Pure-injective modules

Prest, Mike

2008

MIMS EPrint: **2008.83**

ISSN 1749-9097

Pure-injective modules

Mike Prest
School of Mathematics
Alan Turing Building
University of Manchester
Manchester M13 9PL
UK
mprest@manchester.ac.uk

September 26, 2008

Abstract

The pure-injective R-modules are defined easily enough: as those modules which are injective over all pure embeddings, where an embedding $A \to B$ is said to be pure if every finite system of R-linear equations with constants from A and a solution in B has a solution in A. But the definition itself gives no indication of the rich theory around purity and pure-injectivity. The purpose of this survey is to present and illustrate the definitions and a number of the results around pure-injective modules.

1 Introduction

It is pointing out at the outset that pure-injective = algebraically compact (for modules and, indeed, for very general algebraic structures) and some sources use the term "algebraically compact" instead of "pure-injective". The concept of algebraic compactness appears, at first, to be quite different from that of pure-injectivity; in particular the definition of N being pure-injective makes reference to the category containing N whereas that of algebraic compactness refers only to the internal structure of N; nevertheless, they are equivalent.

In this survey I do not make much reference to the history of this topic; for that see, for example, [25], [44], [59], [76], [97]. Also, I do not present proofs or, for the basic results, references; for those one may consult, for example, [1], [22], [25], [45], [64], [85], [99].

2 Purity

Throughout we will work with (right) modules over a ring R but most of what we say holds equally well in more general additive categories (for instance see [84], [15], [64]). Moreover, much of the theory around purity and algebraic compactness was developed for more general algebraic systems (see, for instance, [55], [56] [87], [95], [96], [97]). Here, however, we stay in the perhaps most familiar context, that of modules over a ring. First, let us define purity. Prüfer, [68], defined the notion for abelian groups; the general definition is due to Cohn, [12]; the notion was developed further in, for instance, [83], [84], [91], [23], [72]; also see the notes to Chapter V of [25].

By an (R-linear) equation we mean one of the form $\sum_{i=1}^{n} x_i r_i = 0$ where the x_i are variables and the r_i are elements of the ring R. If we allow some of the variables to take values in a module A then we say that the equation has constants from A and we can re-arrange it to the form $\sum_i x_i r_i = a$ for some $a \in A$. A finite system of equations has the form $\sum_i x_i r_{ij} = 0$ $(j = 1, \ldots, m)$ or, if we allow constants from A, has the form $\sum_i x_i r_{ij} = a_j$ $(j = 1, \ldots, m)$ for some $a_j \in A$. We may also consider systems consisting of infinitely many such equations.

It is useful to have a notation for systems of linear equations: we write θ for a typical finite system of R-linear equations and, if the variables are $\overline{x} = (x_1, \ldots, x_n)$ then we can write $\theta(\overline{x})$ to show the variables; in the case where there are non-zero constants we may write $\theta(\overline{x}, \overline{a})$, that is $\theta(x_1, \ldots, x_n, a_1, \ldots, a_m)$, to specify the constants.

If M is any module then the **solution set** of θ in M is

$$\theta(M) = \{\overline{c} \in M^n : \theta(\overline{c})\}$$

where $\theta(\overline{c})$ is notation for $\sum_i c_i r_{ij} = 0$ $(j = 1, \ldots, m)$. If the system has constants from a module A then, of course, we consider its solution sets just those modules M containing A (at least, containing those constants) and, in this case, we write $\theta(M, \overline{a})$ for the solution set, that is

$$\theta(M, \overline{a}) = \{\overline{c} \in M^n : \sum_i c_i r_{ij} = a_j \ (j = 1, \ldots, m)\}.$$

Clearly $\theta(M)$ is a subgroup of M^n and $\theta(M, \overline{a})$ is either empty or a coset of the solution set, $\theta(M, \overline{0})$, of the system $\theta(\overline{x}, \overline{0})$ obtained by replacing each constant by 0.

An embedding $A \to B$ of modules is **pure** if for every finite system $\theta(\overline{x}, \overline{a})$ of equations with constants from A and a solution in B there is a solution in A; that is, $\theta(B, \overline{a}) \neq \emptyset$ implies $\theta(A, \overline{a}) \neq \emptyset$.

This extends to projected systems of linear equations, in the following sense. Given a system $\theta(x_1, \ldots, x_n)$ of linear equations we may fix some subset $J \subseteq \{1, \ldots, n\}$, let us take $J = \{1, \ldots, n'\}$ for some $n' < n$ for definiteness, and we may consider, for each module M, the projection of the solution set $\theta(M) \le M^n$ to M^J, thus to the first n' coordinates given our choice of J. The condition which describes this set may be defined, using quantifiers, as $\exists x_{n'+1}, \dots, x_n\ \theta(\overline{x})$. Note that some of the variables have been quantified out, so the free variables of this condition are $x_1, \dots, x_{n'}$. Such a condition is said to be a **pp condition** and we use notation such as ϕ, or more precisely $\phi(x_1, \dots, x_{n'})$, for such conditions. Observe, with notation as above, that for any module M, $\phi(M)$ is a subgroup of $M^{n'}$; a subgroup of this form is said to be **pp-definable**. If θ has constants $\overline{a}$ then we also show these in notation such as $\phi(x_1, \dots, x_{n'}, \overline{a})$, for projected systems; clearly $\phi(M, \overline{a})$ is either empty or a coset of $\phi(M, \overline{0})$.

Example 2.1 *The simplest kind of pp condition which is not just an equation is that which expresses divisibility by some element $r \in R$. Take $\theta(x_1, x_2)$ to be the linear equation $x_1 + x_2 r = 0$ and project to the first variable, i.e. quantify out the second variable, to obtain $\phi(x_1)$ which is $\exists x_1\, (x_1 + x_2 r = 0)$; clearly for any module M we have $\phi(M) = Mr$.*

We may write the system $\theta(\overline{x})$ of linear equations using matrix notation, namely as $\overline{x}H = \overline{0}$ where $\overline{x}$ is now treated as a row vector and H is an $n \times m$ matrix with entries from R. Then, if we project to the first n' coordinates, partition $\overline{x}$ as $\overline{x}'\ \overline{x}''$ accordingly (so $\overline{x}' = (x_1, \dots, x_{n'})$, $\overline{x}'' = (x_{n'+1}, \dots, x_n)$) and partition H accordingly as $H = \begin{pmatrix} H' \\ H'' \end{pmatrix}$, then the pp condition $\phi(\overline{x}')$ may be written as $\exists \overline{x}''\, (\overline{x}'\ \overline{x}'') \begin{pmatrix} H' \\ H'' \end{pmatrix} = 0$; equivalently $\exists \overline{x}''\, (\overline{x}'H' + \overline{x}''H'' = \overline{0})$ or, informally, $H'' \mid \overline{x}'H'$ (H'' divides $\overline{x}'H'$). Thus even the most general pp condition may be viewed as a kind of divisibility condition.

