n(\bar{x})) = \underline{F \circ (f_1, ..., f_n)}(\bar{x})\ (F \in C^{\mathrm{pol}}(\mathbb{R}^n), f_1, ..., f_n \in C^{\mathrm{pol}}(\bar{R}^{\bar{x}})).$$

The models of $T_{\Upsilon^{\mathrm{pol}}}$ are called **bounded super real closed rings**.

Observe that the Null ring is also considered as a bounded super real closed ring. Moreover, since all semi-algebraic functions $\mathbb{R}^n \longrightarrow \mathbb{R}$ are polynomially bounded, it is clear that every bounded super real closed ring is real closed.

(2.8) DEFINITION. A homomorphism between $\mathscr{L}_{\Upsilon^{\mathrm{pol}}}$-structures is called a **bounded super homomorphism**. An $\mathscr{L}_{\Upsilon^{\mathrm{pol}}}$-substructure of an $\mathscr{L}_{\Upsilon^{\mathrm{pol}}}$-structure is called a **bounded super substructure**.

(2.9) REMINDER. If we drop the super script "$^{\mathrm{pol}}$" everywhere in (2.7) and (2.8) we get the definition of the language $\mathscr{L}_{\Upsilon}$, the definition of a super real closed ring (cf. [Tr2], (5.1)) and the definition of a super homomorphism.

An Υ-radical ideal of a super real closed ring A is an ideal I of A, which is closed under Υ (by which we mean closed under all the functions s_A, $s \in \Upsilon$). Those are precisely the kernels of super homomorphism (cf. [Tr2]:(6.3)).

If A is super real closed, then $\mathrm{Hol}\, A$ is a super real closed subring of A, as follows immediately from [Tr2]:(9.2)(i).

Bounded super real closed rings arise naturally from super real closed rings as convex subrings:

(2.10) LEMMA. *If B is bounded super real closed (e.g. if B is super real closed) and A is a convex subring of B, then A is a bounded super real closed subring of B.*

PROOF. Take $F \in C^{\mathrm{pol}}(\mathbb{R}^n)$ and $a_1, ..., a_n \in A$. We have to show $F_B(a_1, ..., a_n) \in A$. Since F is polynomially bounded, there is some $P \in \mathbb{R}[X_1, ..., X_n]$ with $|F| \leq P$ on $\mathbb{R}^n$. Let $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $\chi(x) = -x$, if $x \leq 0$ and $\chi(x) = 0$ if $x \geq 0$. Then $|F| \leq P$ reads as $\chi \circ (P - |F|) = 0$ on $\mathbb{R}^n$. Since B is bounded super real closed, also $(\chi \circ (P - |F|))_B = 0$. By definition, this means $\chi_B \circ (P - |F|)_B = 0$. Since B is real closed $\chi_B(b) = 0$ is equivalent to $b \geq 0$ in B ($b \in B$). Hence we have $(P - |F|)_B(a_1, ..., a_n) \geq 0$. In the bounded super real closed ring B, this means $|F_B(a_1, ..., a_n)| \leq P(a_1, ..., a_n) \in A$. Since A is convex in B, $F_B(a_1, ..., a_n) \in A$. □

3. LOCALIZATION OF BOUNDED SUPER REAL CLOSED RINGS

First recall how we can localize super real closed rings:

(3.1) THEOREM. *(cf. [Tr2]:(7.4))*
Let A be a super real closed ring and let $1 \in S \subseteq A$ be closed under multiplication and Υ. Then there is a unique expansion of the localization $S^{-1}A$ to a super real closed ring such that the localization map $A \longrightarrow S^{-1}A$ is a super homomorphism.

The operation of $F \in C(\mathbb{R}^n)$ *on* $(S^{-1}A)^n$ *is given as follows: There are* $t \in \Upsilon$ *and a continuous function* $G \in C(\mathbb{R}^n \times \mathbb{R})$ *with*

$$F(x_1, ..., x_n) \cdot t(y) = G(x_1 \cdot y, ..., x_n \cdot y, y) \ ((\bar{x}, y) \in \mathbb{R}^n \times \mathbb{R}).$$

Then for $f_1, ..., f_n \in A$ *and* $g \in S$

$$F_{S^{-1}A}(\frac{f_1}{g}, ..., \frac{f_n}{g}) := \frac{G_A(f_1, ..., f_n, g)}{t_A(g)}.$$

(3.2) Lemma. *Let* $F, G \in C(\mathbb{R}^n)$ *such that* $\{G = 0\} \subseteq \operatorname{int}\{F = 0\}$*, the interior of the zero set of* F*. Then there is a unique* $H \in C(\mathbb{R}^n)$ *with* $F = H \cdot G$ *such that* $H = 0$ *on* $\{G = 0\}$*.*

If there are a bounded subset B *of* $\mathbb{R}^n$ *and some* $\varepsilon \in \mathbb{R}$*,* $\varepsilon > 0$ *such that* $|G|_{\mathbb{R}^n \setminus B}| \geq \varepsilon$*, then* $|H| \leq c \cdot |F|$ *for some* $c \in \mathbb{R}$*,* $c > 0$*.*

Proof. Existence and uniqueness of H is clear. Assume there are B, ε as stated. Then $K := \overline{B} \setminus \operatorname{int}\{F = 0\}$ is compact and G does not have zeroes on K. Let $c \in \mathbb{R}$, such that $c \geq \frac{1}{\varepsilon}$ and $\frac{1}{c} \leq |G|_K|$. Then for every $x \in \mathbb{R}^n$ we have $|H(x)| \leq c \cdot |F(x)|$: this holds true if $F(x) = 0$, since $F = H \cdot G$ and H vanishes on $\{G = 0\}$. If $F(x) \neq 0$, then $x \notin B$ or $x \in K = B \setminus \operatorname{int}\{F = 0\}$. In both cases we get the assertion by the choice of c. □

(3.3) Corollary. *Let* A *be bounded super real closed. Let* $r \in \mathbb{R}$ *,* $F_1, F_2 \in C^{\mathrm{pol}}(\mathbb{R}^n)$ *and* $a_1, ..., a_n \in A$ *be such that* $|a_i| \leq r$ $(1 \leq i \leq n)$ *and such that* $F_1(x) = F_2(x)$ *for all* $x \in \mathbb{R}^n$ *with* $|x| < r + 1$*.*

Then

$$F_{1,A}(a_1, ..., a_n) = F_{2,A}(a_1, ..., a_n).$$

Proof. Let $G \in C(\mathbb{R}^n)$ be the distance function to the ball with radius r around 0. By assumption, $\{G = 0\} \subseteq \operatorname{int}\{F_1 - F_2 = 0\}$. By (3.2), there is some $H \in C(\mathbb{R}^n)$ with $F_1 - F_2 = H \cdot G$ and since $G \geq 1$ outside $\{|x| \leq r + 1\}$ we know that $|H| \leq c \cdot |F_1 - F_2|$ for some $c \in \mathbb{R}$. Since F_1, F_2 are polynomially bounded, also $H \in C^{\mathrm{pol}}(\mathbb{R}^n)$. Thus

$$F_{1,A}(a_1, ..., a_n) - F_{2,A}(a_1, ..., a_n) = H_A(a_1, ..., a_n) \cdot G_A(a_1, ..., a_n).$$

Since $|a_i| \leq r$ for each i we know that $G_A(a_1, ..., a_n) = 0$, which implies the corollary. □

