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Definable additive categories: purity and model theory

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1 Introduction

Definable additive categories and their model theory are the topic of this paper. We begin with background and preliminary results on additive categories. Then definable categories, their properties and the morphisms between them are investigated, as are certain associated topological spaces ("spectra"). It was in the model theory of modules that these categories were first considered and model theory provides some of the tools for exploring them. Some general model-theoretic background is presented, then various aspects of the model theory of definable categories are considered.

There are new results in this paper but a substantial part is a working up, into a unified form and in a general context, of results which are scattered across the literature and sometimes are just 'folklore'. Primarily, this paper is about the model theory of additive categories but there are various category-theoretic and algebraic results. Indeed, most of the results can be presented and proved using either a model-theoretic or a functor-theoretic approach and, in writing this paper, I have tried to illuminate the relation between these.

Suppose that R is a ring. A subcategory of the category, Mod-R, of right R-modules is said to be definable if it is closed in Mod-R under direct products, direct limits and pure submodules. It was in the model theory of modules that it was first realised that there is a rich theory associated to such subcategories. In that context they arise as the elementary (=axiomatisable) subclasses of modules closed under products and direct summands and they are in bijection with the closed subsets of the Ziegler spectrum, Zg_R, of R (14.2). That is a topological space whose points are certain indecomposable modules and whose topology was originally defined using concepts from model theory but which may also be defined purely in terms of the category Mod-R, alternatively in terms of a certain functor category associated to Mod-R. That functor category, which also arises as the category of model-theoretic pp-imaginaries, could equally be regarded as the real topic of this paper. One purpose of the paper is to explain all this.

Another purpose is to develop everything in what is arguably the correct setting: namely definable subcategories of finitely accessible additive categories. This is what results when the ring R is replaced by any small preadditive category. In this way the context is broadened to encompass functor categories themselves, categories of comodules, certain categories of sheaves of modules and a great variety of particular natural examples.

Yet another purpose of the paper is to update the model theory of modules as presented in [77], more generally the model theory of definable categories, now that it has been reshaped through its interaction with additive category theory. There is a book, [84], which is in a sense a successor of [77], but in that book model theory *per se* has been de-emphasised. There is some overlap, especially with Part III of that book, but this paper, at least the second part, has been written primarily for model-theorists and those interested in the interaction of model theory and additive category theory.

The first part of the paper develops the relevant additive category theory. It is mainly a drawing-together of results which are folklore or are, in some form, in the literature, but 12.10 is new. With that result one has bijections between the following: model-theoretic interpretations of definable categories, one in another; exact functors between their categories of imaginaries, the latter being typical small abelian categories; functors between definable categories which preserve direct products and direct limits. Part of this is contained already in the work, [65], [45], of Makkai and Hu (though the proofs are wildly different) and further development in this direction, encompassing both the additive and non-additive contexts, certainly is an important direction for future research.

Here is an outline of the main results in the first part. Associated to a ring R, rather to its category of modules, is the category fun-$R = (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ of finitely presented additive functors from the category, mod-R, of finitely presented modules to the category, $\mathbf{Ab}$, of abelian groups. This is the functor category which is in fact (22.2) equivalent to the category of pp-defined sorts for R-modules and pp-defined functions between them. The category Mod-R may be recovered from fun-R, *via* evaluation of a functor (or a pp-sort) at a module, as being the category of exact functors from fun-R to $\mathbf{Ab}$ (10.8). The functor category corresponding to the category of left R-modules turns out to be the opposite category, $(\text{fun-}R)^{\text{op}}$ (4.5, 10.10). So, although the categories of right and left (even finitely presented) modules may be very different, there is a perfect duality "at the next representation level up". Serre subcategories of fun-R correspond to definable subcategories of Mod-R (8.1) and there is, altogether, a very rich theory which has been developed, starting with the paper [36] of Gruson and Jensen, around fun-R. It also turns out (4.3) that fun-R is the free abelian category, in the sense of Freyd [26], on R.

It has to be said that, with all the shifting from one representation level to another and the moving to opposite categories, a certain amount of mental gymnastics is involved. This does not get any easier when everything is extended to finitely accessible additive categories (these have also been (mis-)named locally finitely presented additive categories) in place of Mod-R. The paper begins with a summary of the basic facts about these categories. Then other background material is presented: on free abelian categories, purity and localisation. The functor category of an arbitrary definable subcategory is defined, in Section 10, *via* localisation and is shown (12.2) to be independent of representation (a given category may be found, up to natural equivalence, as a definable subcategory of many different categories). The first major goal is the description (12.3), due to Krause [53, 2.9], of a natural bijection between definable categories and small abelian categories. Then we move to the next key result (12.10), showing that exact functors between small abelian categories correspond to those functors between the associated definable categories which commute with direct products and direct limits. These are also the functors which correspond to interpretations in the model-theoretic sense (25.3).

The second part begins by introducing the many-sorted language of a finitely accessible category (§18). I should emphasise that all languages used in this paper are finitary. Then the category of pp-sorts and pp-definable functions is defined: this is the (abelian) category which underlies the pp-imaginaries language of a finitely accessible category. The restriction to pp formulas is natural in the additive situation since then all definable sorts and functions inherit, respectively preserve, the additive structure. In Section 25 the notion of interpretation is cast in the form of a functor and it is seen that these interpretation functors correspond to exact functors between the categories of pp-imaginaries (25.4). The issue of full interpretation is discussed in Section 25: this becomes important when we detach parts of the imaginaries structure and so can no longer necessarily refer definitions back to the "home" sorts. Various results which are used in the first part, concerning pp formulas and pp-types, are proved. In order to make these perhaps accessible to those without model-theoretic background there is some explanation of basic model-theoretic notions, including a short section on ultraproducts.

The remainder of the second part is a rather brief discussion of what various model-theoretic notions look like in this context.

Throughout this paper purity and associated notions are used heavily. The algebraic concepts around purity have been linked with model theory from an early stage (see e.g. [62], [68], [102], [69], also [101] and [1], [66]). That this has been especially so in the context of modules can be seen from the central role played by purity in the model theory of modules, for which one may consult, for example, [77]. It has been known for a long time that both purity and model theory as developed in categories of modules is valid in more general additive categories. A reasonable amount of this has been written down for purity, see e.g. [97] and [19] (also [1], [65], [94] for the general context), but there is no comprehensive account, in particular not one which says much about model theory *per se* (although [66] does treat model theory its emphasis is very different from that here). This paper partly remedies this. Only partly, because, for instance, the basic results of the model theory of modules are not re-proved here in the more general context. That is because the proofs are "really just the same". To see that this is so takes some thought but, once seen, it is obvious and there remains little point in actually writing down the details. A precise meta-theorem like the Mitchell Embedding Theorem would be useful here but whether there is one which is both comprehensive and has a precise formulation, I don't know.

The basic context of this paper is that of finitely accessible additive categories with products. It turns out, from the work of Guil-Asensio and Herzog, see e.g. [39], that a surprising amount may be developed in the additive context without assuming existence of products, but the landscape there is rather different and we will remain in an area where the scenery is very reminiscent of that in categories of modules over a ring. In another direction, requiring existence of direct limits excludes triangulated categories but these may be treated *via* their associated functor categories (see Section 16, and [55] for details). There are also many parallels with the non-additive context ([1] is a useful reference here), especially see [65]. An extension of the link between model theory and functor categories to the non-additive context is made in [90]. The more "geometric"/sheaf-theoretic aspects are developed elsewhere ([86]).

2 Preadditive and additive categories

secpreadd

A category is **preadditive** if every morphism set, (A, B), has an abelian group structure such that composition on either side is linear. A preadditive category with one object is simply a ring. An **additive category** is one which is preadditive, has a zero (=initial and final) object and which is such that every pair of objects has a coproduct: in that case every pair of objects has a direct product, which is canonically isomorphic to their coproduct and which is referred to as their direct sum (see, e.g., [72, 2.1.2]). An additive category is **abelian** if every morphism has a kernel and a cokernel, if every monomorphism is a kernel and if every epimorphism is a cokernel (one says then that every monomorphism and every epimorphism is **regular**). The archetype is the category, **Ab**, of abelian groups. The category of finitely generated abelian groups also is an example.

Start with a small preadditive category $\mathcal{A}$. Denote by $\mathcal{A}^+$ the **additive completion** of $\mathcal{A}$ ([25, p. 60], [71, p. 92]). The objects of $\mathcal{A}^+$ are finite tuples of objects of $\mathcal{A}$ (including the empty tuple, i.e. the zero object of $\mathcal{A}^+$) and maps are matrices with morphisms from $\mathcal{A}$ as entries. This has the universal property that for all additive functors $\mathcal{A} \longrightarrow \mathcal{B}$ with $\mathcal{B}$ additive there is a unique, up to natural equivalence, factorisation through the canonical functor $\mathcal{A} \longrightarrow \mathcal{A}^+$ which takes objects to 1-tuples. (A functor $F : \mathcal{A} \to \mathcal{B}$ is **additive** if for every $A, A' \in \mathcal{A}$ the map $F : (A, A') \to (FA, FA')$, $f \mapsto Ff$ is a homomorphism of abelian groups.)

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\mathrm{can}"] \arrow[dr, "\forall"'] & & \mathcal{A}^+ \arrow[dl, dashed, "\exists!"] \\ & \mathcal{B} & \end{tikzcd}$$

Digression: unique *versus* unique to natural equivalence. Take $\mathcal{A} = \mathbb{F}_2$: a category with one object, whose endomorphism ring is the field of two elements. Then $\mathcal{A}^+$ is a skeletal version of the category of finite-dimensional $\mathbb{F}_2$-vectorspaces. (A category is **skeletal** if there is just one object in each isomorphism class.) Replacing $\mathbb{F}_2$ by an arbitrary ring $\mathcal{A} = R$, we obtain the category of finitely generated free right R-modules, at least a skeletal version but we seldom make such distinctions. Let $\mathcal{B}$ be a version of $\mathcal{A}^+$ with two copies of each vector space and define additive functors $F, G : \mathcal{A}^+ \longrightarrow \mathcal{B}$ to agree on 0 and the 1-dimensional space but to disagree on higher-dimensional spaces. Clearly there is a natural equivalence (even isomorphism) between F and G, induced by the obvious corresponding automorphism of $\mathcal{B}$, but certainly $F \neq G$. Thus the extension of $\mathcal{A} \longrightarrow \mathcal{B}$ to $\mathcal{A}^+ \longrightarrow \mathcal{B}$, though unique to natural equivalence, i.e. to isomorphism in the functor category $(\mathcal{A}^+, \mathcal{B})$, is not literally unique. (We use the notation $(\mathcal{A}, \mathcal{B})$ for the category of additive functors from the preadditive category $\mathcal{A}$ to the preadditive category $\mathcal{B}$. Usually we assume that $\mathcal{A}$ is **skeletally small**, that is, has, up to isomorphism, only a set of objects. This ensures that for all $F, G \in (\mathcal{A}, \mathcal{B})$ the set of morphisms, i.e. natural transformations, from F to G is a set, rather than a proper class.)

Returning to the general case, restriction to $\mathcal{A}$ gives an equivalence, but, as was as illustrated above, not an isomorphism, $(\mathcal{A}^+, \mathcal{B}) \simeq (\mathcal{A}, \mathcal{B})$ (between categories we use the symbol "$\simeq$" for natural equivalence: we never really need isomorphism). For, by the universal property, every object of $(\mathcal{A}, \mathcal{B})$ is the restriction of an object of $(\mathcal{A}^+, \mathcal{B})$ and, by construction o f $\mathcal{A}^+$ from $\mathcal{A}$, it is easy to see that restriction is full and faithful. That is enough (see e.g. [72, 1.5.3]) for equivalence.

Say that **idempotents split** in the preadditive category $\mathcal{A}$ if the following equivalent conditions are satisfied:
(i) for every $A \in \mathcal{A}$ and idempotent $e = e^2 \in \mathrm{End}(A)$ there is an object B of $\mathcal{A}$ and there are morphisms $A \xrightarrow{p} B \xrightarrow{i} A$ such that $pi = 1_B$ and $ip = e$;
(ii) for every $A \in \mathcal{A}$ and idempotent $e = e^2 \in \mathrm{End}(A)$ there is an object B of $\mathcal{A}$ and there are morphisms $A \xrightarrow{p} B \xrightarrow{i} A$ such that $ip = e$, p is an epimorphism and i is a monomorphism (and hence $pi = 1_B$);
(iii) for every $A \in \mathcal{A}$ every idempotent $e = e^2 \in \mathrm{End}(A)$ has a kernel and the

canonical morphism $\ker(e) \oplus \ker(1-e) \longrightarrow A$ is an isomorphism.
For example, to show that (ii) implies (iii) apply the "equational criterion" for a direct sum system ([72, 2.1.2] or [25, p. 50]) to the direct sum of B and the corresponding object for $1-e$.

Denote by $\mathcal{A}^{++}$ the "**idempotent-splitting**" **completion** ("pseudo-abelian" or "Karbouian" completion in the terminology of [51, p. 75]) of $\mathcal{A}^{+}$. The objects are pairs (A, e) with A an object of $\mathcal{A}$ and $e = e^2 \in \mathrm{End}(A)$. The morphisms $(A, e) \longrightarrow (B, f)$ are those morphisms $g : A \longrightarrow B$ such that $fge = g$ (equivalently, the group of those g such that $ge = fg$, modulo the subgroup of those with $ge = fg = 0$). Then $\mathcal{A}^{++}$ is a category in which idempotents split and for every (additive, as always but henceforth seldom stated) functor from $\mathcal{A}$, or $\mathcal{A}^{+}$, to $\mathcal{B}$, where $\mathcal{B}$ is additive with split idempotents, there is a unique, to natural isomorphism, factorisation through $\mathcal{A} \longrightarrow \mathcal{A}^{++}$, respectively $\mathcal{A}^{+} \longrightarrow \mathcal{A}^{++}$, (the obvious functor from $\mathcal{A}$ to $\mathcal{A}^{++}$ which takes A to $(A, 1_A)$). See [25, Exercise 2B, p. 61] or [72, p. 22, Exercise 5(b)] (and [1, Exercise 2b, p. 125] for the non-additive case).