It is easy to check that, although in general $\phi(M)$ is not an R-submodule of $M^{n'}$, it is closed under the (diagonal) action of the endomorphism ring $\mathrm{End}(M)$ so it is, in particular, a module over the centre, $C(R)$, of R.

Let Mod-R denote the category of right R-modules; we will use mod-R for the category of finitely presented modules.

Theorem 2.2 *The following conditions on an embedding $f : A \to B$ of R-modules are equivalent:*
(i) f is a pure embedding;
(ii) for every left R-module L the induced map $f \otimes 1_L : A \otimes_R L \to B \otimes_R L$ is an embedding (of abelian groups);
(iii) for every finitely presented left R-module L the induced map $f \otimes 1_L : A \otimes_R L \to B \otimes_R L$ is an embedding;
(iv) for every $F \in (\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ the map $\overrightarrow{F} f : \overrightarrow{F} A \to \overrightarrow{F} B$ is an embedding;
(v) for every pp condition ϕ with n free variables $\phi(A) = A^n \cap \phi(B)$;
(vi) for every pp condition ϕ with 1 free variable $\phi(A) = A \cap \phi(B)$;

(vii) for every morphism $f' : A' \to B'$ between finitely presented modules and morphisms in Mod-R, *$g : A' \to A$ and $h : B' \to B$ such that $hf' = fg$ there is $k : B' \to A$ such that $kf' = g$.*

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle g}\downarrow & \circlearrowleft \;\; \swarrow_{k} & \downarrow{\scriptstyle h} \\ A & \xrightarrow[f]{} & B \end{array}$$

Condition (iii) needs some explanation: the notation (mod-R, $\mathbf{Ab}$) is for the category whose objects are the additive functors from mod-R to the category $\mathbf{Ab}$ of abelian groups and whose morphisms are the natural transformations. This category of functors is a Grothendieck category which has the representable functors $(C, -)$, for $C \in$ mod-R, as a generating set (up to isomorphism) of finitely generated projective objects. A functor F in (mod-R, $\mathbf{Ab}$) is finitely presented iff it has a presentation of the form $(D, -) \xrightarrow{(g,-)} (C, -) \to F \to 0$ for some morphism $g : C \to D$ in mod-R; we use the notation $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ for the full subcategory of finitely presented functors. Each functor $F \in (\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ can be extended to a functor, which we denote $\overrightarrow{F}$, from Mod-R to $\mathbf{Ab}$: to define $\overrightarrow{F}M$, where $M \in$ Mod-R, we use the fact that M is a direct limit, $M = \varinjlim M_\lambda$, of some directed system $(M_\lambda)_\lambda$ of finitely presented modules and we set $\overrightarrow{F}M = \varinjlim (FM_\lambda)$. It may be checked that $\overrightarrow{F}M$ is well-defined (independent of choice of representation of M as a direct limit) and that, with the obvious extension to an action on morphisms, $\overrightarrow{F}$ is the unique extension of F to a functor on Mod-R which commutes with direct limits.

A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is said to be **pure-exact** or just **pure** if f is a pure embedding, in which case g is said to be a **pure epimorphism**.

Theorem 2.3 *An exact sequence of modules is pure-exact iff it is a direct limit of split short exact sequences.*

Recall that a right R-module M_R is said to be **flat** if whenever $f : {}_RK \to_R L$ is an embedding of left R-modules the induced morphism $1_M \otimes f : M \otimes_R K \to M \otimes_R L$ is an embedding. A module is flat iff it is the direct limit of a directed system of projective modules.

Proposition 2.4 *If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence and C flat then this sequence is pure-exact.*

Theorem 2.5 *The following conditions on an epimorphism $g : B \to C$ are equivalent:*
(i) g is a pure epimorphism;

(ii) for every pp condition ϕ and every $\bar{c} \in \phi(C)$ there is $\bar{b} \in \phi(B)$ such that $g\bar{b} = \bar{c}$;
(iii) for every finitely presented module D, every morphism $h : D \to C$ lifts to a morphism $h' : D \to B$ with $h = gh'$.

Corollary 2.6 *If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is pure-exact and C is finitely presented then this sequence is split.*

An example of a pure, non-split, embedding is that of the group $\mathbb{Z}_{(p)}$, obtained by localising the module $\mathbb{Z}_{\mathbb{Z}}$ at a non-zero prime p, into its p-adic completion, $\overline{\mathbb{Z}_{(p)}}$, the ring of p-adic integers regarded as an abelian group.

Theorem 2.7 *Every short exact sequence of R-modules is pure iff R is a von Neumann regular ring.*

A module M is said to be **absolutely pure** if each embedding in Mod-R with domain M is a pure embedding, so every R-module is absolutely pure iff R is von Neumann regular (recall that the latter condition is right/left symmetric so every right module is absolutely pure iff every left module is absolutely pure). There are the following equivalents for absolute purity, (ii) of which explains the alternative and equivalent term, **fp-injective**, used for these modules.

Theorem 2.8 *For any module M the following are equivalent:*
(i) M is absolutely pure;
(ii) M is fp-injective, that is, M is injective over embeddings with finitely presented cokernel;
(iii) $\mathrm{Ext}^1(C, M) = 0$ for every finitely presented module C;
(iv) M is a pure submodule of an injective module.

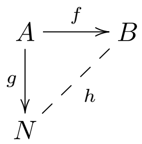
Other notions of purity have been introduced for a variety of reasons, see for instance [10], [91]. Here I mention just one, which in some circumstances reduces to the notion defined in this paper. Say that a ring R is **RD** (for "relatively divisible") if purity for R-modules reduces to simple divisibility, that is, if the condition for an embedding $A \leq B$ to be pure is simply that for every $r \in R$ we have $Ar = A \cap Br$. For abelian groups this is equivalent to purity and there is the following characterisation of the rings with this property.

Theorem 2.9 *[94, 2.6] A ring is RD*
iff every finitely presented right module is a direct summand of a direct sum of cyclically presented modules
iff the same is true for every left module.
In particular the RD condition on one side implies it on the other.

For example, serial rings have this property ([17, Thm.], [93, Thm. 3.3]), as do Prüfer rings ([91, Thm. 1]), as does every hereditary noetherian prime ring with enough invertible right ideals ([19, §2], [20, §3]) - an example is the first Weyl algebra over a field of characteristic 0. A commutative ring is RD iff it is a Prüfer ring ([92, Thm. 3]). A resulting simplified criterion for pure-injectivity over Prüfer rings is [91, Thm. 4].

3 Pure-injective modules

A module N is said to be **pure-injective** if it is injective over pure embeddings, that is, if whenever $f : A \to B$ is a pure embedding and $g : A \to N$ is any morphism then there is a morphism $h : B \to N$ with $hf = g$.



Equivalently N is pure-injective iff every pure embedding in Mod-R with domain N is split.