(3.4) Proposition and Definition. *Let* A *be a bounded super real closed ring. The holomorphy ring* $\operatorname{Hol} A$ *is a bounded super real closed subring of* A *and there is a unique super real closed ring structure on* $\operatorname{Hol} A$*, which expands the bounded super real closed ring structure.*

For $F \in C(\mathbb{R}^n)$ *and* $a_1, ..., a_n \in \operatorname{Hol}(A)$ *we have*

$$(\dagger) \qquad F_{\operatorname{Hol} A}(a_1, ..., a_n) = G_{\operatorname{Hol} A}(a_1, ..., a_n)$$

whenever $G \in C^{\mathrm{pol}}(\mathbb{R}^n)$ *is such that for some* $r \in \mathbb{N}$ *with* $|a_i| \leq r$ *we have* $F(x) = G(x)$ $(x \in \mathbb{R}^n,\ |x| \leq r + 1)$*.*

Proof. $\operatorname{Hol} A$ is a bounded super real closed subring of A, since for all $a_1, ..., a_n \in \operatorname{Hol} A$ and each $F \in C^{\mathrm{pol}}(\mathbb{R}^n)$, there are $r \in \mathbb{R}$ with $|a_i| \leq r$ and a bounded $F^* \in C^*(\mathbb{R}^n)$ such that $F(x) = F^*(x)$ $(|x| \leq r + 1)$; hence by (3.3), $F_A(a_1, ..., a_n) = F^*_A(a_1, ..., a_n) \in \operatorname{Hol} A$.

By (3.3), we may use (†) to define an $\mathscr{L}_\Upsilon$-structure on $\operatorname{Hol}(A)$ which by definition expands the bounded super real closed ring structure on $\operatorname{Hol} A$. It is straightforward (using (3.3)) to check that this defines the unique super real closed ring structure on $\operatorname{Hol} A$ which expands the bounded super real closed ring structure. □

(3.5) THEOREM. *Let $F \in C^{\mathrm{pol}}(\mathbb{R}^n)$ and $P(T) \in \mathbb{R}[T]$, $T = (T_1, ..., T_n)$ of total degree d with $|F| \leq |P|$ on $\mathbb{R}^n$. Then there is a unique continuous function $G \in C(\mathbb{R}^n \times \mathbb{R})$ with*

$$(*) \qquad F(x_1, ..., x_n)\cdot y^{d+1} = G(x_1\cdot y, ..., x_n\cdot y, y) \ ((\bar{x}, y) \in \mathbb{R}^n \times \mathbb{R}).$$

Moreover G is polynomially bounded.

PROOF. Existence is given by [Tr2]:(7.2)(ii). Uniqueness holds, since $G(x_1, ..., x_n, y)$ is uniquely determined by $(*)$ for all $(\bar{x}, y) \in \mathbb{R}^n \times \mathbb{R}$ with $y \neq 0$. Moreover the proof of [Tr2]:(7.2)(ii) shows that G is again polynomially bounded (by a polynomial of total degree d. □

(3.6) THEOREM. *Let A be a bounded super real closed ring and let $1 \in S \subseteq A$ be multiplicatively closed. Then there is a unique expansion of the localization $S^{-1}A$ to a bounded super real closed ring such that the localization map $A \longrightarrow S^{-1}A$ is a bounded super homomorphism.*

The operation of $F \in C^{\mathrm{pol}}(\mathbb{R}^n)$ on $(S^{-1}A)^n$ is given as follows: Pick $d \in \mathbb{N}_0$ such that F is bounded by a polynomial of total degree d and take a polynomially bounded continuous function $G \in C(\mathbb{R}^n \times \mathbb{R})$ with

$$F(x_1, ..., x_n)\cdot y^{d+1} = G(x_1\cdot y, ..., x_n\cdot y, y) \ ((\bar{x}, y) \in \mathbb{R}^n \times \mathbb{R}).$$

Such functions exist by (3.5). Then for $f_1, ..., f_n \in A$ and $g \in S$

$$F_{S^{-1}A}\left(\frac{f_1}{g}, ..., \frac{f_n}{g}\right) := \frac{G_A(f_1, ..., f_n, g)}{g^{d+1}} \in S^{-1}A.$$

PROOF. The proof is parallel to the proof of the localization theorem [Tr2]:(7.4), using (3.5) instead of [Tr2]:(7.2)(i). □

(3.7) COROLLARY. *Let $\varphi : A \longrightarrow B$ be a super homomorphism between bounded super real closed rings and let $1 \in S \subseteq A$ be multiplicatively closed such that $\varphi(S) \subseteq B^\times$. Then the natural map $S^{-1}A \longrightarrow B$ is a super homomorphism, too.*

PROOF. This follows immediately from the explicit definition of the bounded super real closed structure on $S^{-1}\cdot A$ in (3.6). □

4. THE SUPER REAL CLOSED HULL

For a bounded super real closed ring A, we shall now define the smallest super real closed ring containing A as a bounded super real closed subring.

(4.1) THEOREM AND DEFINITION. *Let A be a bounded super real closed ring. Let*

$$\hat{A} = S^{-1}\cdot \mathrm{Hol}\, A,$$

where S is the closure of $A^\times \cap \mathrm{Hol}\, A$ under multiplication and Υ (recall: this means "closed under all the functions $s_{\mathrm{Hol}\, A}$, $s \in \Upsilon$"); here we consider $\mathrm{Hol}(A)$ equipped with the super real closed ring structure defined in (3.4). Then there is a unique $\mathscr{L}_\Upsilon$-structure on $\hat{A}$ such that $\hat{A}$ is a super real closed ring having A as a bounded super real closed subring. $\hat{A}$ is called the **super real closed hull of** *A.*

PROOF. Firstly, as $A^\times \cap \mathrm{Hol}\, A \subseteq S$ we have $A = (A^\times \cap \mathrm{Hol}\, A)^{-1}\cdot \mathrm{Hol}\, A \subseteq S^{-1}\cdot \mathrm{Hol}\, A = \hat{A}$. By (3.4), $\mathrm{Hol}(A)$ is a bounded super real closed subring of A and there is a unique expansion of this structure to a super real closed ring. By definition, S is closed under multiplication and Υ. By (3.1), there is a unique $\mathscr{L}_\Upsilon$-structure on $\hat{A}$ such that $\hat{A}$ is a super real closed ring having $\operatorname{Hol} A$ as a super real closed subring. Since $\hat{A}$ is also the localization of A at S, (3.6) implies that A is a bounded super real closed subring of $\hat{A}$.

It remains to show that $\hat{A}$ with the $\mathscr{L}_\Upsilon$-structure defined above is the unique super real closed ring structure on $\hat{A}$ having A as a bounded super real closed subring. However, any other super real closed ring B expanding the pure ring $\hat{A}$ having A as a bounded super real closed subring, has $\operatorname{Hol} A$ as a super real closed subring (cf. [Tr2]:(9.2)(i)) and the underlying bounded super real closed ring structure is the one induced from A. By (3.4), the super real closed ring structures of B and $\hat{A}$ induced on $\operatorname{Hol} A$ are equal. From the uniqueness property in (3.1) we know that B is the super real closed ring $\hat{A}$. $\square$

(4.2) Corollary. *Let $F, G \in C^{\mathrm{pol}}(\mathbb{R}^n)$.*
(i) If $\{F = 0\} \subseteq \{G = 0\}$, then $T_{\Upsilon^{\mathrm{pol}}} \vdash \forall \bar{x} \;\; \underline{F}(\bar{x}) = 0 \rightarrow \underline{G}(\bar{x}) = 0$.
(ii) If $\{F \geq 0\} \subseteq \{G \geq 0\}$, then $T_{\Upsilon^{\mathrm{pol}}} \vdash \forall \bar{x} \;\; \underline{F}(\bar{x}) \geq 0 \rightarrow \underline{G}(\bar{x}) \geq 0$.