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\mathrm{can}"] \arrow[dr, "\forall"'] & & \mathcal{A}^{++} \arrow[dl, dashed, "\exists!"] \\ & \mathcal{B} & \end{tikzcd}$$

For instance if $\mathcal{A}$ is a ring then $\mathcal{A}^{++}$ is equivalent to the category, proj-$\mathcal{A}$, of finitely generated projective right $\mathcal{A}$-modules.

Again, restriction is an equivalence and we have $(\mathcal{A}^{++}, \mathcal{B}) \simeq (\mathcal{A}^{+}, \mathcal{B}) \simeq (\mathcal{A}, \mathcal{B})$. Therefore, for many results, there is no loss in generality if we assume that a preadditive category $\mathcal{A}$ is actually additive with split idempotents. Of course we will not always do this: if R is a ring then we would normally refer to Mod-R, i.e. $(R^{\mathrm{op}}, \mathbf{Ab})$, rather than Mod-(proj-$R$), i.e. $((\text{proj-}R)^{\mathrm{op}}, \mathbf{Ab})$, even though these categories are equivalent.

We just used the following notation. If $\mathcal{A}$ is a (skeletally) small preadditive category then we often write Mod-$\mathcal{A}$ for $(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$, $\mathcal{A}$-Mod for $(\mathcal{A}, \mathbf{Ab})$, mod-$\mathcal{A}$ for $(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})^{\mathrm{fp}}$ *et cetera*. And $\mathcal{C}^{\mathrm{fp}}$ is our notation for the full subcategory of finitely presented objects of a category $\mathcal{C}$. An object $C \in \mathcal{C}$ is **finitely presented** if the representable functor $(C, -) : \mathcal{C} \longrightarrow \mathbf{Ab}$ commutes with direct limits ("directed colimits" in the more logical terminology), that is, if any morphism from C to the limit of a directed system lifts through a member of the system, that is, if whenever $((D_\lambda)_\lambda, (f_{\lambda\mu} : D_\lambda \longrightarrow D_\mu)_{\lambda \le \mu})$ is a directed system with direct limit $(D, (f_{\lambda\infty} : D_\lambda \longrightarrow D)_\lambda)$ and $g : C \longrightarrow D$ is a morphism, then there is λ and $h : C \longrightarrow D_\lambda$ such that $f_{\lambda\infty} h = g$. This is the definition for arbitrary categories and it does coincide with the usual idea of "finitely generated and finitely related" when the latter makes sense (see, e.g., [47, VI.2.2]).

If we were working with R-modules where R is an algebra over a field k contained in the centre of R then it would be natural to consider functors from Mod-R to the category, Mod-k, of k-vectorspaces, rather than to $\mathbf{Ab}$. This makes no difference since there is the following extension (here we follow [71, p. 92] and [63, §2.3]). Let S be a commutative ring. An S-**preadditive**

category is a preadditive category $\mathcal{A}$, together with a ring morphism $S \longrightarrow Z(\mathcal{A})$ where $Z(\mathcal{A})$ denotes the **centre** of $\mathcal{A}$, by which is meant the ring (small if $\mathcal{A}$ is skeletally small) of natural transformations from $1_{\mathcal{A}}$, the identity functor on $\mathcal{A}$, to itself. If $\mathcal{A}$ is a ring then this is the centre, in the usual sense, of that ring. Thus, to each $s \in S$ there corresponds $\eta_s : 1_{\mathcal{A}} \longrightarrow 1_{\mathcal{A}}$, that is, for each $A \in \mathcal{A}$ there is $(\eta_s)_A \in \mathrm{End}(A)$ such that for every morphism $f : A \longrightarrow B$ in $\mathcal{A}$ one has $(\eta_s)_B f = f\,(\eta_s)_A$.

$$\begin{tikzcd} A \arrow[r, "(\eta_s)_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "(\eta_s)_B"'] & B \end{tikzcd}$$

Thinking of $(\eta_s)_A$ as "multiplication by s" ("on objects of $\mathcal{A}$"), this requirement is that each morphism in $\mathcal{A}$ be S-linear.

An S-preadditive category is equivalently a preadditive category in which each morphism group is equipped with an (S, S)-bimodule structure with S acting centrally. In alternative terminology an S-preadditive category is a category enriched in the category of central S-bimodules (see, e.g., [50]).

It is pointed out in [63] that one may view an S-preadditive category as a non-commutative scheme over the commutative affine base scheme $\mathrm{Spec}(S)$.

If $\mathcal{A}$ is S-preadditive and $F : \mathcal{A} \longrightarrow \mathcal{A}'$, with $\mathcal{A}'$ preadditive, is an additive functor then the image of F is an S-linear category in the obvious way (define $Ff.s$ to be $F(fs)$) and then F is S-linear in the sense that for every $A, B \in \mathcal{A}$ the map $(A, B) \longrightarrow (FA, FB)$ induced by F is a map of (S, S)-bimodules.

The forgetful functor $\text{Mod-}S \longrightarrow \mathbf{Ab}$ induces $(\mathcal{A}, \text{Mod-}S) \longrightarrow (\mathcal{A}, \mathbf{Ab}) = \mathcal{A}\text{-Mod}$ and this, one may check, is an equivalence.

3 Preadditive categories and their ind-completions

secind

A preadditive category $\mathcal{C}$ is **finitely accessible** if it has direct limits (equivalently, see, e.g., [1, 1.5], filtered colimits), if $\mathcal{C}^{\mathrm{fp}}$ is skeletally small and if every object of $\mathcal{C}$ is a direct limit (equivalently, is a filtered colimit) of objects of $\mathcal{C}^{\mathrm{fp}}$, that is, if $C \in \mathcal{C}$ then there is a directed/filtered category $\mathcal{I}$ and a functor/diagram $\mathbb{D} : \mathcal{I} \longrightarrow \mathcal{C}^{\mathrm{fp}}$ such that $C = \varinjlim \mathbb{D}$ (from the proof of [1, 1.5] it follows that the closure of a subcategory under filtered colimits is the same as its closure under direct limits).

Every (preadditive) category $\mathcal{B}$ has an ind-completion, $\mathrm{Ind}\mathcal{B}$, defined as follows. An **ind-object** of $\mathcal{B}$ is a diagram $\mathbb{D} : \mathcal{I} \longrightarrow \mathcal{B}$ where $\mathcal{I}$ is a directed (or filtered) category. A morphism from such a diagram to another, $\mathbb{E} : \mathcal{J} \longrightarrow \mathcal{B}$, is an element of the set $\varprojlim_{i\in\mathcal{I}} \varinjlim_{j\in\mathcal{J}} (\mathbb{D}(i), \mathbb{E}(j))$ (see, for example, the discussion in [47, p. 225] for explication, or [48, Chpt. 6]).

This is a little messy and is simplified by actually adding the objects that such diagrams "represent". Precisely, let $Y : \mathcal{B} \longrightarrow (\mathcal{B}^{\mathrm{op}}, \mathbf{Ab})$ be the Yoneda embedding (on objects, $B \mapsto (-, B)$). Replace a diagram $\mathbb{D}$ as above by the direct limit in $(\mathcal{B}^{\mathrm{op}}, \mathbf{Ab})$ of the composite diagram $Y\mathbb{D} : \mathcal{I} \longrightarrow (\mathcal{B}^{\mathrm{op}}, \mathbf{Ab})$ in the (possibly "large", but $\mathcal{I}$ is assumed to be small) functor category. Then Ind$\mathcal{B}$ is equivalent to the resulting full subcategory of $(\mathcal{B}^{\mathrm{op}}, \mathbf{Ab})$ with objects those functors which are direct limits (over small diagrams) of representable functors.

An object $M \in \text{Mod-}\mathcal{A}$ is **flat** if the functor $M \otimes_{\mathcal{A}} - : \mathcal{A}\text{-Mod} \longrightarrow \mathbf{Ab}$ is exact. It is enough to require that for every monomorphism $f : L \longrightarrow L'$ in $\mathcal{A}$-mod the morphism $1_M \otimes f : M \otimes L \longrightarrow M \otimes L'$ be monic. Tensor product over an arbitrary preadditive category $\mathcal{A}$ is defined by the requirement that the functor $M \otimes_{\mathcal{A}} -$ be right exact and that $M \otimes_{\mathcal{A}} (A, -) \simeq MA$. For then, to compute the value $M \otimes_{\mathcal{A}} L$ where $L \in \mathcal{A}\text{-Mod}$, take a projective presentation, $\bigoplus_j (B_j, -) \longrightarrow \bigoplus_i (A_i, -) \longrightarrow L \longrightarrow 0$, of L (the representable functors $(A, -)$ are, from the Yoneda Lemma, projective objects of Mod-$\mathcal{A}$ and every finitely generated projective is a direct summand of a finite direct sum of such functors). Then the value of $M \otimes_{\mathcal{A}} L$ is determined by exactness of $\bigoplus_j MB_j \longrightarrow \bigoplus_i MA_i \longrightarrow M \otimes_{\mathcal{A}} L \longrightarrow 0$, and one defines the action on morphisms using that $\bigoplus_i (A_i, -)$ is projective, then applying the Yoneda Lemma, then checking well-definedness. A2tensex

Example 3.1. (illustrating tensor product over a small preadditive category with more than one object) Let k be a field and let $\mathcal{A}$ be the k-path category of the quiver Q which is usually (but, in order to avoid a notation clash, not at this point) denoted A_2:

$$\bullet \longrightarrow \bullet$$

there are two objects, A_1, A_2 say; $\mathrm{End}(A_i) = k$, $(A_1, A_2) = k$ and $(A_2, A_1) = 0$. This is the idempotent-splitting completion of the ring, R of 2×2 upper-triangular matrices over k so Mod-$\mathcal{A}$ is quite familiar, being equivalent to Mod-R but this will serve as a minimal illustration of the definition of $\otimes_{\mathcal{A}}$. Let α be a fixed non-zero morphism from A_1 to A_2, so the morphisms in (A_1, A_2) are just scalar multiples of α.

A right $\mathcal{A}$-module M is an additive functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathbf{Ab}$, so is given by two abelian groups, MA_1 and MA_2 and a group morphism $M\alpha : MA_2 \longrightarrow MA_1$ (thus a k-representation of the quiver opposite to Q). A left $\mathcal{A}$-module L is given by the data LA_1, LA_2 and $L\alpha : LA_1 \longrightarrow LA_2$ (thus a k-representation of Q).

One projective presentation of L has the form $(A_2, -)^{(\kappa_3)} \xrightarrow{(f,0)} (A_1, -)^{(\kappa_1)} \oplus (A_2, -)^{(\kappa_2)} \longrightarrow L \longrightarrow 0$ where $\kappa_1 = \dim_k(LA_1)$, $\kappa_2 = \dim(LA_2/\mathrm{im}(L\alpha))$, $\kappa_3 = \dim(\ker(L\alpha))$ (note that $LA_i \simeq ((A_i, -), L))$, and where the k-linear map f corresponds to $\ker(L\alpha)$. Applying $M \otimes_{\mathcal{A}} -$ gives an exact sequence $MA_2^{(\kappa_3)} \longrightarrow MA_1^{(\kappa_1)} \oplus MA_2^{(\kappa_2)} \longrightarrow M \otimes_{\mathcal{A}} L \longrightarrow 0$ which allows one to compute $M \otimes_{\mathcal{A}} L$ from the data M, L. flatlimrep

Theorem 3.2. *(Lazard [60] and Govorov [34] for modules, Oberst and Rohrl [71, 3.2] for functors, Crawley-Boevey [19, 1.3] for a very direct proof) Let $\mathcal{A}$ be a skeletally small preadditive category and let $F \in$ Mod-$\mathcal{A}$. Then the following are equivalent:*
(i) F is flat;

(ii) F is a direct limit (equivalently, a filtered colimit) of representable functors;
(iii) every morphism from a finitely presented $\mathcal{A}$-module to F factors through a finite direct sum of representable functors.

(For the non-additive analogue see [64, p. 386] or [3, Exp. I, 8.3.3].)

In fact the **canonical diagram** of $F \in \text{Flat-}\mathcal{A}$ ($= \text{Flat}(\text{Mod-}\mathcal{A})$, the full subcategory of flat objects of Mod-$\mathcal{A}$) is filtered rather than directed. This is the diagram indexed by the "comma category" $Y\mathcal{A} \downarrow F$, where $Y : \mathcal{A} \longrightarrow (\mathcal{A}^{\text{op}}, -)$ is the Yoneda functor. An object of this category is a morphism $YA \longrightarrow F$ i.e. $(-, A) \longrightarrow F$ i.e. an element of FA; and an arrow is a morphism in Mod-$\mathcal{A}$ which induces a commutative triangle, that is, by fullness of Y, an arrow of $\mathcal{A}$ which induces a commutative triangle. Precisely, a morphism from $f : YA \longrightarrow F$ to $f' : YA' \longrightarrow F$ is a morphism $g : YA \longrightarrow YA'$ (so $g = Y\gamma$ for some $\gamma : A \longrightarrow A'$) such that $f'g = f$. The diagram itself is the functor from $Y\mathcal{A} \downarrow F$ to Mod-$\mathcal{A}$ which takes an object f to the domain of f and an arrow g to itself.