As with any notion of injectivity, any direct product of pure-injective modules is pure-injective, as is every direct summand of a pure-injective module. And, of course, every injective module is pure-injective. Over von Neumann regular rings injective=pure-injective and this characterises these rings.

It is the case that every module purely embeds in a pure-injective module, indeed, the analogy with injectives is complete in the sense that, given any module M, there is a pure embedding of M into a pure-injective module, $H(M)$, the **pure-injective hull** of M, which is minimal in the following senses.

Proposition 3.1 *The pure-injective hull $j : M \to H(M)$ of a module M has the following properties.*
(a) If $f : H(M) \to M'$ is any morphism such that fj is a pure embedding then f is a pure embedding, that is, $M \to H(M)$ is **pure-essential**.[1]
(b) If $j' : M \to N$ is a pure embedding of M into a pure-injective module N then there is a morphism $j'' : H(M) \to N$ such that $j''j = j'$; every such morphism j'' embeds $H(M)$ as a direct summand of N.
(c) The pure-injective hull of M is unique up to isomorphism over M.
(d) The functor $H(M) \otimes -$ is the injective hull of $M \otimes -$.

Part (d) needs explanation. There is a very useful full embedding of Mod-R into the functor category (R-mod, **Ab**), where R-mod denotes the category of finitely presented *left* modules. The functor is given on objects by taking $M \in$ Mod-R to the functor $M \otimes -$ which takes a finitely presented left module L to

[1] A somewhat different notion of pure-essential has been used in some papers, e.g. [47], [91], but that notion is not transitive, see [33], whereas the one just given is.

$M \otimes_R L$ and has the natural action on morphisms. This functor from Mod-R to (R-mod, **Ab**) is full and faithful and induces a correspondence between pure-exact sequences in Mod-R and exact sequences in (R-mod, **Ab**), as well as an equivalence between the category of pure-injective modules in Mod-R and the category of injective objects of (R-mod, **Ab**).

Theorem 3.2 *[34], [35, §1] The functor from* Mod-R *to* $(R\text{-mod}, \mathbf{Ab})$ *which is defined on objects by taking M to $M \otimes_R -$ is a full embedding.*

An exact sequence $0 \to M \to N \to N' \to 0$ in Mod-R *is a pure-exact sequence iff the sequence $0 \to (M \otimes -) \to (N \otimes -) \to (N' \otimes -) \to 0$ is an exact sequence in* $(R\text{-mod}, \mathbf{Ab})$*, iff this sequence is a pure exact sequence in* $(R\text{-mod}, \mathbf{Ab})$.[2]

For every module M the functor $M \otimes -$ is an absolutely pure object of $(R\text{-mod}, \mathbf{Ab})$ *and every absolutely pure object of* $(R\text{-mod}, \mathbf{Ab})$ *is isomorphic to one of the form $M \otimes -$. Furthermore, $M \otimes -$ is an injective object of the functor category iff M is a pure-injective module, and $M \otimes -$ is indecomposable iff M is indecomposable.*

Thus one may transfer results on embeddings and injectives in Grothendieck categories, such as (R-mod, **Ab**), to results on pure embeddings and pure-injectives in module categories. In particular, the existence and basic properties of pure-injective hulls are most easily established *via* this route.

For instance, the corollary below follows immediately from the corresponding result for indecomposable injectives in Grothendieck categories together with the fact that the functor is a full embedding.

Corollary 3.3 *If N is an indecomposable pure-injective R-module then the endomorphism ring of N is local.*

Example 3.4 *A module M is absolutely pure iff its pure-injective hull is its injective hull.*

Of course taking the injective hull in the functor category is not in general a particularly direct way of computing the pure-injective hull, but there are other routes. For example, the hull may also be found as a direct summand of the double dual of a module. Given a right R-module M, let M^* denote the module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ which, note, has a natural structure as a left R-module. Applying the same process to this left module, we obtain the double dual M^{**}.

Theorem 3.5 *Let M be any R-module; then M^{**} is pure-injective. If $0 \to M \to N \to N_1 \to 0$ is a pure-exact sequence then the induced sequence $0 \to M^{**} \to N^{**} \to N_1^{**} \to 0$ is pure, hence split. Furthermore, the natural map $M \to M^{**}$ is a pure embedding of M into a pure-injective module.*

[2] The definition of purity and its various equivalents do make sense in the functor category and are equivalent to each other, see, for instance [64, Chpt. 10].

Therefore the pure-injective hull of M is a direct summand of M^{**}.

Theorem 3.6 *(e.g. [64, 1.3.15]) With notation as before, if* $F \in (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ *then* $\overrightarrow{F}M = 0$ *iff* $\overrightarrow{F}M^{**} = 0$.

It is a consequence of this that M and M^{**} generate the same definable subcategory (in the sense of Section 8 below) of Mod-R.

A module M is said to be **cotorsion** if $\mathrm{Ext}^1(F, M) = 0$ for every flat module F, that is, iff every exact sequence of the form $0 \to M \to X \to F \to 0$, with F flat, is split. In particular every pure-injective module is cotorsion, but not conversely (see [42] for conditions for the converse). The modules F such that $\mathrm{Ext}^1(F, M) = 0$ for every cotorsion module M are exactly the flat modules, in fact, testing with pure-injective modules is enough.

picotor

Proposition 3.7 *[100, 3.4.1] If a module* F *is such that* $\mathrm{Ext}(F, N) = 0$ *for every pure-injective module* N *then* F *must be flat.*

4 Algebraically compact modules

A module M is **algebraically compact** if every collection, $\{a_\lambda + \phi_\lambda(M)\}_\lambda$, of cosets of pp-definable subgroups of M which has the finite intersection property has non-empty intersection; that is, if for every finite $\Lambda' \subseteq \Lambda$, $\bigcap_{\lambda \in \Lambda'} \big(a_\lambda + \phi_\lambda(M)\big) \neq \emptyset$ then $\bigcap_{\lambda \in \Lambda} \big(a_\lambda + \phi_\lambda(M)\big) \neq \emptyset$. For instance take the localisation $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at a prime p, and regard this as a $\mathbb{Z}$-module. The submodules $\mathbb{Z}_{(p)}p^n$ are pp-definable and their cosets form a p-branching tree of infinite depth. Each branch through this tree gives a set of cosets with the finite intersection property so, since there are uncountably many distinct (and incompatible) branches, whereas $\mathbb{Z}_{(p)}$ is countable, it follows that this module is not algebraically compact.

Theorem 4.1 *For any module* M *the following conditions are equivalent:*
(i) M *is algebraically compact;*
(ii) every system of linear equations, in possibly infinitely many variables, with constants from M *and which is finitely solvable in* M*, is solvable in* M*;*
(iii) M *is an algebraic direct summand of some compact Hausdorff topological module*
(iv) M *is pure-injective;*
(v) ([45, 7.1]) for every index set, I*, the summation map* $\Sigma : M^{(I)} \to M$*, given by* $(x_i)_i \mapsto \sum_i x_i$*, factors through the natural embedding of* $M^{(I)}$ *into the corresponding direct product* M^I*.*

$$\begin{array}{ccc} M^{(I)} & \longrightarrow & M^I \\ {\scriptstyle\Sigma}\downarrow & \swarrow & \\ M & & \end{array}$$

The equivalence of algebraic compactness and pure-injectivity is true in much more general algebraic systems and was discovered by various people, see [88].