Proof. (i). Let $A \models T_{\Upsilon^{\mathrm{pol}}}$. By [Tr2]:(5.5)(iv) the super real closed ring $\hat{A}$ is a model of

$$\forall \bar{x} \; \underline{F}(\bar{x}) = 0 \rightarrow \underline{G}(\bar{x}) = 0.$$

Since A is a bounded super real closed subring of $\hat{A}$ (by (4.1)), also A is a model of this sentence.

(ii) follows from (i), since in every real closed ring A, the formula $x \geq 0$ is equivalent to $p_A(x) = 0$, where $p : \mathbb{R} \longrightarrow \mathbb{R}$ is the infimum of the identity function and the constant function 0. $\square$

(4.3) Lemma. *Let A be a bounded super real closed subring of the super real closed ring B. There is a unique A-algebra homomorphism $\hat{A} \longrightarrow B$ and this homomorphism is an embedding of super real closed rings.*

Proof. We have $S_0 := A^\times \cap \operatorname{Hol} A \subseteq T := B^\times \cap \operatorname{Hol} B$. Since B is super real closed, T is closed under Υ: this follows from [Tr2]:(6.12), which says that all maximal ideal of B are Υ-radical.

Since $\operatorname{Hol} A$ is a super real closed subring of $\operatorname{Hol} B$ by (3.4), $T \cap \operatorname{Hol} A$ is Υ-closed as well. Thus the closure S of S_0 under Υ and multiplication is contained in T, too. Hence we get a unique A-algebra homomorphism $\varphi : \hat{A} = S^{-1} \cdot \operatorname{Hol}(A) \longrightarrow T^{-1} \cdot \operatorname{Hol}(B) = B$ and this map is injective. It remains to show that φ is a super homomorphism. This follows immediately from the definition of the $\mathscr{L}_\Upsilon$-structure on both rings in (3.1). $\square$

(4.4) Corollary. *If A is a super real closed ring, then $\hat{A}$ (defined for the underlying bounded super real closed ring) is equal to A. In particular, the $\mathscr{L}_\Upsilon$-structure of A is uniquely determined by the $\mathscr{L}_{\Upsilon^{\mathrm{pol}}}$-structure.* $\square$

(4.5) Corollary. *Let B be a super real closed ring and let A be a bounded super real closed subring of A. Then $B \cong_A \hat{A}$ as (bounded) super real closed rings if and only if B is generated by A as a super real closed ring.*

Proof. Let $C \subseteq \hat{A}$ be the super real closed subring generated by A. By (4.3) there is a super real A-algebra monomorphism $\hat{A} \longrightarrow C$. Composing this map with the inclusion $C \longrightarrow \hat{A}$ and using uniqueness shows that $C = \hat{A}$. Hence $\hat{A}$ is generated by A as a super real closed ring.

Conversely suppose B is generated by A as a super real closed ring. By (4.3), we may view $\hat{A}$ as a super real closed subring of B. Since B is generated by A we get $B = \hat{A}$. $\square$

(4.6) PROPOSITION. *Let A be a bounded super real closed ring. Then A is convex in $\hat{A}$, in other words $\hat{A}$ is a subring of the convex closure B of A. There is a unique super real closed ring-structure on B extending the bounded super real closed ring structure on A. In particular, every bounded super real closed ring which is convexly closed (e.g. a field) has a unique expansion to a super real closed ring.*

PROOF. Since $\mathrm{Hol}(A)$ is convex in A, the convex closure B of $\mathrm{Hol}\,A$ contains A. By [Tr2]:(11.2)(iii) we know that B is the localization of $\mathrm{Hol}\,A$ at the set T of all non zero-divisors t of $\mathrm{Hol}\,A$ with the property that $\mathrm{Hol}\,A$ is convex in $(\mathrm{Hol}\,A)_t$. It follows $A^\times \cap \mathrm{Hol}\,A \subseteq T$. Since T is closed and closed under Υ by [Tr2]:(11.11), the closure S of $A^\times \cap \mathrm{Hol}\,A$ is contained in T. Hence $\hat{A} = S^{-1}\cdot\mathrm{Hol}(A) \subseteq T^{-1}\cdot\mathrm{Hol}(A) = B$, in other words A is convex in $\hat{A}$.

By [Tr2]:(11.12), there is a (unique) expansion of B to a super real closed ring having $\hat{A}$ as super real closed subring. Since B is a localization of $\hat{A}$ we get the uniqueness statement of the proposition from the uniqueness statement in (3.6) together with (4.4). $\square$

(4.7) COROLLARY. *Let A be a bounded super real closed subring of a super real closed ring B. Then A is convex in the super real closed ring generated by A in B.*

PROOF. By (4.5) and (4.6). $\square$

5. SUPER REAL IDEALS

(5.1) DEFINITION. An ideal I of a bounded super real closed ring A is called **super real** if $s_A(I) \subseteq I$ for every $s \in \Upsilon^{\mathrm{pol}}$. Observe that in this case I is a radical ideal, in particular I is convex and satisfies $a \in I \Leftrightarrow |a| \in I$ $(a \in A)$.

Certainly, every ideal I of A is contained in a smallest super real ideal of A, denoted by $\sqrt[\Upsilon]{I}$.

If A is a super real closed ring, then by [Tr2]:(6.10), the super real ideals are precisely the Υ-radical ideals (clearly Υ^{pol} is a set of generalized root functions as defined in [Tr2]:(5.5)).

(5.2) EXAMPLES. Let A be a bounded super real closed ring.

(i) If $F \in C^{\mathrm{pol}}(\mathbb{R})$ is strictly positive everywhere, then in general $F_A(a)$ is not a unit for every unit $a \in A$. For example if A is the bounded super real closed ring $C^{\mathrm{pol}}(\mathbb{R})$, $F = \exp(-x^2)$ and $a = 1 + x^2 \in A$.

(ii) If $a \in A$ is a unit, then in general, there is some $s \in \Upsilon^{\mathrm{pol}}$, which is bounded away from 0 outside $[-1, 1]$ such that $s_A(a)$ is not a unit. For example if A is the bounded super real closed ring $C^{\mathrm{pol}}(\mathbb{R})$, $s = \exp(-\frac{1}{|x|})$ and $a = \frac{1}{1+x^2} \in A$.

Hence in this example, the ideal $I = (s(a))$ of A is proper, but the super real radical of I is not proper. In particular, maximal ideals of bounded super real closed rings are not super real in general. The example also shows that this is not resolved if we replace Υ^{pol} by the set of all $s \in \Upsilon^{\mathrm{pol}}$, which are bounded away from 0 outside a neighborhood of 0: or to replace Υ^{pol} by the set of all bounded $s \in \Upsilon$ such that βs does not have zeroes different from 0 in the Stone-Cech compactification $\beta\mathbb{R}$ of $\mathbb{R}$.

(5.3) REMARK. If $A \subseteq B$ is an extension of rings and I is an ideal of A, then $I\cdot B$ denotes the ideal generated by I in B. Recall that for a convex subring A of a real closed ring B and every radical ideal I of A we have $I\cdot B = \{a\cdot b \mid a \in I,\ b \in B\}$ and this ideal is again radical.