Of course for every object $M \in \text{Mod-}\mathcal{A}$ one may define a canonical diagram as above but this diagram will be filtered iff M is flat ([71, 3.2] and, for the non-additive case, [3, Exp. I, 8.3.3]).

(It is a strong requirement that every flat functor be a direct limit of representable *sub*functors. Oberst and Rohrl, [71, 3.7], give conditions under which this is the case. For example it is true for right R-modules if R is right noetherian and right hereditary.)

Representing a flat object F as a direct limit of such a diagram shows how it can be regarded as a "set (admittedly many-sorted)-with-structure", its "elements" of sort A being the morphisms from $(-, A)$ to F, equivalently, the elements of FA. By taking this seriously, and for general, not just flat, objects, one may develop model theory in categories where the objects are not initially presented as sets with structures, as will be seen later (§18).

indeqflat

Corollary 3.3. *If $\mathcal{A}$ is a skeletally small preadditive category then the ind-completion of $\mathcal{A}$ is* $\text{Ind}\mathcal{A} \simeq \text{Flat-}\mathcal{A}$ *(i.e.* $\text{Flat}(\mathcal{A}^{\text{op}}, \mathbf{Ab})$*).*

The ind-completion of $\mathcal{A}$ has the universal property that every functor from $\mathcal{A}$ to a category with direct limits has a unique (to natural equivalence) extension to a functor from Ind$\mathcal{A}$ which commutes with direct limits ([3, Exp. I, 8.7.3] for the non-additive case). Since Flat-$\mathcal{A}$ also has this property (essentially by the previous discussion) the above equivalence follows.

For example, if R is a ring then the ind-completion of the category, mod-R, of finitely presented right R-modules will, by 3.4 below, be the category, Mod-R, of all R-modules.

Theorem 3.2 yields a bijection, given as 3.5 below, between skeletally small additive categories with split idempotents and finitely accessible additive categories.

addfinacc

Theorem 3.4. *([19, 1.4])*
(1) If $\mathcal{A}$ is a skeletally small preadditive category then Flat-$\mathcal{A}$ *is finitely accessible and* flat-$\mathcal{A}$ *(i.e.* $(\text{Flat-}\mathcal{A})^{\text{fp}}$*) is equivalent to $\mathcal{A}^{++}$, indeed the Yoneda embedding*

$\mathcal{A} \longrightarrow$ flat-$\mathcal{A}$, $A \mapsto (-, A)$ is the idempotent-splitting additive completion $\mathcal{A} \longrightarrow \mathcal{A}^{++}$ hence is an equivalence iff $\mathcal{A}$ is additive with split idempotents.
(2) If $\mathcal{C}$ is an additive finitely accessible category then $\mathcal{C}^{\mathrm{fp}}$ is skeletally small, additive, with split idempotents and $\mathcal{C} \longrightarrow$ Flat-$\mathcal{C}^{\mathrm{fp}}$, $C \mapsto (-, C) \upharpoonright \mathcal{C}^{\mathrm{fp}}$ is an equivalence.

(Note that, although $(-, C)$, for $C \in \mathcal{C}^{\mathrm{fp}}$, is a projective object of Mod-$\mathcal{C}^{\mathrm{fp}}$, it need not be projective in the subcategory $\mathcal{C}$ (rather, in the image of $\mathcal{C}$), the point being that the inclusion of $\mathcal{C} =$ Flat-$\mathcal{C}^{\mathrm{fp}}$ in Mod-$\mathcal{C}^{\mathrm{fp}}$ usually will not preserve epimorphisms. Indeed an epimorphism in $\mathcal{C}$ has epimorphic image in Mod-$\mathcal{C}^{\mathrm{fp}}$ iff, under one definition of purity, it is a pure epimorphism, see Section 5.)

finaccfp

Corollary 3.5. and Theorems *There is a bijection (which extends to a 2-functor between the relevant categories of categories, see [19, p. 1650]) between skeletally small additive categories $\mathcal{A}$ with split idempotents and finitely accessible additive categories $\mathcal{C}$. Under this correspondence:*
- *(Gabriel [28, II.4, Thm. 1]) $\mathcal{C}$ is locally noetherian iff $\mathcal{A} = \mathcal{C}^{\mathrm{fp}}$ is abelian and every object of $\mathcal{C}^{\mathrm{fp}}$ is noetherian (i.e. has the acc on subobjects);*
- *(Roos [93, 2.2]) $\mathcal{C}$ is locally coherent iff $\mathcal{A} = \mathcal{C}^{\mathrm{fp}}$ is abelian;*
- *(essentially Breitsprecher [10, 2.7]) $\mathcal{C}$ is abelian iff $\mathcal{A} = \mathcal{C}^{\mathrm{fp}}$ has cokernels, every epimorphism of $\mathcal{C}^{\mathrm{fp}}$ is a cokernel and for every right exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{C}^{\mathrm{fp}}$, for every morphism $h : B' \longrightarrow B$ in $\mathcal{C}^{\mathrm{fp}}$ with $gh = 0$ there are $A' \in \mathcal{C}^{\mathrm{fp}}$ and morphisms $k : A' \to A$ and $l : A' \to B'$ with l epi and $hl = fk$.*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\ A' \arrow[u, "k"] \arrow[r, "l"'] & B' \arrow[u, "h"'] \end{tikzcd}$$

By a **locally coherent** additive category we mean one which is finitely accessible and in which every finitely presented object is **coherent**, that is, each of its finitely generated subobjects is finitely presented. (An object is **finitely generated** if, whenever it is the sum of a set of subobjects, it is the sum of just finitely many of them. If $\mathcal{C}$ is finitely accessible then this is equivalent to being an epimorphic image of a finitely presented object.) It follows from 3.5 that such a category is abelian (and from 3.10 that it has products). Therefore, see 3.15, such a category is actually Grothendieck.

Crawley-Boevey [19] collects together and extends results linking properties of $\mathcal{A} \simeq \mathcal{C}^{\mathrm{fp}}$ and $\mathcal{C}$. We skip ahead to existence of cokernels (then come back to pseudocokernels, kernels and products) since that allows us to link the formulations above, in terms of flat functors, to formulations in terms of left exact functors.

addcoker

Theorem 3.6. *([19, 2.2], [54, 6.3] for (v)⇒(i)) For $\mathcal{A}$ a skeletally small preadditive category the following are equivalent:*
(i) $\mathcal{A}^{+}$ has cokernels;
(ii) $\mathcal{A}^{++}$ has cokernels;

(iii) Flat-$\mathcal{A}$ *has cokernels;*
(iv) $Y : \mathcal{A}^+ \longrightarrow$ mod-$\mathcal{A}$ *has a left adjoint;*
(v) $Y : \mathcal{A}^{++} \longrightarrow$ mod-$\mathcal{A}$ *has a left adjoint;*
(vi) the inclusion Flat-$\mathcal{A} \longrightarrow$ Mod-$\mathcal{A}$ *has a left adjoint.*

(Note that even if the above conditions are satisfied it might be that $\mathcal{A}^+$ and $\mathcal{A}^{++}$ are not equivalent: for example take $\mathcal{A}$ to be a ring which is the product of two fields.)

If these conditions are satisfied then the left adjoint in (iv) above is defined on an object $M \in$ mod-$\mathcal{A}$ by choosing a projective presentation $(-,A) \xrightarrow{(-,f)} (-,B) \longrightarrow M \longrightarrow 0$ where $f : A \longrightarrow B$ is in $\mathcal{A}^+$ and sending M to $\mathrm{coker}(f)$.

A category is **locally finitely presented** if it is finitely accessible and cocomplete (has all colimits). This, see below, is equivalent to being finitely accessible and complete (having all limits). Since finitely accessible categories already have coproducts, existence of cokernels is enough for a finitely accessible category to be locally finitely presented.

acccoker

Corollary 3.7. *([19, 2.2], [54, 5.7]) If $\mathcal{C}$ is finitely accessible then $\mathcal{C}$ has cokernels iff $\mathcal{C}^{\mathrm{fp}}$ has cokernels iff $\mathcal{C}$ is locally finitely presented.*

In that case the embedding $\mathcal{C}^{\mathrm{fp}} \longrightarrow \mathcal{C}$ is right exact, $\varinjlim$ is right exact on $\mathcal{C}$ and every exact sequence $B \to C \to D \to 0$ in $\mathcal{C}$ is a direct limit of exact such sequences in $\mathcal{C}^{\mathrm{fp}}$.

In particular, a locally coherent additive category $\mathcal{C}$ is locally finitely presented.

flateqlex

Corollary 3.8. *([24, 2.4, 2.9], also [71, 3.4], [19, p. 1646]) If $\mathcal{A}$ is skeletally small preadditive and $\mathcal{A}^+$, equivalently $\mathcal{A}^{++}$, has cokernels then* Flat-$\mathcal{A} \simeq \mathrm{Lex}((\mathcal{A}^+)^{\mathrm{op}}, \mathbf{Ab}) \simeq \mathrm{Lex}((\mathcal{A}^{++})^{\mathrm{op}}, \mathbf{Ab})$*, the category of left exact functors from $(\mathcal{A}^+)^{\mathrm{op}}$, respectively $(\mathcal{A}^{++})^{\mathrm{op}}$, to* $\mathbf{Ab}$.

flateqlex2

Corollary 3.9. *If $\mathcal{C}$ is a locally finitely presented additive category then $\mathcal{C} \simeq \mathrm{Lex}((\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab})$.*

Every locally finitely presented category is complete, indeed a finitely accessible category is complete iff it is cocomplete, [1, 2.47]. Therefore such a category $\mathcal{C}$ has products. For that, however, pseudocokernels suffice: a **pseudocokernel**, also called a **weak cokernel**, of $f : A \to B$ is a morphism $g : B \to C$ with $gf = 0$ such that for every $h : B \to D$ with $hf = 0$ there is at least one $k : C \to D$ with $kg = h$ (for a cokernel we would insist on exactly one).

addpscok

Theorem 3.10. *([16, 2.2, 2.13] for rings, [71, §4], [19, 2.1]) Let $\mathcal{A}$ be skeletally small preadditive. Then the following are equivalent:*
(i) $\mathcal{A}^+$ has pseudocokernels;
(ii) $\mathcal{A}^{++}$ has pseudocokernels;
(iii) Flat-$\mathcal{A}$ *has products;*
(iv) Flat-$\mathcal{A}$ *is closed in* Mod-$\mathcal{A}$ *under products.*

accpscok

Corollary 3.11. *An additive category $\mathcal{C}$ which is finitely accessible has products iff $\mathcal{C}^{\mathrm{fp}}$ has pseudocokernels.*

Crawley-Boevey says [19, 2.3] that a category has the **weak factorisation property** if, given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ with $hg = gf = 0$, then there is E and there are morphisms $B \xrightarrow{k} E$, $E \xrightarrow{l} C$ with $lk = g$ and $kf = 0 = hl$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[rr, "g"] \arrow[dr, dashed, "k"'] & & C \arrow[r, "h"] & D \\ & & E \arrow[ur, dashed, "l"'] & & \end{tikzcd}$$

addker

Theorem 3.12. *([19, 2.3]) Let $\mathcal{A}$ be a skeletally small preadditive category. Then the following are equivalent:*
(i) $\mathcal{A}$ (equivalently $\mathcal{A}^+$, equivalently $\mathcal{A}^{++}$) has the weak factorisation property;
(ii) $\mathrm{Tor}_n^{\mathcal{A}^+}(-,-) = 0$ for all $n \geq 2$;
(iii) Flat-$\mathcal{A}$ *has kernels;*
(iv) Flat-$\mathcal{A}$ *is closed in* Mod-$\mathcal{A}$ *under kernels.*

accker

Corollary 3.13. *([19, 2.3], [54, 5.8]) An additive finitely accessible category $\mathcal{C}$ has kernels iff $\mathcal{C}^{\mathrm{fp}}$ has the weak factorisation property.*

In that case the embedding $\mathcal{C}^{\mathrm{fp}} \longrightarrow \mathcal{C}$ is left exact, $\varinjlim$ is left exact on $\mathcal{C}$ and every exact sequence $0 \to B \to C \to D$ in $\mathcal{C}$ is a direct limit of such exact sequences in $\mathcal{C}^{\mathrm{fp}}$.

limfpseq

Corollary 3.14. *If $\mathcal{C}$ is a locally coherent additive category (so, 3.5, $\mathcal{C}^{\mathrm{fp}}$ is abelian) then the embedding $\mathcal{C}^{\mathrm{fp}} \longrightarrow \mathcal{C}$ is exact, $\varinjlim$ is exact on $\mathcal{C}$ and every exact sequence $0 \to B \to C \to D \to 0$ in $\mathcal{C}$ is a direct limit of such sequences in $\mathcal{C}^{\mathrm{fp}}$.*

Exactness of $\varinjlim$ (hence, since there is a generating set, the Grothendieck property) holds more generally in any abelian finitely accessible additive category.

fpabgroth

Theorem 3.15. *([19, 2.4]) For an additive category $\mathcal{C}$ the following are equivalent:*
(i) $\mathcal{C}$ is finitely accessible and abelian;
(ii) $\mathcal{C}$ is locally finitely presented and abelian;
(iii) $\mathcal{C}$ is locally finitely presented Grothendieck.

extfab

Example 3.16. An example of a locally finitely presented additive but not abelian category is the category, $\mathcal{F}$, of torsionfree abelian groups. This is the finitely accessible category corresponding in the sense of 3.4 to $\mathcal{A} = \mathbb{Z}$. The finitely presented objects are just the free groups of finite rank and the cokernel of a morphism is computed by taking the cokernel as an abelian group and factoring out its torsion subgroup. The category is not abelian since not every monomorphism is a kernel (e.g. consider the inclusion of $2\mathbb{Z}$ in $\mathbb{Z}$).