Example 4.2 *(see [91, Prop. 10]) Let R be a Dedekind domain. If P is a maximal prime ideal of R and if $M = R_{(P)}$ is the localisation of R at P, regarded as an R-module, then the pure-injective hull of M is its P-adic completion. The pure-injective hull of the module R_R is the product, over all maximal primes P, of these completions.*

A module M is said to be **linearly compact** (see [101]) if every collection of cosets of *submodules* of M which has the finite intersection property has non-empty intersection. It is not difficult to see that if M_R is a module which is linearly compact as a module over its endomorphism ring then M is a pure-injective R-module. In particular any linearly compact module over a commutative ring is algebraically compact.

It follows that a commutative linearly compact ring, in particular any complete discrete valuation ring, is algebraically compact as a module over itself. As a partial converse ([91, Prop. 9]), if R is commutative and either noetherian or a valuation ring, then R_R is algebraically compact iff it is linearly compact.

Theorem 4.3 *[101, Prop. 9] If $0 \to A \to B \to C \to 0$ is an exact sequence with A and C linearly compact then B is linearly compact.*

In contrast, an extension of a pure-injective module by a pure-injective module need not be pure-injective; see, for instance, [73, p. 436], also see [100, 3.5.1] for a general criterion.

5 Finiteness conditions

Example 5.1 *Suppose that the module M has the descending chain condition on pp-definable subgroups; then, directly from the definition, M is algebraically compact, hence pure-injective.*

Theorem 5.2 *For any module M the following conditions are equivalent:*
(i) M is Σ-pure-injective (also called Σ-algebraically compact), meaning that for every cardinal κ the direct sum $M^{(\kappa)}$ of κ-many copies of M is pure-injective;
(ii) $M^{(\aleph_0)}$ is pure-injective;
(iii) M has the descending chain condition on pp-definable subgroups.
(iv) M is totally transcendental.

The notion of a structure being totally transcendental is from model theory (if the ring is countable then the term ω-stable is also used). For some time there were rather parallel investigations of these modules by people in model theory and in algebra (cf. [27] and [103]).

An injective module is said to be Σ-injective if it is also Σ-pure-injective. It is well known that a ring is right noetherian iff every injective right module is Σ-injective. A ring R such that every pure-injective right R-module is Σ-pure-injective is said to be **right pure-semisimple** and this condition implies that, in fact, *every* right R-module is (Σ-)pure-injective. It is an open problem (the Pure-semisimplicity Conjecture) whether right pure-semisimple implies left pure-semisimple. It is known that right and left pure-semisimplicity together imply **finite representation type** (i.e. that every module is a direct sum of indecomposable modules and there are, up to isomorphism, only finitely many indecomposable modules).

Since every pp-definable subgroup of M is an $\mathrm{End}(M)$-module, if M has **finite endolength** (i.e. is of finite length over its endomorphism ring) then M is Σ-pure-injective, in particular is pure-injective. So if R is a k-algebra for some field k and if M is an R-module which is finite-dimensional when considered as a k-vectorspace, then M is of finite endolength, hence Σ-pure-injective. Similarly, if R is an **artin algebra**, meaning that the centre of R is artinian and R is finitely generated as a module over its centre, then every R-module of finite length is of finite endolength, hence is (Σ-)pure-injective. Over some rings, for instance over commutative Prüfer rings ([54], [11], [66]), the Σ-pure-injective rings have been classified completely. There is a general theory ([14]) of Crawley-Boevey relating modules of finite endolength to certain additive functions on the category of finitely presented modules.

Over a right pure-semisimple ring (such a ring must be right artinian) every indecomposable pure-injective is of finite length and of finite endolength; furthermore there are, up to isomorphism, only finitely many modules of each finite length ([58, 3.8], [105, Cor. 10], [98, 3.2]); and every finitely presented left module is of finite endolength ([39, 2.3]).

Example 5.3 *If R is a semiprime right and left Goldie ring, with semisimple artinian quotient ring Q, then the Σ-pure-injective R-modules are those which are the sum of a divisible module and a module which is annihilated by a regular element of R. This was proved by Macintyre ([53, Thm. 1]) for the case $R = \mathbb{Z}$ and the general result is from Crawley-Boevey ([13, 1.3]).*

If R is a non-artinian simple noetherian ring then ([13, 1.4]) the only R-modules of finite endolength are the direct sums of copies of the simple modules over the (simple artinian) quotient ring of R.

If R is a non-artinian hereditary noetherian prime ring then ([13, 1.4]) every R-module of finite endolength is a direct sum of R-modules of finite length and copies of the simple module over the simple artinian quotient ring of R. Conversely, every finite direct sum of such modules is of finite endolength.

Modules which are indecomposable and of finite endolength and which are "large" in the sense of not being finitely presented, are often referred to as **generic** modules. The appropriateness of the term may be seen from the main results of [13] which show that for finite-dimensional algebras over an algebraically closed field, the generic modules arise from generic points of finite localisations of the affine line. The results of that paper show how closely the generic modules control the finite-dimensional representation theory of such algebras.

6 The structure of pure-injectives

There is a general structure theorem for pure-injective modules, due originally to Fisher ([24, 7.21] in a somewhat more general context) and proved independently and by various methods by a number of other people. A module is said to be **superdecomposable** if it is non-zero and has no indecomposable direct summand.

Theorem 6.1 *Let N be a pure-injective module. Then $N = H(\bigoplus_\lambda N_\lambda) \oplus N_c$ where each N_λ is indecomposable pure-injective and where N_c is a superdecomposable pure-injective. The modules N_λ, together with their multiplicities, as well as the module N_c, are determined up to isomorphism by N.*

Over some rings, for example Dedekind prime PI rings and tame hereditary algebras, there are no superdecomposable pure-injective modules so, over such rings, the structure of pure-injectives is completely determined by the indecomposable pure-injectives. Ziegler introduced a dimension, which he called "width", defined in terms of the lattice of pp conditions (see the next section) equivalently in terms of localisations of the category $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ of finitely presented functors. This dimension is such that, if there is a superdecomposable pure-injective R-module then this dimension is undefined and, at least if R is countable, if this dimension is undefined then there is a superdecomposable pure-injective ([102, §7]).

An example of a superdecomposable (pure-)injective is given by taking $R = k\langle X, Y\rangle$ to be the free associative k-algebra on two generators and considering the injective hull, $E = E(R_R)$ of the module R_R: there is no non-zero uniform right ideal of R hence no indecomposable direct summand of E.