Our first goal in this section is to show that (5.3) remains valid in the bounded super real closed context. That is, whenever $A \subseteq B$ is a convex extension of bounded super real closed rings and I is a super real ideal of A, then $I{\cdot}B$ is a super real ideal of B (cf. (5.7)). In order to prove this we show that for every $s \in \Upsilon^{\text{pol}}$, there are $t \in \Upsilon^{\text{pol}}$ and $F \in C^{\text{pol}}(\mathbb{R}^2)$ with $s(x{\cdot}y) = t(x){\cdot}F(x,y)$. This is achieved in (5.6) below. First two preparational lemmas from elementary analysis:

(5.4) LEMMA. *Let $A \subseteq \mathbb{R}^2$ be compact with projection $[a,b]$ onto the first coordinate. Let C be the convex hull of A. Then C is again compact and the function $f : [a,b] \longrightarrow \mathbb{R}$ defined by $f(x) = \max C_x$ is continuous, convex and satisfies $f(a) = \max A_a$, $f(b) = \max A_b$. Here C_x denotes the set $\{y \in \mathbb{R} \mid (x,y) \in C\}$ and similarly for A_a, A_b.*

Moreover, if A is the graph of a strictly increasing function $[a,b] \longrightarrow \mathbb{R}$, then also f is strictly increasing.

PROOF. C is compact by classical convex geometry (hint: C is the union of 3-simplices with vertices in A, now use compactness of A). Hence f is well-defined. Since C is convex, the function f is clearly convex, i.e. satisfies $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$ for all $0 \leq \lambda \leq 1$. We have $f(a) = \max A_a$, since for every $y > \max A_a$, there is obviously a convex set containing A but not containing the point (a,y). Similarly $f(b) = \max A_b$.

Suppose f is not continuous. Take a point $x \in [a,b]$, some $\varepsilon > 0$ and a sequence $(x_n) \subseteq [a,b]$ converging to x such that $|f(x_n) - f(x)| \geq \varepsilon$. Since C is compact we may assume that $(f(x_n))$ converges as well, say with limit y. Then $(x,y) \in C$, hence by definition of $f(x)$, $y \leq f(x)$ and by choice of ε and the x_n we have $y + \varepsilon < f(x)$. Hence there are x_n, arbitrary close to x such that $f(x_n) + \frac{\varepsilon}{2} < f(x)$. But this contradicts the convexity inequality for f.

Finally assume that A is the graph of a strictly increasing function $g : [a,b] \longrightarrow \mathbb{R}$. For $a \leq x < y \leq b$ we show $f(x) < f(y)$. Since g is increasing, C is contained in $[a,b] \times [g(a), g(b)]$ and so $f \leq g(b)$ everywhere with $f(b) = g(b)$.
If $f(x) < g(b)$, then by convexity of f, $f(y) > f(x)$. On the other hand $f(x) = g(b)$ is not possible, since C is the union of the convex hulls of finite subsets of $A =$Graph(g): no such set contains $(x, g(b))$ as g is strictly increasing. □

(5.5) LEMMA. *Let $s_n \in \Upsilon$ $(n \in \mathbb{N})$. Then there is some $t \in \Upsilon$, $0 \leq t \leq 1$, symmetric (i.e. $t(-x) = t(x)$), non-decreasing and convex in $[0,\infty)$ such that for every $n \in \mathbb{N}$ there is some $\delta > 0$ with*

$$t(x) \geq |s_n(x)| \ (|x| < \delta).$$

PROOF. We may replace $s_n(x)$ by

$$|x| + \max\{|s_k(y)| \mid |y| \leq |x|,\ 1 \leq k \leq n\}.$$

Then the new s_n is symmetric (i.e. $s_n(x) = s_n(-x)$), continuous, its restriction to $[0,\infty)$ is a strictly increasing homeomorphism $[0,\infty) \longrightarrow [0,\infty)$ and we have $s_1 \leq s_2 \leq \ldots$.

Let $\rho_n := s_n^{-1}(\frac{1}{n})$, where s_n^{-1} denotes the compositional inverse of $s_n \restriction [0,\infty)$. Then $\rho_n \leq s_1^{-1}(\frac{1}{n})$, otherwise $\frac{1}{n} = s_1(s_1^{-1}(\frac{1}{n})) < s_1(\rho_n) \leq s_n(\rho_n) = \frac{1}{n}$, a contradiction. Since $\lim_{n\to\infty} s_1^{-1}(\frac{1}{n}) = 0$ also $\lim_{n\to\infty} \rho_n = 0$.

Moreover $\rho_{n+1} < \rho_n$, otherwise $\frac{1}{n} = s_n(s_n^{-1}(\frac{1}{n})) = s_n(\rho_n) \leq s_n(\rho_{n+1}) \leq s_{n+1}(\rho_{n+1}) = \frac{1}{n+1}$, a contradiction.

Hence (ρ_n) is a strictly decreasing sequence with $\lim_{n\to\infty} \rho_n = 0$ and we define a function $\tau : (0,\rho_1] \longrightarrow \mathbb{R}$ as follows: Given $x \in (0,\rho_1]$ there is a unique $n \in \mathbb{N}$ such that $\rho_{n+1} < x \leq$

ρ_n. Take the unique $\lambda \in (0,1]$ with $x = \lambda\rho_n + (1-\lambda)\rho_{n+1}$ and define

$$\tau(x) = \max\{s_n(x), \lambda\frac{1}{n} + (1-\lambda)\frac{1}{n+1}\}.$$

Then $\tau(\rho_n) = \frac{1}{n}$ $(n \in \mathbb{N})$ and

$$\lim_{x\to\rho_{n+1}, x>\rho_{n+1}} \tau(x) = \max\{s_n(\rho_{n+1}), \frac{1}{n+1}\} = \frac{1}{n+1} \text{ as } s_n(\rho_{n+1}) \leq s_{n+1}(\rho_{n+1}) = \frac{1}{n+1}.$$

Hence τ is continuous. For $x \in (\rho_{n+1}, \rho_n]$ we have $\tau(x) \leq \max\{s_n(\rho_n), \frac{1}{n}\} = \frac{1}{n}$, which shows that $\lim_{x\to 0}\tau(x) = 0$ and τ has a continuous extension to $[0,\rho_1]$ via $\tau(0) = 0$.

Since each s_n is strictly increasing, the function τ is increasing in each interval $[\rho_{n+1}, \rho_n]$. Since $\tau(\rho_n) = \frac{1}{n}$ we get that τ is a strictly increasing homeomorphism $[0,\rho_1] \longrightarrow [0,1]$.

We now define a function $t : [0,\rho_1] \longrightarrow [0,1]$. Let C be the convex hull of the graph of τ and let

$$t(x) := \sup C_x \ (0 \leq x \leq \rho_1),$$

where $C_x := \{y \in \mathbb{R} \mid (x,y) \in C\}$. By (5.4), t is a strictly increasing and convex homeomorphism $[0,\rho_1] \longrightarrow [0,1]$. We extend t to $\mathbb{R}$ via $t(x) = 1$ if $x \geq \rho_1$ and $t(x) = t(-x)$ if $x < 0$. Then t is symmetric, $0 \leq t \leq 1$ and it is straightforward to check that t is still convex in $[0,\infty)$.