Example 3.17. Let M be an R-module, more generally let $M \in$ Mod-$\mathcal{A}$ for some small preadditive category $\mathcal{A}$. Denote by $\sigma[M]$ the subcategory of Mod-$\mathcal{A}$ generated by M under subobjects, quotient objects and direct limits, see [103, §15]. Note that $\sigma[M]$ has products: if $(M_\lambda)_\lambda$ is a collection of objects in $\sigma[M]$ then the product of these in $\sigma[M]$ is $T^M(\prod_\lambda M_\lambda)$ where $T^M(N)$ denotes the largest subobject of $N \in$ Mod-$\mathcal{A}$ which is in $\sigma[M]$ (see [103, 15.1(6)]). Note that, by 3.7, a category of the form $\sigma[M]$ is finitely accessible iff it is locally finitely presented. The question of when this is so is addressed in [88]. sigM

For example, if k is a field and C is a coalgebra then the category of C-comodules has the form $\sigma[M]$ for a suitable C^*-module M, where C^* is the dual algebra of C, and this category is locally finitely presented (e.g. see [88, after 1.4]). More generally this holds if C is an R-coalgebra where R is right noetherian and C_R is projective, see [104].

4 The free abelian category of a preadditive category

secfreeab

Let $\mathcal{A}$ be a skeletally small preadditive category.

A **free abelian category** on $\mathcal{A}$ is a functor $\mathcal{A} \longrightarrow \mathrm{Ab}(\mathcal{A})$ where $\mathrm{Ab}(\mathcal{A})$ is abelian and has the universal property that for every additive functor $\mathcal{A} \longrightarrow \mathcal{B}$ where $\mathcal{B}$ is abelian there is an extension to an exact functor from $\mathrm{Ab}(\mathcal{A})$ to $\mathcal{B}$ and there is, up to natural equivalence, just one such exact functor.

$$\begin{tikzcd} \mathcal{A} \arrow[rr] \arrow[dr, "\forall"'] & & \mathrm{Ab}(\mathcal{A}) \arrow[dl, dashed, "\exists!\text{exact}"] \\ & \mathcal{B} & \end{tikzcd}$$

Clearly, if the free abelian category on $\mathcal{A}$ exists, it is unique up to natural equivalence. Existence follows from a very general result of Freyd. An **exact category** is a preadditive category $\mathcal{A}$ with a distinguished class, $\mathcal{E}$, of pairs of composable maps satisfying certain closure conditions which we don't need to go into here: equivalently (cf. 4.1 below), they are the extension-closed full subcategories of abelian categories with, as additional data, the set of sequences which are short exact in the containing abelian category. A functor $F : (\mathcal{A}, \mathcal{E}) \longrightarrow (\mathcal{A}', \mathcal{E}')$ between exact categories is **exact** if for every $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{E}$ the image, $FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$, is in $\mathcal{E}'$. If $\mathcal{A}$ is abelian and no exact structure is specified then it is to be understood that it has the natural structure of an exact category, that is, $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{E}$ iff $\ker(g) = \mathrm{im}(f)$.

abAexact

Theorem 4.1. *([26, 4.1]) Let $(\mathcal{A}, \mathcal{E})$ be a skeletally small preadditive exact category. Then there is an abelian category $\mathrm{Ab}(\mathcal{A}, \mathcal{E})$, and exact functor $(\mathcal{A}, \mathcal{E}) \longrightarrow \mathrm{Ab}(\mathcal{A}, \mathcal{E})$ (i.e. to $(\mathrm{Ab}(\mathcal{A}, \mathcal{E}), \mathcal{E}_{\mathrm{natural}})$) such that for every exact functor $(\mathcal{A}, \mathcal{E}) \longrightarrow \mathcal{B}$ with $\mathcal{B}$ abelian there is an extension to an exact functor, unique up to natural equivalence, from $\mathrm{Ab}(\mathcal{A}, \mathcal{E})$ to $\mathcal{B}$.*

$$\begin{tikzcd}
(\mathcal{A},\mathcal{E}) \arrow[rr] \arrow[dr, "\forall"'] & & \mathrm{Ab}(\mathcal{A},\mathcal{E}) \arrow[dl, dashed, "\exists!\text{exact}"] \\
& \mathcal{B} &
\end{tikzcd}$$

Taking $\mathcal{E} = \emptyset$ one obtains the existence of free abelian categories.

abA

Corollary 4.2. *Let $\mathcal{A}$ be a skeletally small preadditive category. Then the free abelian category on $\mathcal{A}$ exists: there is an abelian category* $\mathrm{Ab}(\mathcal{A})$ *and an additive functor $\mathcal{A} \longrightarrow \mathrm{Ab}(\mathcal{A})$ such that for every additive functor $\mathcal{A} \longrightarrow \mathcal{B}$ with $\mathcal{B}$ abelian there is a unique-to-natural equivalence exact extension* $\mathrm{Ab}(\mathcal{A}) \longrightarrow \mathcal{B}$ *through $\mathcal{A} \longrightarrow \mathrm{Ab}(\mathcal{A})$.*

The proof of the general result in [26] is quite long but the corollary can be obtained more directly once it is realised that $\mathrm{Ab}(\mathcal{A})$ is $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. This is stated (for $\mathcal{A}$ a ring) in [35] and there one is referred to [26] to extract a proof. Such a proof is given in [54, 2.10] (for rings in [56, 1.2]). Also see Adelman's direct construction [2]. We will see later that $\mathrm{Ab}(\mathcal{A})$ is the category of pp-imaginaries for *left* $\mathcal{A}$-modules (and that is the opposite of the category of pp-imaginaries for right $\mathcal{A}$-modules). That observation is due to Herzog (I learned it from him in a coffee shop in Bielefeld, mid 90s).

freeabcat

Theorem 4.3. *([35, Lemma 1] for $\mathcal{A}$ a ring) Let $\mathcal{A}$ be a small preadditive category. Then $Y^2 : \mathcal{A} \longrightarrow (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$, $A \mapsto ((A,-),-)$ is the free abelian category on $\mathcal{A}$. That is, in the notation which we will introduce below,* $\mathrm{Ab}(\mathcal{A}) \simeq \mathrm{fun}^{\mathrm{d}}\text{-}\mathcal{A}$.

Proof. Any additive functor from $\mathcal{A}$ to an abelian category $\mathcal{B}$ has a unique extension to an additive functor from $\mathcal{A}^{++}$ to $\mathcal{B}$ (see Section 2) so we may assume that $\mathcal{A} = \mathcal{A}^{++}$

An additive functor $f : \mathcal{A} \longrightarrow \mathcal{B}$ where $\mathcal{B}$ is abelian is extended to a left exact functor, f', from $(\mathcal{A}\text{-mod})^{\mathrm{op}}$, that is, to a right exact functor from $\mathcal{A}$-mod, by sending $M \in \mathcal{A}\text{-mod}$ to $\ker(fA \xrightarrow{f\alpha} fA_1)$ where $A \xrightarrow{\alpha} A_1$ in $\mathcal{A}$ is such that $(A_1,-) \xrightarrow{(\alpha,-)} (A,-) \to M \to 0$ is exact in $\mathcal{A}$-mod. One checks independence of presentation and extends this in the obvious way to an action on morphisms, checking that this also is well-defined.

Then f' is extended to a right exact and, one checks, exact, in addition to well-defined, functor, f'', from $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ to $\mathcal{B}$ by sending $F \in (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ to $\mathrm{coker}(f'M^{\mathrm{o}} \xrightarrow{f'g^{\mathrm{o}}} f'L^{\mathrm{o}})$ where $L \xrightarrow{g} M$ in $\mathcal{A}$-mod is such that $(M,-) \xrightarrow{(g,-)} (L,-) \to F \to 0$ is exact in $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ (superscript $^{\mathrm{o}}$ indicates objects and morphisms in the opposite category).

See the diagram after 4.8. $\square$

If $\mathcal{A}$ is a skeletally small preadditive category we will, following (and adapting) notation of Benson [8], set $\mathrm{Fun}\text{-}\mathcal{A} = (\mathrm{mod}\text{-}\mathcal{A}, \mathbf{Ab})$ and $\mathrm{fun}\text{-}\mathcal{A} = (\mathrm{Fun}\text{-}\mathcal{A})^{\mathrm{fp}} = (\mathrm{mod}\text{-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$. For reasons which shortly will become clear we also set $\mathrm{Fun}^{\mathrm{d}}\text{-}\mathcal{A} = (\mathcal{A}\text{-mod}, \mathbf{Ab})$ and $\mathrm{fun}^{\mathrm{d}}\text{-}\mathcal{A} = (\mathrm{Fun}^{\mathrm{d}}\text{-}\mathcal{A})^{\mathrm{fp}} = (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. So $\mathrm{Ab}(\mathcal{A}) = \mathrm{fun}^{\mathrm{d}}\text{-}\mathcal{A}$.

Often we will write the notation for this category out in full but it is useful to have the shorter notation. Gruson and Jensen used $D(\mathcal{A})$ and $C(\mathcal{A})$ for what we have written as $\mathrm{Fun}^{\mathrm{d}}$-$\mathcal{A}$ and $\mathrm{fun}^{\mathrm{d}}$-$\mathcal{A}$ respectively and many authors have followed them but we feel that such an important object deserves a more distinctive notation and Benson's is well-chosen. Informally, any of these will be referred to as "the functor category of $\mathcal{A}$", with "finitely presented" and/or "dual" being added as necessary.

Example 4.4. Let R be a finite-dimensional algebra over a field k. Suppose that R is of **finite representation type** (i.e. there are only finitely many indecomposable R-modules up to isomorphism). The **Auslander algebra** of R is $\mathrm{Aus}(R) = \mathrm{End}\big(\bigoplus_{i=1}^{k} N_i\big)$ where $N_1, \ldots, N_k$ are the distinct indecomposable R-modules. If $F \in \mathrm{Fun}$-R then there is a natural left action, *via* F, of $\mathrm{Aus}(R)$ on $\bigoplus_i FN_i$ which, in fact (see, e.g. [8, §4.9] for a brief account), gives an equivalence Fun-$R \simeq \mathrm{Aus}(R)$-Mod. This is therefore a source of many examples where Fun-R , hence fun-R, may be computed explicitly.

Replacing $\mathcal{A}$ by $\mathcal{A}^{\mathrm{op}}$ in 4.2 and 4.3 one obtains statements for contravariant functors from $\mathcal{A}$, so $\mathrm{Ab}(\mathcal{A}^{\mathrm{op}}) = (\mathcal{A}^{\mathrm{op}}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} = (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$ and since, clearly (by the universal property), $\mathrm{Ab}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ab}(\mathcal{A})^{\mathrm{op}}$, one deduces the existence (though not immediately the description) of a duality $((\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}} \simeq (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. The duality, due to Gruson and Jensen [38] and independently to Auslander [6], between fun-$\mathcal{A}$ and $\mathrm{fun}^{\mathrm{d}}$-$\mathcal{A}$ is described explicitly as follows. If $F \in (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$ then $dF \in (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ is defined on objects $L \in \mathcal{A}$-mod by

$$dF(L) = (F, - \otimes_{\mathcal{A}} L)$$

and the definition of d on morphisms is what it must be. We record the existence of this duality for later reference.

fundual

Theorem 4.5. *([6], [38, 5.6]) Let $\mathcal{A}$ be a skeletally small preadditive category. Then d is a duality:* $\mathrm{fun}^{\mathrm{d}}\text{-}\mathcal{A} \simeq (\text{fun-}\mathcal{A})^{\mathrm{op}}$.

For example if $M \in$ mod-$\mathcal{A}$ then, [4, 6.1], both $(M, -)$ and $M \otimes -$ are finitely presented objects of Fun-$\mathcal{A}$, respectively $\mathrm{Fun}^{\mathrm{d}}$-$\mathcal{A}$ and each is the dual of the other. For instance, $d(M, -) \cdot L = ((M, -), - \otimes_{\mathcal{A}} L) \simeq M \otimes_{\mathcal{A}} L$ (by Yoneda) $\simeq (M \otimes_{\mathcal{A}} -) \cdot L$. In particular $d((-, A), -) = (-, A) \otimes_{\mathcal{A}} - = (-)(A)$ is evaluation of a left module at A for $A \in \mathcal{A}$, by definition of $\otimes_{\mathcal{A}}$ plus the fact, e.g. [25, pp. 86, 87], that the definition we gave is "symmetric".

This duality extends that between $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ and, as pointed out in [54, 2.11], it follows immediately from the universal property of the free abelian category that d is the unique-to-natural-equivalence such extension to a duality between fun-$\mathcal{A}$ and $\mathrm{fun}^{\mathrm{d}}$-$\mathcal{A}$. This same duality was also found model-theoretically [76], [40], though in a very different form (in terms of pp formulas and imaginaries, see Section 23). When it was realised that this model-theoretic duality was equivalent to a duality between the functor categories Auslander remarked (in a discussion over coffee, Trondheim, early 90s) that there could not be two different such natural dualities and, indeed, it is easy to check directly that they are the same.