Classification of indecomposable pure-injectives began with Kaplansky ([46], for abelian groups) and received fresh impetus from work, especially that of Ziegler [102], in the model theory of modules. See [64] for a current account.

We make the general point that over most rings it is impossible to classify all modules: even algebras of tame representation type typically are "wild" when their infinitely generated representations are considered. In practice one is interested in the classification of certain "significant" modules rather than in arbitrary modules; the pure-injective modules seem to form such a class of modules which arise in practice and where there is hope of some kind of classification.

7 Pure-injective modules in model theory

Pure-injective modules play a central role in the model theory of modules. This is not the place to go into details about that subject but there are short introductions such as [62], [63] as well as relevant sections in texts such as [43], [77].

The compactness condition which is the definition of algebraic compactness is a special case of a notion, "saturation", from model theory (more precisely it is saturation for pp conditions). In particular the pure-injective modules are exactly those which are direct summands of κ-saturated modules where κ may be taken to be the smallest cardinal strictly greater than $\mathrm{card}(R) + \aleph_0$.

Sabbagh ([78, Cor. 4 to Thm. 4]) showed that pure-injective modules are, model-theoretically, ubiquitous in the sense that every R-module M is an elementary substructure of its pure-injective hull $H(M)$. That means that any property which can be expressed by a first order sentence in the language of R-modules, allowing constants from M, holds for M iff it holds for $H(M)$. In particular, every theory of modules is a theory of pure-injective modules. This consequence was extended by Ziegler as follows.

Theorem 7.1 *Every module is elementarily equivalent to a direct sum of indecomposable pure-injective modules.*

There is, up to isomorphism, just a set of indecomposable pure-injective R-modules and Ziegler introduced a topology on this set and a corresponding notion of support for modules. Namely, to a module M associate the set of indecomposable pure-injective direct summands of modules elementarily equivalent to M. This is the same as the set of indecomposable pure-injectives in the definable subcategory of Mod-R generated by M and this set, the **support** of M, is a closed subset of Ziegler's space. Every closed subset arises in this way and one has a bijection between the closed subsets of this space and those complete theories of modules with class of models closed under products (in many cases that is all complete theories), equivalently between closed subsets and theories whose class of models is closed under products and direct summands (that is, definable subcategories, in the sense of the next section). This space, now called the (right) **Ziegler spectrum** of R, has since played a central role in the model theory of modules.

The set of all pp conditions (in, say, one free variable) is naturally ordered by implication (that is, inclusion of solution sets) and forms a modular lattice, which is, in fact, naturally isomorphic to the lattice of finitely generated subfunctors of the forgetul functor $(R_R, -)$ in the category $(\text{mod-}R, \mathbf{Ab})$. The lattice of pp-definable subgroups of any module is a factor lattice of this. Garavaglia, especially in [26], studied finiteness conditions, extending the descending chain condition, in these lattices (as we have seen, the dcc is equivalent to Σ-pure-injectivity). He considered the relation between the Krull dimension (in the sense of Gabriel and Rentschler) of these lattices and the structure of models (especially the pure-injective ones). The work of Ziegler carried this considerably further and it turns out that various dimensions on these lattices correspond exactly to dimensions defined *via* certain localisations in the functor category (mod-R, **Ab**) (rather like the relation between Krull dimension of the lattice of right ideals and Gabriel dimension of the module category). The existence of these dimensions (i.e. having an ordinal value rather than being undefined) is reflected in the structure of pure-injectives.

For instance, the m-dimension of a modular lattice is defined by inductively collapsing the intervals of finite length (see [59] or [64]) and the Krull-Gabriel dimension of a Grothendieck category is defined by inductively localising away the finitely presented objects of finite length (see [45], also [64]). The Krull-Gabriel dimension of a ring R is defined to be that of the functor category (mod-R, **Ab**).

Theorem 7.2 *[102, §7] Suppose that N is a pure-injective module such that some non-trivial interval in the lattice of pp-definable subgroups of N has m-dimension. Then there is an indecomposable direct summand of N.* *wdthnotspdec*

Corollary 7.3 *If the width of the lattice of pp conditions over R is defined, in particular if the m-dimension of this lattice (equivalently the Krull-Gabriel dimension of R) is defined, then there are no superdecomposable pure-injective R-modules.*

If R is countable then the converse is true ([102, 7.8(2)]).

It is not known whether the converse is true without the assumption of countability.

The analysis of the lattice of pp-definable subgroups by m-dimension (which is a refinement of Krull dimension, using lattices of finite length rather than artinian lattices as the base of the inductive definition) exactly corresponds to the Cantor-Bendixson analysis of the corresponding closed subset of the Ziegler spectrum. The latter is the process where one removes all the isolated points from a topological space, then repeats the process in what is left, inductively and transfinitely.

8 Definable subcategories

Ziegler's result that, over any ring, each pure-injective module is elementarily equivalent to a direct sum of indecomposable pure-injective modules may be rephrased in terms of definable subcategories. We say that a full subcategory (or a subclass) $\mathcal{D}$ of Mod-R is a **definable subcategory** if it is closed in Mod-R under direct products, direct limits and pure submodules.

Theorem 8.1 *The following conditions on a subclass $\mathcal{D}$ of* Mod-R *which is closed under isomorphism, are equivalent:*

(i) $\mathcal{D}$ is definable;
(ii) $\mathcal{D}$ is closed under direct products, direct limits and pure submodules;
(iii) $\mathcal{D}$ is closed under finite direct sums, reduced products and pure submodules;
(iv) $\mathcal{D}$ is closed under direct products, ultrapowers and pure submodules.
Any such subclass is closed under arbitrary direct sums as well as direct summands and, more generally, pure-epimorphic images.

Theorem 8.2 *If $\mathcal{D}$ is a definable subclass of* Mod-R *then $\mathcal{D}$ is closed under pure-injective hulls.*

The definable subcategory **generated by** a module or a set of modules is the smallest definable subcategory containing that module or set of modules.

If modules are elementarily equivalent then they generate the same definable subcategory and the converse is almost true (it is literally true if each quotient $\phi(M)/\psi(M)$ is either trivial or infinite; the precise statement is that M and N generate the same definable subcategory iff $M^{(\aleph_0)}$ and $N^{(\aleph_0)}$ are elementarily equivalent).

If M is a module of finite endolength then the modules which are direct summands of direct sums of copies of M form a definable subcategory. The next result gives some more examples.

Theorem 8.3 *[21, 3.16, 3.23], [79, Thm. 4] The following conditions on a ring R are equivalent:*
(i) R is right coherent
(ii) the class of absolutely pure right R-modules is definable
(iii) the class of flat left R-modules is definable.

Those papers of Eklof and Sabbagh also show that the class of injective right modules, respectively of projective left modules, is definable iff R is right noetherian, respectively right coherent and left perfect. There is also a theorem of Herzog ([38, 4.4]) which gives a perfect duality between definable subcategories of Mod-R and those of R-Mod; in this sense, the classes appearing in the theorem above are dual ([38, 9.3]).