It remains to show that for $n \in \mathbb{N}$, there is some $\delta > 0$ with $t(x) \geq s_n(x)$ $(|x| < \delta)$: We take $\delta = \rho_n$ and we may assume that $x \in (0,\delta)$. Pick $m \geq n$ with $\rho_{m+1} < x \leq \rho_m$. By definition of τ, $\tau(x) \geq s_m(x)$, thus $\tau(x) \geq s_m(x)$ as $s_m \geq s_n$. By definition of t we have $t(x) \geq \tau(x)$ which shows the claim. ☐

(5.6) PROPOSITION. *Let $s \in \Upsilon$. There are $t \in \Upsilon$ with $0 \leq t \leq 1$, $c \in \mathbb{R}$ and $F \in C(\mathbb{R}^2)$ such that*

$$s(x\cdot y) = t(x)\cdot F(x,y) \quad \textit{and } |F(x,y)| \leq c\cdot\Big(1 + (1+|y|)\cdot|s(x\cdot y)|\Big) \ ((x,y) \in \mathbb{R}^2)$$

PROOF. Let $s_0(x) = x$ and for $n > 0$, $s_n(x) := \max_{|y|\leq n} n\cdot|s(y\cdot x)|$. Then $s_n \in \Upsilon$ and from (5.5) we get some $t \in \Upsilon$, symmetric with $0 \leq t \leq 1$, non-decreasing and convex in $[0,\infty)$ such that for every $n \in \mathbb{N}_0$ there is some $\delta > 0$ with

$$t(x) \geq |s_n(x)| \ (|x| < \delta).$$

By definition of s_n for $n \geq 1$ this means

$$(*) \qquad t(x) \geq n\cdot|s(yx)| \ (|x| < \delta, |y| \leq n).$$

We first show that the function $\frac{s(xy)}{t(x)}$, defined on $(\mathbb{R}\setminus\{0\})\times\mathbb{R}$ has a continuous extension F through 0 on $\mathbb{R}\times\mathbb{R}$:

Pick $b \in \mathbb{R}$. For $n \in \mathbb{N}$ we have to find some $\delta > 0$ with $|\frac{s(xy)}{t(x)}| < \frac{1}{n}$ for all $x \in (-\delta,\delta)$, $x \neq 0$ and all y with $|b-y| < \delta$. Enlarge n if necessary such that $|b| < n$ and take $\delta > 0$ with $|b| + \delta < n$ such that $(*)$ holds. Let $0 < |x| < \delta$ and $|b-y| < \delta$. Then $|y| < |b| + \delta < n$. Thus $|s(xy)| \leq \frac{1}{n}t(x)$, as desired.

It remains to find $c \in \mathbb{R}$ such that for all $(x,y) \in \mathbb{R}^2$, $x \neq 0$ we have

$$(\dagger) \qquad |\frac{s(x\cdot y)}{t(x)}| \leq c\cdot(1 + (1+|y|)\cdot|s(x\cdot y)|).$$

By choice of t there is some $\delta > 0$ such that $t(x) \geq |s(x)|$ and $t(x) \geq |x|$ for all x with $|x| < \delta$. It is enough to find an element c satisfying (†) separately on each of the the following four subsets of $\mathbb{R}^2$, covering $\mathbb{R}^2$:

<u>Case 1.</u> $|x| \geq \delta$.
Then $t(x) = t(|x|) \geq t(\delta) > 0$ since t is symmetric and increasing in $[0,\infty)$. Hence $|\frac{s(xy)}{t(x)}| \leq |\frac{s(xy)}{t(\delta)}|$ and we may choose $c := \frac{1}{t(\delta)}$.
<u>Case 2.</u> $|x| < \delta$ and $|y| \leq 1$.
As F is continuous we may choose c as the maximum of $|F|$ on the rectangle $[-\delta,\delta]\times[-1,1]$.
<u>Case 3.</u> $|x| < \delta$ and $|y| \geq 1$ and $|x{\cdot}y| \geq \delta$.
Then by the choice of δ we have $t(x) = t(|x|) \geq |x|$, hence $|\frac{s(xy)}{t(x)}| \leq |\frac{s(xy)}{x}| \leq |y{\cdot}\frac{s(xy)}{\delta}|$, since $\frac{1}{|x|} \leq |\frac{y}{\delta}|$ by assumption in case 3. Hence we may choose $c = \frac{1}{\delta}$.
<u>Case 4.</u> $|x| < \delta$ and $|y| \geq 1$ and $|x{\cdot}y| < \delta$.
Since t is convex in $[0,\infty)$ and $\frac{1}{|y|} \leq 1$ we have $t(x) = t(|x|) = t(\frac{1}{|y|}{\cdot}|x{\cdot}y|) \geq \frac{1}{|y|}{\cdot}t(|x{\cdot}y|) = \frac{1}{|y|}{\cdot}t(x{\cdot}y)$. Since $|x{\cdot}y| < \delta$ we have $t(x{\cdot}y) \geq |s(x{\cdot}y)|$ by the choice of δ. Hence

$$|\frac{s(x{\cdot}y)}{t(x)}| \leq |y|{\cdot}\frac{t(x{\cdot}y)}{t(x{\cdot}y)} = |y|$$

and we may choose $c = 1$. □

(5.7) Proposition. *Let A be a convex subring of a bounded super real closed ring B. If I is a super real ideal of A then $I{\cdot}B$ is super real, too.*

Proof. For $a \in I$, $b \in B$ and $s \in \Upsilon^{\mathrm{pol}}$ we have to show that $s_B(a{\cdot}b) \in I{\cdot}B$. By (5.6) there are $t \in \Upsilon^{\mathrm{pol}}$, $c > 0$ and $F \in C(\mathbb{R}^2)$ with

$$s(x{\cdot}y) = t(x){\cdot}F(x,y) \ ((x,y) \in \mathbb{R}^2)$$

such that $|F(x,y)| \leq c{\cdot}\Big(1+(1+|y|){\cdot}|s(x{\cdot}y)|\Big)$ everywhere. Since s is polynomially bounded also F is polynomially bounded. Hence $s_B(a{\cdot}b) = t_B(a){\cdot}F_B(a,b)$. Since $t_B(a) = t_A(a) \in I$ we get the claim. □

If A is a super real closed ring and I is an ideal of A, then there is a largest super real ideal I^Υ of A contained in I and $I^\Upsilon = \{a \in I \mid s_A(a) \in I \text{ for all } s \in \Upsilon\}$. (cf. [Tr2]:(6.7)).
With the aid of (5.7), this can be extended to bounded super real closed rings:

(5.8) Proposition and Definition. *Let A be a bounded super real closed ring. If I is an ideal of A, then there is a largest super real ideal I^Υ contained in I. We have*

$$I^\Upsilon = \{a \in I \mid s_A(a) \in I \textit{ for all } s \in \Upsilon^{\mathrm{pol}}\} = (I \cap \mathrm{Hol}\, A)^\Upsilon{\cdot}A.$$

Proof. Let $J := (I \cap \mathrm{Hol}\, A)^\Upsilon$. By (5.7) we know that $J{\cdot}A$ is super real. Moreover it is clear that every super real ideal of A contained in I has to be contained in $K := \{a \in I \mid s_A(a) \in I \text{ for all } s \in \Upsilon^{\mathrm{pol}}\}$. In particular $J{\cdot}A \subseteq K$ and it remains to show that $K \subseteq J{\cdot}A$.