The formula below turns out to be extremely useful; it is formulated model-theoretically at 23.3 and is extended at 23.4.

dualhomfmlafun

Theorem 4.6. *(see [78, p. 193, Thm.]) Suppose that $\mathcal{A}$ is a small preadditive category, let $F \in (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\text{fp}}$ and let $\overrightarrow{F}$ denote the extension of F to a functor on* Mod-$\mathcal{A}$ *which commutes with direct limits. Let $M \in \text{Mod-}\mathcal{A}$. Then $\overrightarrow{F}M \simeq (dF, M \otimes_{\mathcal{A}} -)$.*

abAex1

Example 4.7. A left $\mathcal{A}$-module L is a functor $L : \mathcal{A} \longrightarrow \mathbf{Ab}$ so there is a unique exact extension, $\text{ev}_L : \text{Ab}(\mathcal{A}) \longrightarrow \mathbf{Ab}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[dr, "L"'] & \text{Ab}(\mathcal{A}) \arrow[d, "\text{ev}_L"] \\ & \mathbf{Ab} \end{tikzcd}$$

It is easy to check (and this explains the notation) that, regarding $\text{Ab}(\mathcal{A})$ as $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\text{fp}}$, ev_L is the functor "evaluation at L" which takes $F \in (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\text{fp}}$ to $\overrightarrow{F}L$. For, if $K \in \mathcal{A}$-mod and if $f : A_1 \longrightarrow A_2$ in $\mathcal{A}$ is such that $(A_2, -) \xrightarrow{(f,-)} (A_1, -) \longrightarrow K \longrightarrow 0$ is exact then, by definition, the extension, L', of L to $(\mathcal{A}\text{-mod})^{\text{op}}$ is given by exactness of $0 \longrightarrow L'K \longrightarrow LA_1 \xrightarrow{Lf} LA_2$. But also $0 \longrightarrow (K, L) \longrightarrow ((A_1, -), L) \simeq LA_1 \xrightarrow{((f,-),L) \simeq Lf} ((A_2, -), L) \simeq LA_2$ is exact, so $L'K \simeq (K, L)$. Then, if $F \in (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\text{fp}}$ and $g : K \longrightarrow K'$ in $\mathcal{A}$-mod is such that $(K', -) \xrightarrow{(g,-)} (K, -) \longrightarrow F \longrightarrow 0$ is exact, we have that $L'K' \xrightarrow{L'g} L'K \longrightarrow \text{ev}_L F \longrightarrow 0$, that is, $(K', L) \longrightarrow (K, L) \longrightarrow \text{ev}_L F \longrightarrow 0$, is exact. So, since, as is easily seen, $(K', -) \longrightarrow (K, -) \longrightarrow \overrightarrow{F} \longrightarrow 0$ is an exact sequence of functors in $(\mathcal{A}\text{-Mod}, \mathbf{Ab})$, we do have $\text{ev}_L F = \overrightarrow{F}$.

If, instead, we were to view $\text{Ab}(\mathcal{A})$ as the model-theoretic imaginary category $\mathbb{L}^{\text{eq+}}_{\mathcal{A}^{\text{op}}}$ (22.1) then it would be natural to write $L^{\text{eq+}}$ in place of ev_L. In fact we may take this as a (non-model-theoretic) definition of $L^{\text{eq+}}$ (the usual model-theoretic definition is at the end of Section 22).

modeqex

Theorem 4.8. *The association $L (\in \mathcal{A}\text{-Mod}) \mapsto \text{ev}_L (= L^{\text{eq+}}) \in \text{Ex}(\text{Ab}(\mathcal{A}), \mathbf{Ab})$ extends to a functor which is an equivalence $\mathcal{A}\text{-Mod} \simeq \text{Ex}(\text{Ab}(\mathcal{A}), \mathbf{Ab})$.*
(This could be (monstrously) named the model-theoretic "imaginarification functor".)

This will be proved, and extended, at 10.8.

For module categories (and definable subcategories of them) the above result was first shown to me by Herzog (at a conference in Notre Dame, early 90s). The general case was developed by Krause in [53], [54]. The result is also contained, though not obviously, in the work of Makkai and Hu, [65, 5.1], [45, 5.10(ii)].

The canonical functor, Y^2, from $\mathcal{A}$ to $\text{Ab}(\mathcal{A})$ factors through $(\mathcal{A}\text{-mod})^{\text{op}}$ as the composition of two Yoneda functors so one always has (as in the proof of 4.3), given $\mathcal{A} \longrightarrow \mathcal{B}$ with $\mathcal{B}$ abelian, the following diagram, where f'' is exact and f' is the restriction of f''.

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[dr, "f"'] & (\mathcal{A}\text{-mod})^{\mathrm{op}} \arrow[r, "Y^{\mathrm{o}}"] \arrow[d, "f'"] & \mathrm{Ab}(\mathcal{A}) = (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \arrow[dl, "f''"] \\
& \mathcal{B} &
\end{tikzcd}$$

Note that $\mathcal{A}$-mod has cokernels and Y takes right exact sequences in $\mathcal{A}$-mod to left exact sequences in $\mathrm{Ab}(\mathcal{A})$, hence Y^{o} is left exact and, therefore, f' is left exact. abAex2

Example 4.9. Consider the diagram shown, where i is the Yoneda map $A \mapsto (-, A)$ and i', i'' are corresponding functors as above.

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[dr, "i"'] & (\mathcal{A}\text{-mod})^{\mathrm{op}} \arrow[r, "Y^{\mathrm{o}}"] \arrow[d, "i'"] & \mathrm{Ab}(\mathcal{A}) = (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \arrow[dl, "i''"] \\
& \mathrm{Mod}\text{-}\mathcal{A} &
\end{tikzcd}$$

If mod-$\mathcal{A}$ is abelian (but this is not always the case, see 6.3) then it may replace Mod-$\mathcal{A}$ in the diagram.

Let us go through this (essentially the proof of 4.3) in the case that the initial category $\mathcal{A}$ is a ring R.

First we define $i' : (R\text{-mod})^{\mathrm{op}} \longrightarrow \mathrm{Mod}\text{-}R$. Let $L \in R$-mod: there is an exact sequence $R^m \to R^n \to L \to 0$ in R-mod where, in this sequence, "R" means the projective left module $(R, -) = {}_RR$. The functor $(-, {}_RR)$ takes right exact sequences in R-mod to left exact sequences in Mod-R and agrees with i' on the projective generator ${}_RR$ ($i'({}_RR) = i(R) = R_R = ({}_RR, {}_RR)$). Hence i' is the functor which on objects is $L \mapsto L^\star = (L, R)$. (Regarding the case where mod-R is abelian, since the matrix of the map from R^m to R^n is arbitrary, so is the kernel of the transposed map, therefore, by [100, I.13.3] for example, $L^\star$ is finitely presented for all L iff R is right coherent.)

Now for i''. Take $F \in \mathrm{Ab}(R)$. Then there is a morphism $g : K \to L$ in R-mod and an exact sequence $(L, -) \xrightarrow{(g,-)} (K, -) \to F \to 0$ in $\mathrm{Ab}(R)$. Since i'' is exact this gives the exact sequence $L^\star \xrightarrow{g^\star} K^\star \to i''F \to 0$ in Mod-R and so we deduce that, if $F = \mathrm{coker}(g, -)$, then $i''F = \mathrm{coker}(g^\star)$

In the general case if $f : A_1 \longrightarrow A_2$ in $\mathcal{A}(= \mathcal{A}^{++})$ is such that $(A_2, -) \xrightarrow{(f,-)} (A_1, -) \longrightarrow L \longrightarrow 0$ is exact then, applying $(-, (A, -))$ to this sequence, we see that i' is the functor which takes $L \in \mathcal{A}$-mod to the functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathbf{Ab}$ which takes $A \in \mathcal{A}$ to $(L, (A, -))$ so if, as is reasonable, we denote the latter functor by $L^\star$, then the description of i'' is as above. abAex3

Example 4.10. The functor $i : \mathcal{A} \longrightarrow ((\mathrm{mod}\text{-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})$ given on objects by $A \mapsto (-, (-, A))$ induces the outer part of the next diagram, with the interior parts coming from the previous example, where the functor $\mathrm{Mod}\text{-}\mathcal{A} \longrightarrow ((\mathrm{mod}\text{-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})$ is given by $M \mapsto (-, M) \restriction \mathrm{mod}\text{-}\mathcal{A}$.

$$\begin{tikzcd}
 & & (\mathcal{A}\text{-mod})^{\mathrm{op}} \arrow[dl] \arrow[dd, "i'"] \arrow[dr, "Y^{\circ}"] & \\
\mathcal{A} \arrow[urr, "Y^{\circ}"] \arrow[r] \arrow[drr, "i"'] & \text{Mod-}\mathcal{A} \arrow[dr] & & \mathrm{Ab}(\mathcal{A}) = (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \arrow[dl, "i''"] \\
 & & ((\text{mod-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab}) &
\end{tikzcd}$$

That functor is left exact, as are the other two in the interior of the left-pointing triangle so, since there is agreement on representables, we have commutativity, hence $i'L = (-, L^\star)$ (restricted to mod-$\mathcal{A}$) for $L \in \mathcal{A}$-mod. Next, if $g : K \to L$ in $\mathcal{A}$-mod is such that $(L, -) \xrightarrow{(g,-)} (K, -) \to F \to 0$ is exact in $\mathrm{Ab}(\mathcal{A})$, then $i''F = \mathrm{coker}((-, L^\star) \xrightarrow{(-,g^\star)} (-, K^\star))$.

If Mod-$\mathcal{A}$ is locally coherent then $(\text{mod-}\mathcal{A})^{(\mathrm{op})}$ is abelian therefore, (see 6.3 below), the category $((\text{mod-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})$ is locally coherent. As seen in 4.9, Mod-$\mathcal{A}$ may, in this case, be replaced by mod-$\mathcal{A}$. It also follows that $((\text{mod-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})^{\mathrm{fp}}$ is abelian and so, in this case, the latter category can replace $((\text{mod-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})$ in the diagram. It is also easy to see directly that $(-, L^\star)$ is in this case a finitely presented functor.

Thus, by 6.3, if $\mathcal{A}^+$, or $\mathcal{A}^{++}$, has pseudocokernels then categories of finitely presented objects may be used throughout the diagram.

Even if $((\text{mod-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})^{\mathrm{fp}}$, equivalently mod-$\mathcal{A}$, is abelian the functor i'' will not in general induce an equivalence with $((\text{mod-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})^{\mathrm{fp}}$. For instance, if $L \in \mathcal{A}$-mod is non-zero but is such that $L^\star = 0$ then $i'L = (-, L^\star) = 0$, so i'' takes the image, $(L, -) \neq 0$, of L in $\mathrm{Ab}(\mathcal{A})$ to 0. In particular, even if $(\text{mod-}\mathcal{A})^{\mathrm{op}} \simeq \mathcal{A}$-mod, so $((\text{mod-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})^{\mathrm{fp}} \simeq (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$, this equivalence is not i''. For example, if $\mathcal{A} = \mathbb{Z}$ then i'' takes $({}_{\mathbb{Z}}\mathbb{Z}, -)$ to $(-, \mathbb{Z}_{\mathbb{Z}})$.

For more on the comparison of the functor categories $(R\text{-mod}, \mathbf{Ab})$ and $((\text{mod-}R)^{\mathrm{op}}, \mathbf{Ab})$, see [43]. abAex4

Example 4.11. By 6.3 the category $\mathcal{A}^{++}$ has pseudocokernels iff $\mathcal{A}$-Mod is locally coherent iff $\mathcal{A}$-mod, hence $(\mathcal{A}\text{-mod})^{\mathrm{op}}$, is abelian. In that case the Yoneda inclusion $(\mathcal{A}\text{-mod})^{\mathrm{op}} \longrightarrow (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ has an exact right inverse $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \longrightarrow (\mathcal{A}\text{-mod})^{\mathrm{op}}$, described as follows. If $F \in (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ let $g : K \to L$ be such that $(L, -) \xrightarrow{(g,-)} (K, -) \to F \to 0$ is exact. With notation as in 4.9, this sequence is taken by i'' to the exact sequence $i'L^{\circ} \xrightarrow{i'g^{\circ}} i'K^{\circ} \longrightarrow i''F \longrightarrow 0$, that is, to the image under i' of the exact sequence $0 \longrightarrow \ker(g) \longrightarrow K \xrightarrow{f} L$ in $\mathcal{A}$-mod ($\ker(g)$ is finitely presented since $\mathcal{A}$-Mod is locally coherent). So define the inverse by sending F to $\ker(g)$ (cf. comments after 3.6).

We add an observation which becomes relevant in Section 25. abAsmallest

Lemma 4.12. *Let $\mathcal{A}_0$ be a small preadditive category and consider the embedding $i : \mathcal{A}_0 \longrightarrow \mathcal{A} = \mathrm{Ab}(\mathcal{A}_0)$ into its free abelian category. Suppose that $\mathcal{B}$ is* *an abelian subcategory (in particular an exact subcategory) of $\mathcal{A}$ which contains $\mathcal{A}_0$. Then $\mathcal{B} = \mathcal{A}$ (up to equivalence).*

Proof. Let $j : \mathcal{A}_0 \longrightarrow \mathcal{B}$ and $j' : \mathcal{B} \longrightarrow \mathcal{A}$ be the inclusions, so $i = j'j$. By definition of free abelian category there is an exact functor $f : \mathcal{A} \longrightarrow \mathcal{B}$ such that $j = fi$. Set $\mathcal{B}_0$ to be the image of f. This, being the image of $\mathcal{A}$ by an exact functor, is an abelian category. Any exact sequence in $\mathcal{B}_0$ is the image of an exact sequence in $\mathcal{A}$ (e.g. [72, 4.3.10]) so, since f is exact, the first sequence is exact as a sequence of $\mathcal{B}$. So $\mathcal{B}_0$ is an abelian subcategory of $\mathcal{A}$ and hence, without loss of generality, we may suppose that $\mathcal{B}$ is the image of f. We show that $\mathcal{B}$ has the properties of the free abelian category on $\mathcal{A}_0$.