Definable subcategories of the class, Mod-$\mathbb{Z}$, of abelian groups include: the class of all torsionfree groups; the class of all divisible groups; the intersection of these two classes; the class of all groups of exponent bounded by a given integer n. The class of torsion abelian groups is clearly not a definable subcategory (though it is a definable category in the wider sense described at the end of this section).

Theorem 8.4 *([102, §4]) Every definable subcategory is generated by the set of indecomposable pure-injective modules which it contains.*

Set pinj_R to be the set of isomorphism types of indecomposable pure-injective modules (that this is a set follows, for example, from [102, 4.2(10]; it also follows from 3.2 and the corresponding fact for injectives in Grothendieck categories). If $\mathcal{D}$ is a definable subcategory then we set $\mathcal{D} \cap \mathrm{pinj}_R$ to the the set of (isomorphism types of) indecomposable pure-injectives contained in $\mathcal{D}$. By the result above there is a bijection between definable subcategories of Mod-R and sets of this form. As mentioned in the previous section, Ziegler proved that these sets are exactly the closed subsets for the topology he introduced on pinj_R and which we define next.

A topology may be defined by specifying a basis of open sets; Ziegler took those of the form (ϕ/ψ) where ϕ and ψ are pp conditions such that $\phi \geq \psi$ in the sense that for every module M the solution set of ψ is contained in that of ϕ and where $(\phi/\psi) = \{N \in \mathrm{pinj}_R : \phi(N) > \psi(N)\}$ - the set of indecomposable pure-injectives N where the solutions to ψ form a *proper* subgroup of the set of solutions to ϕ. There is a variety of alternative ways of defining this topology; for example the same basis is obtained by taking the sets $(F) = \{N \in \mathrm{pinj}_R : \overrightarrow{F}N \neq 0\}$ where F ranges over finitely presented functors in (mod-R, $\mathbf{Ab}$). To say that these sets form a basis is to say that for every point $N \in \mathrm{pinj}_R$ and every pair (ϕ/ψ) and (ϕ'/ψ') of basic open sets there is some basic open set $(\phi''/\psi'') \subseteq (\phi/\psi) \cap (\phi'/\psi')$ with $N \in (\phi''/\psi'')$. The topological space obtained in this way is now called the **Ziegler spectrum** of R.

Herzog [38] showed that there is a natural bijection between definable subcategories of Mod-R and those of R-Mod and that this induces a "homeomorphism at the level of topology" between the right and left Ziegler spectra of R (that is, an isomorphism between the lattices of open sets of these two spaces).

In fact, see [64] or [65], the theory of purity applies in the general context of definable categories: these are defined to be the categories which arise as definable subcategories of categories of the form Mod-$\mathcal{R} = (\mathcal{R}^{\mathrm{op}}, \mathbf{Ab})$ where $\mathcal{R}$ is any skeletally small preadditive category. Equivalently these are the exactly definable categories in the sense of [48]: those of the form $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ where $\mathcal{A}$ is a skeletally small abelian category and Ex denotes the full subcategory of all exact functors.

9 Cotorsion and Cotilting

The module M is said to be **cotilting** if $\mathrm{cog}(M) = {}^{\perp_1}M$ where the latter denotes $\{A \in \text{mod-}R : \mathrm{Ext}^1(A, M) = 0\}$. These modules arise for example in the context of cotorsion theories, which are defined rather like not-necessarily-hereditary torsion theories but using Ext-orthogonality rather than Hom-orthogonality, see [89]. It was observed that all known examples of cotilting modules were pure-injective and for a while it was open whether this is always true; that this *is* so was proved by Bazzoni [7]. She also proved, [7, 3.2], that if M is cotilting then ${}^{\perp_1}M$ is a definable subcategory. There are also generalisations of these notions, this question, and, see [86], answer, using Ext^n.

For more around this topic see various of the articles in [2], [32], and references therin.

Baldwin, Eklof and Trlifaj ([5]) investigate when classes of modules of the form ${}^{\perp_\infty}N = \{A : \mathrm{Ext}^i(A, N) = 0 \ \forall i > 0\}$ are abstract elementary classes; such classes arise from cotilting and cotorsion theories. They show that if ${}^{\perp_\infty}N$ is an abstract elementary class then N must be cotorsion and, in the other direction, if N satisfies the stronger condition of being pure-injective, then ${}^{\perp_\infty}N$ is an abstract elementary class. Also see [90] where it is shown that every cotilting class gives an abstract elementary class of finite character.

In conclusion: we have been able just to skim the surface and point in a few directions; in particular we have not touched on the detailed classification results which are in the literature. For more I refer the reader to the existing surveys and detailed expositions, as well as the developing literature, a sampling of which is included among the references.

References

[1] J. Adámek and J. Rosický, Locally Presentable and Accessible Categories, London Math. Soc. Lect. Note Ser., Vol. 189, Cambridge University Press, 1994.

[2] L. Angeleri Hügel, D. Happel and H. Krause, Handbook of Tilting Theory, London Math. Soc. Lect. Note Ser., Vol. 332, Cambridge University Press, 2007.

[3] M. Auslander, Coherent functors, pp. 189-231 *in* Proceedings of the Conference on Categorical Algebra, La Jolla, 1965, Springer-Verlag, 1966.

[4] S. Balcerzyk, On the algebraically compact groups of I. Kaplansky, Fund. Math., 44 (1957), 91-93.

[5] J. T. Baldwin, P. C. Eklof and J. Trlifaj, ${}^{\perp}N$ as an abstract elementary class, Ann. Pure Appl. Logic, 149(1-3) (2007), 25-39.

[6] W. Baur, On the elementary theory of quadruples of vector spaces, Ann. Math. Logic, 19(3) (1980), 243-262.

[7] S. Bazzoni, Cotilting modules are pure-injective, Proc. Amer. Math. Soc., 131(12) (2003), 3665-3672.

[8] F. Borceux and J. Rosický, Purity in algebra, Algebra Universalis, 56(1) (2007), 17-35.

[9] K. Burke and M. Prest, The Ziegler and Zariski spectra of some domestic string algebras, Algebras and Representation Theory, 5(3) (2002), 211-234.

[10] M. C. R. Butler and G. Horrocks, Classes of extensions and resolutions, Phil. Trans. Roy. Soc. London Ser. A, 254(1039) (1961), 155-222.

[11] R. Camps and A. Facchini, The Prüfer rings that are endomorphism rings of artinian modules, Comm. Algebra, 22 (1994), 3133-3157.

[12] P. M. Cohn, On the free product of associative rings, Math. Z. 71(1) (1959), 380-398.

[13] W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc., 63(2) (1991), 241-265.

[14] W. Crawley-Boevey, Modules of finite length over their endomorphism rings, pp. 127-184 *in* Representations of Algebras and Related Topics, London Math. Soc. Lect. Note Ser. Vol. 168, Cambridge University Press, 1992.