Pick $a \in K$. Since $\frac{1}{1+a^2}, \frac{a}{1+a^2} \in \mathrm{Hol}\, A$ we have $\frac{a}{1+a^2} \in I \cap \mathrm{Hol}\, A$ and it remains to show that $\frac{a}{1+a^2} \in (I \cap \mathrm{Hol}\, A)^\Upsilon$. It suffices to show $s_A(|\frac{a}{1+a^2}|) \in I$ for every strictly increasing $s \in \Upsilon$ and indeed by [Tr2]:(6.7) it suffices to take $s \in \Upsilon^{\mathrm{pol}}$. Since $|\frac{a}{1+a^2}| \leq |a|$ we have $\sqrt{s}_A(|\frac{a}{1+a^2}|) \leq \sqrt{s}_A(|a|) \in I$ by our choice of a in K. Now the convexity condition for real closed rings [*] implies that $\sqrt{s}_A(|a|)$ divides $s_A(|\frac{a}{1+a^2}|)$ in A. Hence $s_A(|\frac{a}{1+a^2}|) \in I$ as desired. $\square$

(5.9) THEOREM. *Let I be an ideal of a bounded super real closed ring A. Then*

$$\sqrt[\Upsilon]{I}\cdot\hat{A} = \sqrt[\Upsilon]{I\cdot\hat{A}} \text{ and } \sqrt[\Upsilon]{I} = \sqrt[\Upsilon]{I\cdot\hat{A}} \cap A.$$

PROOF. The inclusion $\sqrt[\Upsilon]{I}\cdot\hat{A} \subseteq \sqrt[\Upsilon]{I\cdot\hat{A}}$ follows from $\sqrt[\Upsilon]{I} \subseteq \sqrt[\Upsilon]{I\cdot\hat{A}}$ and the inclusion $\sqrt[\Upsilon]{I}\cdot\hat{A} \supseteq \sqrt[\Upsilon]{I\cdot\hat{A}}$ holds, since by (5.7), $\sqrt[\Upsilon]{I}\cdot\hat{A}$ is super real.
Clearly $(\sqrt[\Upsilon]{I}\cdot\hat{A}) \cap A$ contains $\sqrt[\Upsilon]{I}$ and it remains to show that

$$(\sqrt[\Upsilon]{I}\cdot\hat{A}) \cap A \subseteq \sqrt[\Upsilon]{I}.$$

We may assume that $I = \sqrt[\Upsilon]{I}$. Take $b \in (I\cdot\hat{A}) \cap A$. In order to show $b \in I$ we may replace b by b^2, hence we may assume that $b \geq 0$. Since $1+b^2$ is a unit in A we have $\frac{b}{1+b^2} \in (I\cdot\hat{A}) \cap A$. Since $b = \frac{b}{1+b^2}\cdot(1+b^2)$ we may replace b with $\frac{b}{1+b^2}$ and we may assume that $0 \leq b \leq 1$. Since $b \in I\cdot\hat{A}$, there are $a \in I$ and $c \in \hat{A}$ with $b = a\cdot c$. As $b \geq 0$, $b = |b| = |c|\cdot|a|$ and we may assume that $a, c \geq 0$, too (observe that I is radical, hence $|a| \in I$). By (4.5), $\hat{A}$ is generated by A as a super real closed ring. Thus there are $F \in C(\mathbb{R}^n)$ and $a_1, ..., a_n \in A$ with $c = F_{\hat{A}}(a_1, ..., a_n)$.

Pick $\varphi : [0, \infty) \longrightarrow [1, \infty)$ continuous and strictly increasing with $|F(\bar{x})| \leq \varphi(|\bar{x}|)$ ($\bar{x} \in \mathbb{R}^n$). Define $t : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$t(y) = \begin{cases} \frac{|y|}{\varphi(\frac{1}{y^2})} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Using [Tr2]:(7.2)(i) with $s(x) = x$ we get $t \in \Upsilon$ and some $G \in C(\mathbb{R}^n \times \mathbb{R})$ with

$$(*) \qquad F(x_1, ..., x_n)\cdot t(y) = G(x_1y, ..., x_ny, y) \text{ on } \mathbb{R}^n \times \mathbb{R}.$$

Since φ is strictly increasing and ≥ 1 everywhere it is straightforward to see that $t|_{[0.\infty)} : [0.\infty) \longrightarrow [0.\infty)$ is an homeomorphism which is polynomially bounded and whose compositional inverse is polynomially bounded, too. Hence $t \in \Upsilon^{\mathrm{pol}}$ and there is some $t_1 \in \Upsilon^{\mathrm{pol}}$ with $t \circ t_1(y) = t_1 \circ t(y) = y$ for all $y \geq 0$. As $a \geq 0$ we get $a = t_A(t_{1,A}(a))$ from (4.2)(ii).

Since I is a super real ideal, also $a_0 := t_{1,A}(a) \in I$. From $(*)$ we then get

$$b = c\cdot a = F_{\hat{A}}(a_1, ..., a_n)\cdot t_{\hat{A}}(a_0) = G_{\hat{A}}(a_1\cdot a_0, ..., a_n\cdot a_0, a_0).$$

Let $H := (G \vee 0) \wedge 1$. Then in $\mathbb{R}$ we have

$$\forall(\bar{x}, y) :\ 0 \leq G(x_1y, ..., x_ny, y) \leq 1 \Rightarrow G(x_1y, ..., x_ny, y) = H(x_1y, ..., x_ny, y).$$

Since this sentence is also valid in $\hat{A}$ and $0 \leq b \leq 1$ we get

$$b = H_{\hat{A}}(a_1\cdot a_0, ..., a_n\cdot a_0, a_0).$$

Since H is bounded it follows $b = H_A(a_1\cdot a_0, ..., a_n\cdot a_0, a_0)$. As $H(0) = G(0) = 0$, there is some $s \in \Upsilon^{\mathrm{pol}}$ with $H(z_1, ..., z_{n+1}) \leq s(z_1^2 + ... + z_{n+1}^2)$ for all $z_1, ..., z_{n+1} \in \mathbb{R}$: choose s so that $s(t) \geq \max\{H(z) \mid \sum z_i^2 \leq t\}$ $(t \geq 0)$.

It follows $0 \leq b = H_A(a_1\cdot a_0, ..., a_n\cdot a_0, a_0) \leq s_A((a_1a_0)^2 + ... + (a_na_0)^2 + a_0^2)$. Since $a_0 \in I$ and I is super real, we get $b \in I$ as desired. $\square$

Note that in general for a proper ideal I of a bounded super real closed ring A, the ideal $I^\Upsilon\cdot\hat{A}$ is properly contained in $(I\cdot\hat{A})^\Upsilon$ (e.g. if $I\cdot\hat{A} = \hat{A}$, cf. (5.2)(ii))

[*] The convexity condition says: $0 \leq a \leq b \Rightarrow b|a^2$.

(5.10) Scholium. *Let A be bounded super real closed ring. An ideal of A is super real if and only if I is the kernel of a bounded super homomorphism $A \longrightarrow B$ into a bounded super real closed ring.*

Proof. If $\varphi : A \longrightarrow B$ is such a homomorphism and $a \in I$, then $s_A(a) \in I$, since $\varphi(s_A(a)) = s_B(\varphi(a)) = s_B(0) = 0$.