Suppose that $k : \mathcal{A}_0 \longrightarrow \mathcal{B}'$ is a functor to an abelian category.

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[r, "j"] \arrow[dr, "k"'] & \mathcal{B} \arrow[d, "h"] \arrow[r, bend right, "j'"'] & \mathcal{A} \arrow[l, bend right, "f"'] \arrow[dl, "g"] \\
& \mathcal{B}' &
\end{tikzcd}$$

Then there is an exact functor g as shown with $gj'j = k$. So gj' is an exact functor from $\mathcal{B}$ to $\mathcal{B}'$ making the left-hand triangle commute. Suppose that $h : \mathcal{B} \longrightarrow \mathcal{B}'$ is an exact functor with $hj = k$; we must show that h is naturally equivalent to gj'. Now $hfi = hj = k$ so uniqueness of g gives hf naturally equivalent to g. Also $gj'fi = gj'j = gi = k$ so $gj'f$ also is naturally equivalent to g and hence to hf. Since $\mathcal{B}$ is the image of f it follows that gj' is naturally equivalent to h, as claimed. Thus $j : \mathcal{A}_0 \longrightarrow \mathcal{B}$ also is the free abelian category of $\mathcal{A}_0$ so it follows that j' and f are equivalences, as required. □

5 Purity

secpur

The notion of purity was introduced by Prüfer [89] for abelian groups and by Cohn [17] for modules over rings. Cohn's definition is that seen in condition (vii) of 5.2 and he showed that this is equivalent to the formulation, (vi), in terms of solutions of systems of linear equations. There are many equivalents in the context of modules and they do remain equivalent in locally finitely presented additive categories (in particular in functor categories, see [99]). They do, however, seem to diverge in finitely accessible additive categories so, in that context, we must choose which definition to use (we don't spend time here trying to sort out the relations between the various definitions in this generality).

We will start with the definition of pure monomorphism, saying that a morphism $f : L \longrightarrow M$ in a finitely accessible category is **pure** if it satisfies the condition (iii) of 5.2 below (this is the definition for general finitely accessible categories, see [1, p. 85]). It follows that f must be a monomorphism (see [1, 2.29] or 5.1 and the proof of 5.2). If there is an exact sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ with f a pure monomorphism then g is a **pure epimorphism** and the sequence is **pure-exact**.

This definition of pure monomorphism coincides with that in terms of solutions of systems of linear equations (i.e. the model-theoretic definition), see 18.3. Of course to justify that one must say what is meant by a system of linear equations in a finitely accessible category: for that see Section 18.

In [19, §3] Crawley-Boevey defines, in the context of finitely accessible categories, a composable pair of maps $L \xrightarrow{f} M \xrightarrow{g} N$ with $gf = 0$ to be pure-exact if for every finitely presented $A \in \mathcal{C}$ the induced sequence $0 \to (A, L) \to (A, M) \to (A, N) \to 0$ is exact (condition (i) of 5.2). We note that this does imply that f is monic, g is epi and the sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is "locally exact" in the following sense.

wkpes

Lemma 5.1. *Let $\mathcal{C}$ be finitely accessible and suppose that $L \xrightarrow{f} M \xrightarrow{g} N$ with $gf = 0$ is such that for every finitely presented $A \in \mathcal{C}$ the induced sequence $0 \to (A, L) \to (A, M) \to (A, N) \to 0$ is exact. Then f is monic and g is an epimorphism. Also, given $g' : M \longrightarrow N'$ such that $g'f = 0$ there is, for every $A \in \mathcal{C}^{\mathrm{fp}}$, a morphism $h_A : N \longrightarrow N'$ such that for every morphism $k' : A \longrightarrow N'$ there is a lift $k : A \longrightarrow N$ with $h_A k = k'$. There is a dual sense in which f is a "local pseudokernel" for g.*

Proof. If $L' \rightrightarrows L$ are distinct morphisms then, because L' is a direct limit of finitely presented objects, there is $A \in \mathcal{C}^{\mathrm{fp}}$ such that the compositions $A \longrightarrow L' \rightrightarrows L$ are distinct. Since $(A, f) : (A, L) \longrightarrow (A, M)$ is monic the postcompositions with f also are distinct, hence so are the compositions $L' \rightrightarrows L \xrightarrow{f} M$, as required.

A dual argument shows that g is epi.

The other statements follow directly from the facts that (A, g) is a cokernel for (A, f) and (A, f) is a kernel for (A, g). □

In the next result we assume that the category is locally finitely presented equivalently, by 3.7, $\mathcal{C}$, equivalently $\mathcal{C}^{\mathrm{fp}}$, has cokernels.

charpes

Theorem 5.2. *If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an exact sequence in the locally finitely presented additive category $\mathcal{C}$ then the following are equivalent:*
(i) for every finitely presented $A \in \mathcal{C}$ the induced sequence $0 \to (A, L) \to (A, M) \to (A, N) \to 0$ is exact;
(ii) for every $A \in \mathcal{C}^{\mathrm{fp}}$ and morphism $h : A \to N$ there is $k : A \to M$ such that $gk = h$;
(iii) for every morphism $h : A \to B$ in $\mathcal{C}^{\mathrm{fp}}$ and commutative diagram as shown

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "k'"] \arrow[dl, dashed] \\ L \arrow[r, "f"'] & M \end{tikzcd}$$

there is $l : B \to L$ with $lh = k$;
(iv) as (iii) but requiring of $h \in \mathcal{C}$ only that $\mathrm{coker}(h) \in \mathcal{C}^{\mathrm{fp}}$.
(v) the sequence is a direct limit of split exact sequences;

(vi) every finite system of linear equations with constants from L and a solution in M already has a solution in L.

If $\mathcal{C} \simeq$ Mod-$\mathcal{A}$ for some small preadditive category $\mathcal{A}$ then further equivalent conditions are the following:
(vii) for every $X \in \mathcal{A}$-Mod the sequence $0 \to L\otimes_{\mathcal{A}}X \to M\otimes_{\mathcal{A}}X \to N\otimes_{\mathcal{A}}X \to 0$ is exact;
(viii) as (vii) but for every $X \in \mathcal{A}$-mod.

If, further, $\mathcal{A}$ is a ring R then a further equivalent condition is:
(ix) if E is an injective cogenerator for Mod-R then the sequence $0 \to N^* \to M^* \to L^* \to 0$ of left R-modules is split exact where, for N any right R-module, $N^* = (N, E)$ is given the natural structure of a left R-module.

Proof. These equivalences, at some levels of generality, may be found variously in, for example, [1], [19], [100]. I include some, but not all, details.

Of course (ii) is a re-phrasing of (i). Now assume that (i) holds and that $C \in \mathcal{C}^{\mathrm{fp}}$ is a (pseudo)cokernel for h as in (iv), complete the diagram in (iv) and note that since $gk'h = 0$ there is p with $p\pi = gk'$ so, by assumption on C there is q with $p = gq$.

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[r, "\pi"] \arrow[d, "k'"] \arrow[dl, dashed, "l"'] & C \arrow[r] \arrow[d, "p"] \arrow[dl, "q"] & 0 \\
L \arrow[r, "f"'] & M \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

Then $g(q\pi - k') = 0$ so $q\pi - k'$ factors as $q\pi - k' = fl$ for some $l : B \longrightarrow L$. It follows that $flh = -fk$ so, since f is monic (by 5.1), $k = (-l)h$. This shows that (i) implies (iv) even in finitely accessible categories. Most of the other proofs, however, seem to require existence of some finite (co)limits (sometimes the weak/pseudo version suffices).

Let us also show that (iv) implies (i). So consider a morphism $p : C \longrightarrow N$: it must be shown that this factors through g. Since $\mathcal{C}$ is complete we can form the pullback exact sequence (e.g. [25, 2.52]).

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "1_L"'] & P \arrow[r, "\pi"] \arrow[d, "\pi'"] \arrow[dl, dashed, "l"'] & C \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"'] & M \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

By assumption there is $l : P \longrightarrow L$ with $li = 1_L$ (so the top sequence splits but let us continue to accommodate slightly weaker hypotheses). Therefore $\pi' i - fli = 0$ so, since C is a (pseudo)cokernel of i, there is $q : C \longrightarrow M$ with $q\pi = \pi' - fl$. We obtain $gq\pi = p\pi$ and hence, since π is epi, $g = pq$, as required.

The fact that (iii) implies (v) is true even for general finitely accessible (not necessarily additive) categories and a proof is given at [1, 2.30(ii)]. It is easily checked that a direct limit of pure-exact sequences is pure so the converse also is true. □

We will want to be able to refer to pure-exact sequences also in definable subcategories (§10) of finitely accessible categories with products. Such subcategories need not be finitely accessible so none of the above equivalents is directly usable as a definition. There is, however, one further equivalent which needs only ultraproducts hence, since these are certain direct limits of certain products, it is one which makes sense in any category with direct limits and products.

uppessplit

Theorem 5.3. *If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an exact sequence in the locally finitely presented additive category $\mathcal{C}$ then the following are equivalent:*
(i) the sequence is pure-exact;
(ii) some ultrapower of the sequence is split, that is, there is an index set I and an ultrafilter $\mathcal{F}$ on I such that $0 \to L^I/\mathcal{F} \xrightarrow{f^I/\mathcal{F}} M^I/\mathcal{F} \xrightarrow{g^I/\mathcal{F}} N^I/\mathcal{F} \to 0$ is split. (Both I and $\mathcal{F}$ may be chosen independently of the sequence and the cardinality of I may be bounded in terms of the number of arrows in a skeletal version of $\mathcal{C}^{\mathrm{fp}}$.)

This follows immediately from a standard result in model theory: see the comments after 21.3. It follows that a short exact sequence is pure-exact iff it is elementarily equivalent (in the category of short exact sequences from $\mathcal{C}$) to a split exact sequence.

Therefore, if $\mathcal{D}$ is a subcategory of a locally finitely presented category $\mathcal{C}$ with products and if $\mathcal{D}$ is closed in $\mathcal{C}$ under products and direct limits, condition (ii) of 5.3 will be used as a definition of purity in $\mathcal{D}$ and this depends only on the category structure of $\mathcal{D}$, that is, it is independent of the representation of $\mathcal{D}$ as such a subcategory. In particular, if a finitely accessible category $\mathcal{C}$ with products is embedded as a definable subcategory of a locally finitely presented category $\mathcal{C}'$ then the above "internal" definition of purity in $\mathcal{C}$ coincides with that induced by purity in $\mathcal{C}'$ (that is an embedding in $\mathcal{C}$ is pure in $\mathcal{C}$ iff it is pure when regarded as a morphism in $\mathcal{C}'$). One may check that this coincides with that corresponding to condition (iii), equivalently (vi), in 5.2.

An object of $\mathcal{C}$ is **pure-injective** (also called **algebraically compact**, see 21.2) if it is injective over pure embeddings. Again, there are many equivalents but they coincide in locally finitely presented additive categories.

charpi

Theorem 5.4. *For any object N of a locally finitely presented additive category, $\mathcal{C}$, with products the following are equivalent:*
(i) N is pure-injective: if $L \xrightarrow{i} M$ is a pure monomorphism and $f : L \to N$ is any morphism then there is $g : M \to N$ with $gi = f$;
(ii) if $N \to M$ is a pure monomorphism then it is split;
(iii) for any index set I the summation morphism $N^{(I)} \to N$ factors through the canonical embedding $N^{(I)} \to N^I$ (and it is enough to check for a large enough I as in 5.3).

$$\begin{tikzcd} N^{(I)} \arrow[r, "\mathrm{can}"] \arrow[d, "\Sigma"'] & N^I \arrow[dl, dashed, "\exists"] \\ N & \end{tikzcd}$$

If $\mathcal{C} = \text{Mod-}\mathcal{A}$ for some small preadditive category $\mathcal{A}$ then a further equivalent is:
(iv) the functor $(N \otimes_{\mathcal{A}} -) \in (\mathcal{A}\text{-mod}, \mathbf{Ab})$ is injective.

The proofs for this in [46, 7.1], given for categories of modules, easily adapt to this more general case. In connection with the last equivalent, see 5.12. We will refer to (iii) as the Jensen-Lenzing criterion for pure-injectivity.

There are yet more equivalents, which reflect the alternative terminology "algebraically compact" (see 21.2), in terms of solution of systems of (projections of) linear equations, hence which first require the setting up of a language for doing model theory in finitely accessible categories (§18).