[15] W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra, 22(5) (1994), 1641-1674.

[16] W. Crawley-Boevey, Infinite-dimensional modules in the representation theory of finite-dimensional algebras, pp. 29-54 *in* I. Reiten, S. Smaløand Ø. Solberg (Eds.), Algebras and Modules I, Canadian Math. Soc. Conf. Proc., Vol. 23, Amer. Math. Soc, 1998.

[17] Yu. Drozd, On generalised uniserial rings, Math. Zametki, 18(5) (1975), 705-710.

[18] S. Ebrahimi Atani, On pure-injective modules over pullback rings, Comm. Algebra, 28(9) (2000), 4037-4069.

[19] D. Eisenbud and J. C. Robson, Modules over Dedekind prime rings, J. Algebra, 16(1) (1970), 67-85.

[20] D. Eisenbud and J. C. Robson, Hereditary noetherian prime rings, J. Algebra, 16(1) (1970), 86-104

[21] P. Eklof and G. Sabbagh, Model-completions and modules, Ann. Math. Logic, 2(3) (1971), 251-295.

[22] A. Facchini, Module Theory: Endomorphism rings and direct sum decompositions in some classes of modules, Progress in Math., Vol. 167, Birkhäuser, Basel, 1998.

[23] D. J. Fieldhouse, Pure theories, Math. Ann., 184(1) (1969), 1-18.

[24] E. Fisher, Abelian Structures, Yale University, 1974/5, partly published as Abelian Structures I, pp. 270-322 *in* Abelian Group Theory, Lecture Notes in Mathematics, Vol. 616, Springer-Verlag, 1977.

[25] L. Fuchs, Infinite Abelian Groups, Vol. I, Academic Press. 1970.

[26] S. Garavaglia, Dimension and rank in the model theory of modules, *preprint*, University of Michigan, 1979, *revised* 1980.

[27] S. Garavaglia, Decomposition of totally transcendental modules, J. Symbolic Logic, 45(1) (1980), 155-164.

[28] G. A. Garkusha, Grothendieck categories (*in Russian*), pp. 1-68 *in* Algebra and Analysis (2), 13(2) (2001), *English transl. in* St. Petersburg Math. J., 13(2) (2002).

[29] G. Garkusha and M. Prest, Triangulated categories and the Ziegler spectrum, Algebras and Representation Theory, 8(4) (2005), 499-523.

[30] G. Garkusha and M. Prest, Classifying Serre subcategories of finitely presented modules, Proc. Amer. Math. Soc., 136(3) (2008), 761-770.

[31] W. Geigle, The Krull-Gabriel dimension of the representation theory of a tame hereditary artin algebra and applications to the structure of exact sequences, Manus. Math., 54(1-2) (1985), 83-106.

[32] R. Göbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules, W. de Gruyter, 2006.

[33] J. L. Gómez Pardo and P. A. Guil Asensio, Chain conditions on direct summands and pure quotient modules, pp. 195-203 *in* Interactions between Ring Theory and Representations of Algebras (Murcia), Lecture Notes in Pure Appl. Math, 210, Marcel Dekker, New York, 2000.

[34] L. Gruson and C. U. Jensen, Modules algébriquement compact et foncteurs $\varprojlim^{(i)}$, C. R. Acad. Sci. Paris, 276 (1973), 1651-1653.

[35] L. Gruson and C. U. Jensen, Dimensions cohomologiques reliées aux foncteurs $\varprojlim^{(i)}$, pp. 243-294 *in* Lecture Notes in Mathematics, Vol. 867, Springer-Verlag, 1981.

P. A. Guil Asensio and D. Simson, Indecomposable decompositions of pure-injective objects and the pure-semisimplicity. J. Algebra 244(2) (2001), 478-491.

[36] P. A. Guil Asensio and I. Herzog, Indecomposable flat cotorsion modules, J. London Math. Soc., 76(3) (2007), 797-811.

[37] P. A. Guil Asensio and I. Herzog, Pure-injectivity in the category of flat modules, Contemp. Math., 419 (2006), 155-165.

[38] I. Herzog, Elementary duality of modules, Trans. Amer. Math. Soc., 340(1) (1993), 37-69.

[39] I. Herzog, Finitely presented right modules over a left pure-semisimple ring, Bull. London Math. Soc., 26(4) (1994), 333-338.

[40] I. Herzog, The Ziegler spectrum of a locally coherent Grothendieck category, Proc. London Math. Soc., 74(3) (1997), 503-558.

[41] I. Herzog, Pure injective envelopes, J. Algebra Appl., 2(4) (2003), 397-402.

[42] I. Herzog and Ph. Rothmaler, When cotorsion modules are pure-injective, *preprint*.

[43] W. Hodges, Model Theory, Encyclopedia of Mathematics, Vol. 42, Cambridge University Press, 1993.

[44] B. Huisgen-Zimmermann, Purity, algebraic compactness, direct sum decompositions and representation type, pp. 331-367 *in* [51].

[45] C. U. Jensen and H. Lenzing, Model Theoretic Algebra; with particular emphasis on Fields, Rings and Modules, Gordon and Breach, 1989.

[46] I. Kaplansky, Infinite Abelian Groups, Univ. of Michigan Press, Ann Arbor, 1954. *Also revised edition*, Ann Arbor, 1969.

[47] R. Kielpinski, On Γ-pure injective modules, Bull. Polon. Acad. Sci. Math., 15 (1967), 127-131.

[48] H. Krause, Exactly definable categories, J. Algebra, 201(2) (1998), 456-492.

[49] H. Krause, Smashing subcatgories and the telescope conjecture - an algebraic approach, Invent. Math., 139(1) (2000), 99-133.

[50] H. Krause, The Spectrum of a Module Category, Habilitationsschrift, Universität Bielefeld, 1997, *published as* Mem. Amer. Math. Soc., No. 707, 2001.

[51] H. Krause and C. M. Ringel (Eds.), Infinite Length Modules, Birkhäuser, 2000.

[52] A. Laradji, A generalization of algebraic compactness, Comm. Algebra, 23(10) (1995), 3589-3600.

[53] A. J. Macintyre, On ω_1-categorical theories of abelian groups, Fund. Math., 70 (1971), 253-270.

[54] E. Monari-Martinez, On Σ-pure-injective modules over valuation domains, Arch. Math., 46(1) (1986), 26-32.

[55] J. Mycielski, Some compactifications of general algebras, Colloq. Math., 13 ((1964), 1-9.

[56] J. Mycielski and C. Ryll-Nardzewski, Equationally compact algebras II, Fund. Math., 61 (1968), 271-281.

[57] F. Okoh, Indecomposable pure-injective modules over hereditary Artin algebras of tame type, Comm. Algebra, 8(20) (1980), 1939-1941.

[58] M. Prest, Duality and pure-semisimple rings, J. London Math. Soc. (2), 38 (1988), 403-409.