Conversely suppose I is super real. By (5.7), $I\cdot\hat{A}$ is super real, too. Together with (5.9) it follows that $I\cdot\hat{A}$ is a super real ideal of $\hat{A}$ lying over I. By [Tr2]:(6.3), super real ideals of $\hat{A}$ are kernels of super homomorphisms. Hence we can compose $A \longrightarrow \hat{A}$ with $\hat{A} \longrightarrow \hat{A}/I\cdot\hat{A}$ and we get that I is the kernel of a bounded super homomorphism. □

(5.11) Corollary. *Let A be bounded super real closed and let $I \subseteq A$ be a super real ideal. There is a unique $\mathscr{L}_{\Upsilon\mathrm{pol}}$-structure on A/I such that A/I is a bounded super real closed ring and the residue map $A \longrightarrow A/I$ is a bounded super real homomorphism.*

Moreover, there is a unique A-algebra homomorphism $\hat{A} \longrightarrow \widehat{A/I}$ and this homomorphism is super real with kernel $I\cdot\hat{A}$. In particular, there is a unique A-algebra isomorphism of super real closed rings

$$\hat{A}/(I\cdot\hat{A}) \xrightarrow{\cong} \widehat{A/I}$$

Proof. By (5.7) we know that $I\cdot\hat{A}$ is a super real ideal of $\hat{A}$ lying over I. Since super real ideals are kernels of super homomorphisms by (5.10), we can compose $A \longrightarrow \hat{A}$ with $\hat{A} \longrightarrow \hat{A}/I\cdot\hat{A}$ and get that I is the kernel of a bounded super homomorphism. The image is A/I and it is clear that the $\mathscr{L}_{\Upsilon\mathrm{pol}}$-structure on A/I is uniquely determined by saying that the residue map $A \longrightarrow A/I$ is a bounded super real homomorphism.

We get an embedding of rings $A/I \longrightarrow \hat{A}/I\cdot\hat{A}$, which is a bounded super real homomorphism. By (4.3), we may view $\widehat{A/I}$ as a super real closed subring of $\hat{A}/I\cdot\hat{A}$. Since $\hat{A}$ is generated by A as a super real closed ring, also $\hat{A}/I\cdot\hat{A}$ is generated by A/I as a super real closed ring, thus $\hat{A}/I\cdot\hat{A} = \widehat{A/I}$. Hence we have a super real homomorphism $\varphi : \hat{A} \longrightarrow \widehat{A/I}$ with kernel $I\cdot\hat{A}$. There can only be one such A-algebra homomorphism, since $\hat{A}$ is the localization of A at $(\hat{A})^\times \cap A$. □

(5.12) Theorem. *If $\varphi : A \longrightarrow B$ is a bounded super homomorphism between bounded super real closed rings A and B, then there is a unique extension of φ to a ring homomorphism $\hat{\varphi} : \hat{A} \longrightarrow \hat{B}$ and this extension is super real.*

The functor F from bounded super real closed rings to super real closed rings, which maps A to $\hat{A}$ and φ to $\hat{\varphi}$ is an idempotent mono-reflector. This means: F is left adjoint to the inclusion from the category of super real closed rings into the category of bounded super real closed rings, $F \circ F = F$ and the adjoint morphism $A \longrightarrow \hat{A}$ is a monomorphism.

Proof. First we prove the assertion about φ. Uniqueness again follows from the fact that $\hat{A}$ is the localization of A at $(\hat{A})^\times \cap A$. Existence of $\hat{\varphi}$ follows from (5.11) and (4.3).

Hence the functor F is well defined. By (4.4), $F \circ F = F$, which also shows that F is a reflector. F is a mono-reflector, since $A \longrightarrow \hat{A}$ is a monomorphism. □

We conclude this section by showing that the reflector $A \mapsto \hat{A}$ is also well-behaved with respect to localization:

(5.13) Proposition. *Let A be a bounded super real closed ring, let $1 \in S \subseteq A$ be multiplicatively closed and let T be the closure of S in $\hat{A}$ under multiplication and Υ. Recall from (3.1) that there is a unique super real closed ring structure on $T^{-1}\cdot\hat{A}$ such that the localization map $\hat{A} \longrightarrow T^{-1}\cdot\hat{A}$ is a super homomorphism.*

The natural morphism $\hat{\varphi}:\hat{A}\longrightarrow\widehat{S^{-1}\cdot A}$ induced by the localization map $\varphi:A\longrightarrow S^{-1}\cdot A$, sends T into $(\widehat{S^{-1}\cdot A})^{\times}$ and the induced map

$$T^{-1}\cdot\hat{A}\longrightarrow\widehat{S^{-1}\cdot A}$$

is an A-algebra isomorphism of super real closed rings.

PROOF. Since $\varphi(S)$ consists of units of $S^{-1}\cdot A$ also $\hat{\varphi}(S)$ consists of units of $\widehat{S^{-1}\cdot A}$. Since T is the closure of S under multiplication and Υ, $\hat{\varphi}(T)$ is the closure of $\hat{\varphi}(S)$ under multiplication and Υ. Since $\widehat{S^{-1}\cdot A}$ is super real closed, every maximal ideal of $\widehat{S^{-1}\cdot A}$ is super real (cf. [Tr2]:(6.12)), hence for every for every $s\in\Upsilon$ and each element $b\in\widehat{S^{-1}\cdot A}$, b is a unit in $\widehat{S^{-1}\cdot A}$ if and only if $s(b)$ is a unit in $\widehat{S^{-1}\cdot A}$. This proves that indeed $\hat{\varphi}(T)\subseteq(\widehat{S^{-1}\cdot A})^{\times}$.

In order to show that the induced map $T^{-1}\cdot\hat{A}\longrightarrow\widehat{S^{-1}\cdot A}$ is an isomorphism it now suffices to verify the universal condition defining $T^{-1}\cdot\hat{A}$ in the category of super real closed rings for $\widehat{S^{-1}\cdot A}$, more precisely for the morphism $\hat{A}\longrightarrow\widehat{S^{-1}\cdot A}$. Let $\psi:\hat{A}\longrightarrow B$ be a super homomorphism into a super real closed ring B with $\psi(T)\subseteq B^{\times}$. Then $\psi\restriction A:A\longrightarrow B$ is a super homomorphism with $\psi|_A(S)\subseteq B^{\times}$ and by (3.7) there is a unique super homomorphism $h:S^{-1}\cdot A\longrightarrow B$ such that $\psi|_A=h\circ\varphi$. By (5.12), $\hat{h}:\widehat{S^{-1}\cdot A}\longrightarrow B$ is the unique super homomorphism extending h with $\psi=\hat{h}\circ\hat{\varphi}$. □

6. THE SUPER REAL CORE

(6.1) PROPOSITION. *Let A_0 be a convex subring of the super real closed ring A. Then there is a largest super real closed subring of A that is contained in A_0.*

PROOF. By [Tr2]:(9.2)(i), the convex hull of a super real closed subring of A is itself a super real closed subring of A. Hence, by using Zorn, it is enough to show for convex super real closed subrings B,C of A, that the ring D generated by B and C in A is again a super real closed subring of A. By [Tr2]:(10.5) we know that D is a convex subring of A and by [Tr2]:(9.2)(i), it is enough to show that D is closed under Υ: Let $b_1,...,b_n\in B$ and $c_1,...,c_n\in C$. Pick $s\in\Upsilon$. It is enough to show $|s_A(b_1c_1+...+b_nc_n)|\leq d$ for some $d\in D$. We may certainly assume that s is symmetric (i.e. $s(-x)=s(x)$) and strictly increasing on $(0,\infty)$. The Cauchy-Schwarz inequality implies $s(x_1y_1+...+x_ny_n)\leq s(|x||y|)$ for all $x=(x_1,...,x_n),y=(y_1,...,y_n)\in\mathbb{R}^n$, where $|x|,|y|$ denote the euclidean norm of x,y respectively. Since $s(|x||y|)\leq s(|x|^2)+s(|y|^2)$ we get $s(x_1y_1+...+x_ny_n)\leq s(|x|^2)+s(|y|^2)$ on $\mathbb{R}^n\times\mathbb{R}^n$. Thus $s_A(b_1c_1+...+b_nc_n)\leq s_A(b_1^2+...+b_n^2)+s_A(c_1^2+...+c_n^2)\in D$ as desired. □