In any functor category indeed, see 10.1, in any finitely accessible category with products and, more generally, in any definable category (for these, see later), there are minimal pure pure-injective extensions. That is, given an object M of a category $\mathcal{C}$ (of any of the above kinds), a **pure-injective hull** (or **pure-injective envelope**) of M, is a pure embedding $M \to N$ with N pure-injective and such that every pure monomorphism $M \to N'$ to a pure-injective object N' extends to a monomorphism (necessarily pure, e.g. [77, 4.14]) $N \to N'$. In particular, there is no direct summand of N strictly between M and N. In such categories every object does have a pure-injective hull (the term is often used just for the object, that is for the codomain, N), denoted $H(M)$ and this is unique to isomorphism (not necessarily unique) over M.

pihulls

Theorem 5.5. *(see, e.g., [77, p. 77], [46, 7.6] for a proof and for references) Let $\mathcal{A}$ be a small preadditive category. Then every object M of $\text{Mod-}\mathcal{A}$ has a pure-injective hull, $H(M)$. More generally, if $\mathcal{D}$ is any definable subcategory of a category of the form $\text{Mod-}\mathcal{A}$ and $M \in \mathcal{D}$ then the pure-injective hull of M in $\text{Mod-}\mathcal{A}$ is also a pure-injective hull of M in $\mathcal{D}$.*

This is most easily proved by pulling back existence of injective hulls in $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ to existence of pure-injective hulls in $\text{Mod-}\mathcal{A}$, see 5.12.

The statement for definable subcategories follows from that for functor categories since purity/pure-injectivity in the subcategory is just the restriction of purity/pure-injectivity in the larger category.

In fact, objects in finitely accessible additive categories (not necessarily with products) have pure-injective hulls ([42, Thm. 6]).

An object $M \in \mathcal{C}$ is **absolutely pure** if every monomorphism $M \to N \in \mathcal{C}$ with domain M is pure.

charabspur

Proposition 5.6. *(cf. [19, 2.5]) For an object M of a locally finitely presented additive category $\mathcal{C}$ the following are equivalent:*
(i) M is absolutely pure;
(ii) if $f : A \to B$ is a monomorphism of $\mathcal{C}$ with $\text{coker}(f)$ finitely presented then every morphism $g : A \to M$ factors through f;
(iii) if $f : A \to B$ is a monomorphism of $\mathcal{C}$ with B finitely presented and A finitely generated then every morphism $g : A \to M$ factors through f;
(iv) $\text{Ext}^1(C, M) = 0$ for $C \in \mathcal{C}^{\text{fp}}$.
(v) the pure-injective hull of M is an injective object of $\mathcal{C}$.

Proof. For (i)⇒(ii) form the pushout of f and g and then apply condition (iv) of 5.2. Condition (iv) is obtained by applying (ii) to each extension $0 \longrightarrow M \longrightarrow M' \longrightarrow C \longrightarrow 0$ with $g = 1_M$. For (iv)⇒(i) take an exact sequence $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ and, to show that it must be pure, take a morphism $h : C \longrightarrow M''$ with C finitely presented. Form the pullback exact sequence which, by assumption, must be split, so there is a lift of h to a morphism $C \longrightarrow M'$.

Assume that (iii) holds and let $f : M \longrightarrow N$ be monic. We check condition (iii) of 5.2 to show that it is pure. Since f is monic, A may be replaced by its image in B, which is finitely generated, then we apply the assumed property (iii) to deduce that f is pure.

Finally, if M is absolutely pure and $H(M) \longrightarrow M'$ is monic then so is $H(M) \longrightarrow H(M')$. By assumption the composite $M \longrightarrow H(M')$ is pure so, as stated in the definition of pure-injective hull, the morphism $H(M) \longrightarrow H(M')$ is pure, hence split. So $H(M) \longrightarrow M'$ is split. Conversely, if $H(M)$ is injective then, given a monomorphism $M \longrightarrow M'$, form the pushout.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & H(M) \arrow[d] \\ M' \arrow[r] & P \end{tikzcd}$$

By assumption $H(M) \longrightarrow P$ is split, in particular is pure, so the composition $M \longrightarrow P$ is pure. It follows (easily) that $M \longrightarrow M'$ is pure. □

The conditions (ii), (iii) and (iv), give rise to the equivalent (in locally finitely presented categories) terminology **fp-injective** for absolutely pure.

abspurex

Corollary 5.7. *If $\mathcal{C}$ is locally coherent then $M \in \mathcal{C}$ is absolutely pure iff $(-, M)$ is an exact functor on the abelian category $\mathcal{C}^{\mathrm{fp}}$.*

The next result was proved by Eklof and Sabbagh [23, 3.16] for module categories (they also proved the converse: if the class of absolutely pure modules is closed under direct limits then the module category is locally coherent).

cohpur

Proposition 5.8. *Suppose that $\mathcal{C}$ is a locally finitely presented category. Then the class of absolutely pure objects is closed under products and pure subobjects. If $\mathcal{C}$ is locally coherent then the class of absolutely pure objects is closed under direct limits, hence is a definable category (in the terminology of §10).*

Proof. Closure under products follows immediately by checking property (ii) of 5.6, as does the fact that a pure subobject, M_0, of an absolutely pure object, M, is absolutely pure (see the diagram, use (iv) of 5.2 to obtain h' from h).

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, dashed, "h"] \arrow[dl, dashed, "h'"] \\ M_0 \arrow[r, "i"'] & M \end{tikzcd}$$

Finally, if $M = \varinjlim_\lambda M_\lambda$ with the M_λ absolutely pure, hence with the $(-, M_\lambda)$ exact on $\mathcal{C}^{\mathrm{fp}}$ then, by exactness of direct limits in **Ab** and the definition of finitely presented, $(-, M)$ also is exact on $\mathcal{C}^{\mathrm{fp}}$ so, 5.7, M is absolutely pure. □

purinflat

Proposition 5.9. *Let $\mathcal{A}$ be a skeletally small preadditive category and suppose that the embedding $M \leq F \in$ Mod-$\mathcal{A}$ is pure. If F is flat then M is flat.*

Proof. We check the condition of 3.2(iii). Let $C \in$ mod-$\mathcal{A}$ and let $f : C \to M$. By 3.2 the composition if factors through a representable functor, say we have the diagram as shown where we have completed the embedding of M into F to a pure-exact sequence and where $D = \mathrm{coker}(g)$ (whence we obtain k).

$$\begin{tikzcd}
& C \arrow[r, "g"] \arrow[d, "f"'] & (-,A) \arrow[r, "q"] \arrow[d, "h"] & D = (-,A)/C \arrow[r] \arrow[d, "k"] \arrow[dl, dashed, "l"] & 0 \\
0 \arrow[r] & M \arrow[r, "i"'] & F \arrow[r, "p"'] & N \arrow[r] & 0
\end{tikzcd}$$

Since p is a pure epimorphism there is (5.2) $l : D \to F$ with $pl = k$. Consider $h - lq : (-, A) \to N$. We have $p(h - lq) = ph - kq = 0$ so $h - lq = il'$ for some $l' : (-, A) \longrightarrow M$. Also $il'g = (h - lq)g = hg = if$ so f factors through g, as required. □

endflatpur

Proposition 5.10. *([16, 2.2]) Suppose that $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence in* Mod-$\mathcal{A}$ *and suppose that N is flat. Then the sequence is pure-exact.*

Proof. Let $A \in \mathcal{C}^{\mathrm{fp}}$. By 3.2 any morphism from A to N factors through a representable, hence a projective, object and, therefore, lifts through $M \longrightarrow N$, as required. □

inclpur

Corollary 5.11. *Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in the locally finitely presented category $\mathcal{C}$. Then the image of this sequence under the (left-exact) Yoneda map $\mathcal{C} \longrightarrow$* Mod-$\mathcal{C}^{\mathrm{fp}}$ *is exact iff this image is pure-exact in* Mod-$\mathcal{C}^{\mathrm{fp}}$ *iff the original sequence is pure-exact in $\mathcal{C}$.*

Proof. By 5.2 exactness of the image of this sequence is precisely the condition that the original sequence be pure-exact. Since $(-, N)$ is flat exactness of the image sequence is, by 5.10, equivalent to its being pure-exact. □

Gruson and Jensen [36] defined a full embedding of any module category Mod-R into a locally coherent abelian category, what they denoted $D(R)$ but what we have denoted Fun$^{\mathrm{d}}$-R (§4), in such a way as to turn the theory around purity into the usual theory around injectivity: the injective objects of the larger category are, up to isomorphism, exactly the images of pure-injective R-modules, the absolutely pure objects of the larger category are the images of the R-modules and the image of an exact sequence in Mod-R is exact iff the original sequence is pure-exact. The construction, which on objects takes M to $(M \otimes_R -) \restriction R$-mod, works just as well if we start with a functor category

Mod-$\mathcal{A}$ rather than a module category over a ring. The original reference has few details but more are in [38], also see [46] and [84, §12.1.1].

pinjtoinj

Theorem 5.12. *Let $\mathcal{A}$ be a skeletally small abelian category and consider the functor ϵ : Mod-$\mathcal{A} \longrightarrow (\mathcal{A}\text{-mod}, \mathbf{Ab})$ which takes M to $M \otimes_{\mathcal{A}} -$ and which has the obvious action on morphisms. Then ϵ is a full embedding. An exact sequence in* Mod-$\mathcal{A}$ *is pure-exact iff its image is exact iff its image is pure-exact. Each functor $M \otimes -$ is absolutely pure and every absolutely pure object of $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ is isomorphic to a functor of this form. The functor $M \otimes_{\mathcal{A}} -$ is (indecomposable) injective iff M is (indecomposable) pure-injective.*

Since $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ is a locally coherent category (by 6.1), the image of this embedding, the subcategory of absolutely pure objects, is a definable subcategory (by 5.8), that is, it is closed under products, pure subobjects and direct limits.

Crawley-Boevey [19, 3.3] used a different embedding when dealing with finitely accessible categories $\mathcal{C}$ with products, defining $D(\mathcal{C})$ to be $(\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$-Flat $(= \mathrm{Lex}((\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}, \mathbf{Ab})$ if $\mathcal{C}^{\mathrm{fp}}$ has cokernels). This is, like $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ a locally coherent category (that follows from 6.1) and it is shown in [19, 3.3, Lemma 2] that there is an embedding of $\mathcal{C}$ into this which has the properties of the embedding discussed above. The absolutely pure objects of this category are precisely the exact functors [19, 3.3, Lemma 1], hence this representation of $\mathcal{C}$ is as $\mathrm{Ex}((\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}, \mathbf{Ab})$ that is, as $\mathrm{Ex}(\mathrm{fun}(\mathcal{C}), \mathbf{Ab})$. This is a representation of $\mathcal{C}$ as an exactly definable category (see §11) compared with the Gruson-Jensen representation of $\mathcal{C}$ as a definable subcategory.

6 Locally coherent categories

secloccoh

We expand and re-phrase 3.10. By a definable subcategory we mean one which is closed under direct limits, direct products and pure subobjects.

addaccprod

Theorem 6.1. *Let $\mathcal{A}$ be a skeletally small preadditive category and let $\mathcal{C}$ be a finitely accessible additive category. Then the following are equivalent.*

(a) (i) $\mathcal{A}^{+}$, equivalently $\mathcal{A}^{++}$, has pseudocokernels;
(ii) ("$\mathcal{C}$"=) Flat-$\mathcal{A}$ *has products;*
(iii) Flat-$\mathcal{A}$ *is closed under products in* Mod-$\mathcal{A}$*;*
(iv) $\mathcal{A}$-Mod *is locally coherent;*
(v) Flat-$\mathcal{A}$ *is a definable subcategory of* Mod-$\mathcal{A}$*;*
(vi) $\mathcal{A}$-Abs *is a definable subcategory of* $\mathcal{A}$-Mod.
(b) (i) ("$\mathcal{A}$"=)$\mathcal{C}^{\mathrm{fp}}$ has pseudocokernels;
(ii) $\mathcal{C}$ has products;
(iii) $\mathcal{C}$ is closed under products in Mod-$\mathcal{C}^{\mathrm{fp}}$*;*
(iv) $\mathcal{C}^{\mathrm{fp}}$-Mod *is locally coherent;*
(v) $\mathcal{C}$ is a definable subcategory of Mod-$\mathcal{C}^{\mathrm{fp}}$*;*
(vi) $\mathcal{C}^{\mathrm{fp}}$-Abs *is a definable subcategory of* $\mathcal{C}^{\mathrm{fp}}$-Mod.

Equivalence of the first three conditions is 3.10, the equivalence of (iii) and (v) is by 5.9 (and since any direct limit of flat objects is flat, see 3.2). The equivalence of (iii) and (iv) is [71, 4.1]. The equivalence of the last three conditions was proved by Eklof and Sabbagh [96, Thm. 4], [23, 3.16] when $\mathcal{A}$ is a ring, in which case these are equivalents to the ring being left coherent. The extension to general $\mathcal{A}$ is straightforward, see 5.8 and, for (vi)⇒(iv), [54, 9.3] (also [49, Thm. 7] for the equivalence of (iv) and (v)). For generalisation to the non-additive context, see [7].

funloccoh

Corollary 6.2. *For any skeletally small preadditive $\mathcal{A}$ the categories* $(\text{mod-}\mathcal{A}, \mathbf{Ab})$ *and* $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ *are locally coherent (since each of* mod-$\mathcal{A}$ *and* $\mathcal{A}$-mod *has cokernels).*

modloccoh

Theorem 6.3. *Let $\mathcal{A}$ be skeletally small preadditive. The following are equivalent:*
(i) $((\text{mod-}\mathcal{A})^{\text{op}}, \mathbf{Ab})^{\text{fp}}$ *is abelian;*
(ii) $((\text{mod-}\mathcal{A})^{\text{op}}, \mathbf{Ab})$ *is locally coherent;*
(iii) mod-$\mathcal{A}$ *has pseudokernels;*
(iv) mod-$\mathcal{A}$ *has kernels;*
(v) mod-$\mathcal{A}$ *is abelian;*
(vi) Mod-$\mathcal{A}$ *is locally coherent;*
(vii) $\mathcal{A}^+$*, respectively* $\mathcal{A}^{++}$*, has pseudokernels.*

Proof. The only part which is not immediate from what has been said already (*viz.* 6.1 and 3.5) is (iii)⇒(iv). So suppose that $f : L \to M$ is a morphism in mod-$\mathcal{A}$, let $(i : K \to L) = \ker(f)$ in Mod-$\mathcal{A}$ and let $g : K_0 \to L$ be a pseudokernel of f in mod-$\mathcal{A}$. Since $K = \ker(f)$ there is $h : K_0 \to K$ such that $g = ih$.