[59] M. Prest, Model Theory and Modules, London Math. Soc. Lect. Note Ser., Vol. 130, Cambridge University Press, 1988.

[60] M. Prest, Ziegler spectra of tame hereditary algebras, J. Algebra, 207(1) (1998), 146-164.

[61] M. Prest, Topological and geometric aspects of the Ziegler spectrum, pp. 369-392 *in* H. Krause and C. M. Ringel (Eds.), Infinite Length Modules, Birkhäuser, 2000.

[62] M. Prest, Model theory for algebra, pp. 199-226 *in* M. Hazewinkel (ed.), Handbook of Algebra, Vol. 3, Elsevier, 2003.

[63] M. Prest, Model theory and modules, pp. 227-253 *in* M. Hazewinkel (ed.), Handbook of Algebra, Vol. 3, Elsevier, 2003.

[64] M. Prest, Purity, Spectra and Localisation, Cambridge University Press, 2009.

[65] M. Prest, Definable additive categories: purity and model theory, Mem. Amer. Math. Soc., *to appear*.

[66] M. Prest and G. Puninski, Σ-pure-injective modules over a commutative Prüfer ring, Comm. Algebra, 27(2) (1999), 961-971.

[67] M. Prest and G. Puninski, Pure injective envelopes of finite length modules over a Generalized Weyl Algebra, J. Algebra, 251(1) (2002), 150-177.

[68] H. Prüfer, Untersuchungen über die Zerlegbarkeit der abzählbaren primären abelschen Gruppen, Math. Z., 17(1) (1923), 35-61.

[69] G. Puninski, Superdecomposable pure-injective modules exist over some string algebras, Proc. Amer. Math. Soc., 132(7) (2004), 1891-1898.

[70] G. Puninski, V. Puninskaya and C. Toffalori, Superdecomposable pure-injective modules and integral group rings, J. London Math. Soc. (2), 73(1) (2006), 48-64.

[71] G. Puninski, V. Puninskaya, and C. Toffalori, Krull-Gabriel dimension and the model-theoretic complexity of the category of modules over group rings of finite groups, J. London Math. Soc., 78(1) (2008), 125-142.

[72] M. Raynaud and L. Gruson, Critères de platitude et de projectivité, Seconde partie, Invent. Math., 13(1-2) (1971), 52-89.

[73] C. M. Ringel, Some algebraically compact modules I, pp. 419-439 *in* Abelian Groups and Modules, Eds. A. Facchini and C. Menini, Kluwer, 1995.

[74] C. M. Ringel, The Ziegler spectrum of a tame hereditary algebra, Colloq. Math., 76 (1998), 105-115.

[75] C. M. Ringel, Infinite length modules: some examples as introduction, pp. 1-73 *in* [51].

[76] Ph. Rothmaler, Purity in model theory, pp. 445-469 *in* R. Göbel and M. Droste (Eds.), Advances in Algebra and Model Theory, Gordon and Breach, 1997.

[77] Ph. Rothmaler, Introduction to Model Theory, *transl./revised from German original* (1995), Algebra, Logic and Appl., Vol. 15., Gordon and Breach, 2000.

[78] G. Sabbagh, Aspects logiques de la pureté dans les modules, C. R. Acad. Sci. Paris, 271 (1970), 909-912.

[79] G. Sabbagh and P. Eklof, Definability problems for rings and modules, J. Symbolic Logic, 36(4) (1971), 623-649.

[80] D. Simson, A class of potential counterexamples to the pure semisimplicity conjecture, pp. 345-373 *in* M. Droste and R. Göbel (Eds.), Advances in Algebra and Model Theory, Gordon and Breach, 1997.

[81] J. Schröer, On the Krull-Gabriel dimension of an algebra, Math. Z. 233(2) (2000), 287-303.

[82] D. Simson, On pure global dimension of locally finitely presented Grothendieck categories, Fund. Math., 96 (1977), 91-116.

[83] B. Stenström, Pure submodules, Arkiv Mat., 7(2) (1967), 159-171.

[84] B. Stenström, Purity in functor categories, J. Algebra, 8(3) (1968), 352-361.

[85] B. Stenström, Rings of Quotients, Springer-Verlag, 1975.

[86] J. Šťovíček, All n-cotilting modules are pure-injective, Proc. Amer. Math. Soc., 134(7) (2006), 1891-1897.

[87] W. Taylor, Some constructions of compact algebras, Ann. Math. Logic, 3(4) (1971), 395-435.

[88] W. Taylor, Review of, "B. Banaschewski and E. Nelson, Equational compactness in equational classes of algebras. Algebra Universalis 2 (1972), 152–165", Mathematical Reviews MR0308010 (46 no.7125).

[89] J. Trlifaj, Infinite dimensional tilting modules and cotorsion pairs, pp. 279-321 *in* [2].

[90] J. Trlifaj, Abstract elementary classes induced by tilting and cotilting modules have finite character, Proc. Amer. Math. Soc., *to appear.*

[91] R. B. Warfield, Purity and algebraic compactness for modules, Pacific J. Math., 28(3) (1969), 699-719.

[92] R. B. Warfield, Decomposability of finitely presented modules, Proc. Amer. Math. Soc., 25(1) (1970), 167-172.

[93] R. B. Warfield, Exchange rings and decomposition of modules, Math. Ann., 199(1) (1972), 31-36.

[94] R. B. Warfield, Serial rings and finitely presented modules, J. Algebra, 37(2) (1975), 187-222.

[95] B. Weglorz, Equationally compact algebras (I), Fund. Math., 59 (1966), 289-298.

[96] B. Weglorz, Equationally compact algebras (III), Fund. Math., 60 (1966), 89-93.

[97] G. H. Wenzel, Equational compactness, Appendix 6 (pp. 417-447) *in* G. Gratzer, Universal Algebra (2nd Edn.), Springer-Verlag, 1979.

[98] R. Wisbauer, On modules with the Kulikov property and pure semisimple modules and rings, J. Pure Appl. Algebra, 70(3) (1991), 315-320.

[99] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, 1991.

[100] J. Xu, Flat Covers of Modules, Lecture Notes in Mathematics, Vol. 1634, Springer, 1996.

[101] D. Zelinsky, Linearly compact rings and modules, Amer. J. Math., 75(1) (1953), 79-90.

[102] M. Ziegler, Model theory of modules, Ann. Pure Appl. Logic, 26(2) (1984), 149-213.

[103] W. Zimmermann, Rein injektive direkte Summen von Moduln, Comm. Algebra, 5(10) (1977), 1083-1117.

[104] B. Zimmermann-Huisgen and W. Zimmermann, Algebraically compact rings and modules, Math. Z., 161(1) (1978), 81-93.

[105] B. Zimmermann-Huisgen and W. Zimmermann, On the sparsity of representations of rings of pure global dimension zero, Trans. Amer. Math. Soc., 320(2) (1990), 695-711.