(6.2) COROLLARY AND DEFINITION. *For any bounded super real closed ring A there is a largest bounded super real closed subring, denoted by A^{Υ} with the property $\widehat{A^{\Upsilon}}=A^{\Upsilon}$. We call A^{Υ} the* **super real core** *of A.*

PROOF. By (6.1), A^{Υ} is the largest super real closed subring of $\hat{A}$, which is contained in A. □

Observe that A^{Υ} is convex in A, since $\operatorname{Hol}A\subseteq A^{\Upsilon}$. For a proper ideal I of A we know $I^{\Upsilon}=(I\cap\operatorname{Hol}A)^{\Upsilon}\cdot A$ from (5.8). Hence $I^{\Upsilon}=(I\cap A^{\Upsilon})^{\Upsilon}\cdot A$ as well.

On the other hand $\sqrt[\Upsilon]{I}\cap A^{\Upsilon}$ in general properly contains $\sqrt[\Upsilon]{I\cap A^{\Upsilon}}$ (e.g. if $\sqrt[\Upsilon]{I}=A$).

(6.3) Corollary. *For any bounded super real closed ring A we have*

$$A^{\Upsilon} = \{a \in A \mid s_{\hat{A}}(a) \in A \text{ for all } s \in \Upsilon\}.$$

Proof. Since A^{Υ} is a super real closed subring of $\hat{A}$ we have "$\subseteq$". Conversely take $a \in A$ with $s_{\hat{A}}(a) \in A$ for all $s \in \Upsilon$. Let B be the super real closed subring generated by a in $\hat{A}$. Thus $B = \{F_{\hat{A}}(a) \mid F \in C(\mathbb{R})\}$. Certainly every element of B is bounded in absolute value by some $s_{\hat{A}}(a)$ for some $s \in \Upsilon$. Hence by choice of a, the convex hull C of B in $\hat{A}$ is contained in A. C is a super real closed subring of B by [Tr2]:(9.2)(i). Hence $a \in C \subseteq A^{\Upsilon}$.□

(6.4) Observation. *If B is a real closed ring and $A \subseteq B$ is a convex subring, then A is a domain if and only if B is a domain, and A is local if and only if B is local (as follows from the Gelfand-Kolmogorov Theorem). In particular for every bounded super real closed ring A we have*
(i) A is a domain $\Leftrightarrow$ $\hat{A}$ is a domain $\Leftrightarrow$ A^{Υ} is a domain $\Leftrightarrow$ $\operatorname{Hol} A$ is a domain.
(ii) A is local $\Leftrightarrow$ $\hat{A}$ is local $\Leftrightarrow$ A^{Υ} is local $\Leftrightarrow$ $\operatorname{Hol} A$ is local.

(6.5) Examples. Let

$$A := \{f \in C(\mathbb{R}^2) \mid f \text{ is polynomially bounded in the second coordinate}\}.$$

Hence $f \in A$ if and only if for every $x \in \mathbb{R}$, the function $f(x, \_\,) : \mathbb{R} \longrightarrow \mathbb{R}$ is polynomially bounded. Clearly A is a convex subring of $C(\mathbb{R}^2)$, hence A is a bounded super real closed subring of $C(\mathbb{R}^2)$. We have $\hat{A} = C(\mathbb{R}^2)$ and

$$A^{\Upsilon} = \{f \in C(\mathbb{R}^2) \mid f \text{ is bounded in the second coordinate}\}.$$

Here a super real closed ring properly between $C^*(\mathbb{R})$ and $C(\mathbb{R})$: Take

$$A = \{f \in C(\mathbb{R}) \mid f \text{ is bounded on } (0, \infty)\}.$$

Also note that there are many super real closed ring properly between $C^*([0, \infty))$ and $C([0, \infty))$, e.g

$$A = \{f \in C(\mathbb{R}) \mid f \text{ is bounded on } \mathbb{N}\}$$

has this property since $x \cdot \operatorname{dist}_{\mathbb{N}}(x) \in A \setminus C^*(\mathbb{R})$.

The formation of the super real core is functorial: If $\varphi : A \longrightarrow B$ is a bounded super homomorphism between bounded super real closed rings, then $\varphi|_{A^{\Upsilon}}$ is a super homomorphism $A^{\Upsilon} \longrightarrow B^{\Upsilon}$: since $\hat{\varphi}$ respects the $\mathscr{L}_{\Upsilon}$-structure on $\hat{A}$ by (5.12), $\varphi(A^{\Upsilon})$ is a super real closed subring of $\hat{B}$ contained in B, i.e. $\varphi(A^{\Upsilon}) \subseteq B^{\Upsilon}$. Hence the assignment $A \longrightarrow A^{\Upsilon}$ is functorial, by sending φ to $\varphi|_{A^{\Upsilon}}$. We shall not make use of this here. Instead, we state another description of the super real core.

Since $\operatorname{Hol} A \subseteq A^{\Upsilon} \subseteq A$, there are subsets S of $\operatorname{Hol} A$ with $A^{\Upsilon} = S^{-1} \cdot \operatorname{Hol} A$. We can compute the largest such set upon input A:

(6.6) Proposition. *For any bounded super real closed ring A, the largest multiplicatively closed subset S of $\operatorname{Hol} A$ satisfying $A^{\Upsilon} = S^{-1} \cdot \operatorname{Hol} A$ is*

$$S = \{a \in \operatorname{Hol} A \mid s_{\operatorname{Hol} A}(a) \in A^{\times} \text{ for all } s \in \Upsilon\}.$$

Proof. The super real closed subrings of $\hat{A}$ contained in A are all of the form $T^{-1} \cdot \operatorname{Hol} A$, where $T \subseteq A^{\times} \cap \operatorname{Hol} A$. Since A^{Υ} is the largest super real closed subring of $\hat{A}$ contained in A, the set $T := (A^{\Upsilon})^{\times} \cap \operatorname{Hol} A$ is the largest among all of them. It remains to show $T = S$.

If $a \in T$, then a is a unit in A^Υ and since A^Υ is super real closed, all elements $s_{A^\Upsilon}(a)$ are units of A^Υ as well. Since $(A^\Upsilon)^\times \subseteq A^\times$ we get $a \in S$.

Conversely let $a \in S$. The set $T_0 := \{s_{\mathrm{Hol}\,A}(a) \mid s \in \Upsilon\}$ is closed under multiplication and Υ (note that $s_1, s_2 \in \Upsilon$ implies $s_1(x)\cdot s_2(x) \in \Upsilon$ and $s_1 \circ s_2 \in \Upsilon$). Therefore $T_0^{-1}\cdot\mathrm{Hol}\,A$ has a unique super real closed ring structure (induced from $\hat{A}$). Since $a \in S$ we know $T_0 \subseteq A^\times$ and therefore $T_0^{-1}\cdot\mathrm{Hol}\,A \subseteq A$. So by the choice of T we obtain $T_0 \subseteq T$. Thus $a \in T_0 \subseteq T$. $\square$

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