If h were not epi there would be (since Mod-$\mathcal{A}$ is locally finitely presented) $K_1 \in \text{mod-}\mathcal{A}$ and $k : K_1 \to K$ such that $\text{im}(k) \not\leq \text{im}(h)$. Now $f(ik) = 0$ so there is $k' : K_1 \to K_0$ such that $ik = gk'$, which equals ihk'. Since i is monic $k = hk'$ - contrary to choice of K_1 and k.

$$\begin{tikzcd}
K_1 \arrow[r, "k'"] \arrow[dr, "k"'] & K_0 \arrow[d, "h"] \arrow[dr, "g"] & & \\
0 \arrow[r] & K \arrow[r, "i"'] & L \arrow[r, "f"'] & M
\end{tikzcd}$$

Therefore h is epi and K is finitely generated. That is, the kernel in Mod-$\mathcal{A}$ of each map in mod-$\mathcal{A}$ is finitely generated, which is enough (for then $\ker(h) = \ker(ih)$ is finitely generated so K is finitely presented). □

We may extend the terminology used for rings by saying that $\mathcal{A}$ as in 6.1 is left coherent and $\mathcal{A}$ as in 6.3 is right coherent.

We have, therefore, the following definable subcategories of Mod-$\mathcal{A}$:
(1) Flat-$\mathcal{A}$ is definable iff $\mathcal{A}$-Mod is locally coherent iff $\mathcal{A}^+$, equivalently $\mathcal{A}^{++}$, has pseudocokernels and, in the case that $\mathcal{A}^+$, equivalently $\mathcal{A}^{++}$, has cokernels, Flat-$\mathcal{A} = \text{Lex}(\mathcal{A}^{\text{op}}, \mathbf{Ab})$ ($\mathcal{A}$ is left coherent);

(2) Abs-$\mathcal{A}$ is definable iff Mod-$\mathcal{A}$ is locally coherent iff $\mathcal{A}^+$, equivalently $\mathcal{A}^{++}$, has pseudokernels ($\mathcal{A}$ is right coherent);
(3) if $\mathcal{A}^+$, equivalently $\mathcal{A}^{++}$, has pseudokernels and pseudocokernels, in particular if this category is abelian, then all of $\mathrm{Lex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$, $\mathrm{Lex}(\mathcal{A}, \mathbf{Ab})$, $\mathrm{Ex}(\mathcal{A}^{(\mathrm{op})}, \mathbf{Ab})$ are definable subcategories.

7 Localisation

secloc

Here the basic definitions and some results around localisation are recalled (for more see [100], [72] though note that the terminology in the latter is rather different). Suppose that $\mathcal{C}$ is an abelian category. A full subcategory, $\mathcal{S}$, of $\mathcal{C}$ is a **Serre subcategory** if for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ the middle term, B, is in $\mathcal{S}$ iff the outside terms, A and C, are in $\mathcal{S}$.

serrquot

Theorem 7.1. *(see [72, 4.3.11, 4.3.12]) Let $\mathcal{S}$ be a Serre subcategory of the skeletally small abelian category $\mathcal{C}$. Then there is a category, $\mathcal{C}/\mathcal{S}$, the* **quotient category***, and a functor, $Q : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{S}$, the* **localisation functor***, such that for every abelian category $\mathcal{B}$ and exact functor $F : \mathcal{C} \longrightarrow \mathcal{B}$ such that $F\mathcal{S} = 0$ (i.e. $FC = 0$ for all $C \in \mathcal{S}$) there is a unique exact factorisation of F through Q. The quotient category is unique to natural equivalence and the localisation functor is exact.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathcal{C}/\mathcal{S} \arrow[d, dashed, "\exists!"] \\ & \mathcal{B} \end{tikzcd}$$

We will apply this where $\mathcal{C}$ is the category of finitely presented objects of a locally coherent abelian category.

If $\mathcal{C}$ is an abelian category with arbitrary direct sums then a **localising subcategory** of $\mathcal{C}$ is a Serre subcategory which is closed under arbitrary direct sums, equivalently under direct limits. We will consider such categories only when $\mathcal{C}$ is actually Grothendieck, in which case the equivalent term, hereditary torsion subcategory, is also used.

So let $\mathcal{C}$ be a Grothendieck category. A full subcategory $\mathcal{T}$ is a **torsion subcategory** if it is closed under extensions, quotient objects and direct sums. If it is also closed under subobjects then it is a **hereditary** torsion subcategory (or subclass). The objects of $\mathcal{T}$ are referred to as **torsion** and those in the corresponding **torsionfree class/subcategory** $\mathcal{F} = \{D : (\mathcal{T}, D) = 0\}$ are the **torsionfree** objects. It is then the case that $\mathcal{T} = \{C : (C, \mathcal{F}) = 0\}$ and the pair $\tau = (\mathcal{T}, \mathcal{F})$ is referred to as a **torsion theory**, which is said to be **hereditary** if $\mathcal{T}$ is closed under subobjects, equivalently if $\mathcal{F}$ is closed under injective hulls. It may be checked that torsionfree subclasses are precisely those closed under extensions, subobjects and, in the hereditary case, injective hulls (and then closure under extensions follows from the other two conditions). A set $\mathcal{E}$ of injective objects **cogenerates** the torsion theory with $\mathcal{T} = \{C : (C, \mathcal{E}) = 0\}$

and $\mathcal{F}$ being the class of objects which embed in some product of copies of objects in $\mathcal{E}$. All torsion theories considered here will be hereditary, so that adjective sometimes will be dropped.

Let τ be a (hereditary) torsion theory on $\mathcal{C}$. For each object $C \in \mathcal{C}$ there is a maximal subobject, τC, the **torsion subobject**, in $\mathcal{T}$ and clearly $C/\tau C$ is the largest torsionfree quotient of C. The action of the localisation functor $Q = Q_\tau$, corresponding to $\mathcal{T} = \mathcal{T}_\tau$, on C can be described as follows. Let C_τ be the inverse image in the injective envelope, $E(C/\tau C)$, of $C/\tau C$, of the torsion subobject, $\tau\big(E(C/\tau C)/(C/\tau C)\big)$, of the quotient $E(C/\tau C)/(C/\tau C)$. Then $Q_\tau C = C_\tau$. This means that, as well as having $(\mathcal{T}, C_\tau) = 0$ one has $\mathrm{Ext}^1(\mathcal{T}, C_\tau) = 0$ (that is, C_τ is **τ-divisible**), reflecting the fact that the localisation functor has kernel $\mathcal{T}$. It is always the case that if $\mathcal{C}$ is a Grothendieck category and τ is a hereditary torsion theory then the localised, or **quotient**, category, which we normally denote $\mathcal{C}_\tau$ rather than $\mathcal{C}/\mathcal{T}_\tau$, is Grothendieck (e.g. [72, 6.23]).

(There is an equivalent definition of the quotient category which changes the morphism sets rather than the objects and which emphasises the similarity with localisation at a Grothendieck site (see [72, p. 167], [100]). The above two-stage localisation process is analogous to the non-additive process of sheafification at a site.)

locsum

Theorem 7.2. *(see [100, §§ IX.1, X.1], [72, 4.3.8, §4.4, 4.6.2]) Let τ be a hereditary torsion theory on the Grothendieck category $\mathcal{C}$. Then the localised category $\mathcal{C}_\tau$ also is Grothendieck, the localisation functor Q_τ is exact and, if $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is any exact functor to a Grothendieck category $\mathcal{C}'$ such that $F\mathcal{T}_\tau = 0$ then F factors uniquely through Q_τ.*

The localisation functor $Q_\tau : \mathcal{C} \longrightarrow \mathcal{C}_\tau$ has a right adjoint, namely the inclusion, i, of $\mathcal{C}_\tau$ in $\mathcal{C}$: $\mathcal{C}(C, iD) \simeq \mathcal{C}_\tau(C_\tau, D)$ for every $C \in \mathcal{C}$ and $D \in \mathcal{C}_\tau$.

For any τ-torsionfree, τ-injective object C, one has $Q_\tau C \simeq C$ and the injective objects of $(i)\mathcal{C}_\tau$ are exactly the τ-torsionfree injective objects of $\mathcal{C}$.

If $\mathcal{G}$ is a generating set of objects for $\mathcal{C}$ then $Q_\tau\mathcal{G}$ is a generating set for $\mathcal{C}_\tau$.

It will be convenient terminologically to refer to "the torsion theory τ" and notationally to use any one of $\mathcal{T}$, $\mathcal{F}$, τ to refer to the localisation so, for instance, $C_\mathcal{T}$ will also be used for C_τ.

A hereditary torsion theory τ is **of finite type** if it is determined by the *finitely presented* torsion objects, that is, if $\mathcal{T}_\tau$ is the smallest localising subcategory of $\mathcal{C}$ which contains $\mathcal{T}_\tau \cap \mathcal{C}^{\mathrm{fp}}$. If $\mathcal{C}^{\mathrm{fp}}$ is abelian then the latter is a Serre subcategory of $\mathcal{C}^{\mathrm{fp}}$ and $\mathcal{T}_\tau$ is then the closure of this Serre subcategory under direct limits. Thus for locally coherent $\mathcal{C}$ there is a bijection between Serre subcategories of $\mathcal{C}^{\mathrm{fp}}$ and hereditary torsion theories of finite type on $\mathcal{C}$.

ftloccoh

Theorem 7.3. *([41, 2.16], [52, §2], [79, A3.16] see [84, 11.1.33]) Suppose that $\mathcal{C}$ is a locally coherent abelian category and that τ is a torsion theory of finite type on $\mathcal{C}$. Then the localised category, $\mathcal{C}_\tau$, also is locally coherent. Furthermore $(\mathcal{C}_\tau)^{\mathrm{fp}} = \mathcal{C}^{\mathrm{fp}}/(\mathcal{T}_\tau \cap \mathcal{C}^{\mathrm{fp}})$ (the second we denote simply as $(\mathcal{C}^{\mathrm{fp}})_\tau$). That is, the finitely presented objects of the localisation are the images under localisation of the finitely presented objects of $\mathcal{C}$.*

8 Serre subcategories of the functor category

Let $\mathcal{A}$ be a small preadditive category and consider the associated, locally coherent (6.2), functor category Fun-$\mathcal{A} = (\text{mod-}\mathcal{A}, \mathbf{Ab})$. As described in the previous section, hereditary torsion theories, τ, of finite type on Fun-$\mathcal{A}$ correspond bijectively to Serre subcategories, $\mathcal{S}$, of the finitely presented functor category, fun-$\mathcal{A} = (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\text{fp}}$, the maps being $\tau \mapsto \mathcal{T}_\tau \cap \text{fun-}\mathcal{A}$ and $\mathcal{S} \mapsto \overrightarrow{\mathcal{S}}$, the closure of $\mathcal{S}$ under direct limits.

Given a Serre subcatgory, $\mathcal{S}$, of fun-$\mathcal{A}$, set $\mathcal{D} = \mathcal{D}_\mathcal{S} = \{M \in \text{Mod-}\mathcal{A} : \overrightarrow{F}M = 0 \text{ for all } F \in \mathcal{D}\}$ to be the subcategory/subclass of Mod-$\mathcal{A}$ consisting of those objects annihilated by $\mathcal{S}$ (given the equivalent datum of a torsion theory, τ, of finite type on Fun-$\mathcal{A}$, we write $\mathcal{D}_\tau$). Here $\overrightarrow{F}$ is the extension of the functor F (on mod-$\mathcal{A}$) to a functor on Mod-$\mathcal{A}$ which commutes with direct limits, defined in the obvious way by using that $\text{Mod-}\mathcal{A} = \overrightarrow{\text{mod-}\mathcal{A}}$ (see [5, pp. 4,5]): it is an exercise to check well-definedness. It is easy to check that $\mathcal{D}$ is closed under products, direct limits and pure subobjects, that is, $\mathcal{D}$ is a definable subcategory of Mod-$\mathcal{A}$. In fact there is the following theorem. In model-theoretic form (that is, modulo the identification, 22.1, of pairs of pp formulas and finitely presented functors) this goes back to Ziegler's paper [105] and, for the left/right duality, to [40]. So 19.4 is a rephrasing of part of this. The functorial form was noted by Herzog and developed by him and also by Krause in the early 90s, see [41, 2.8], [52, 2.10] (also [79, A3.4]).

serdefbij

Theorem 8.1. *Let $\mathcal{A}$ be a skeletally small preadditive category. Then there is a bijection between the following:*
(i) Serre subcategories of fun-$\mathcal{A}$*;*
(ii) hereditary torsion theories of finite type on Fun-$\mathcal{A}$*;*
(iii) Serre subcategories of fun^{d}-$\mathcal{A}$*;*
(iv) hereditary torsion theories of finite type on Fun^{d}-$\mathcal{A}$*;*
(v) definable subcategories of Mod-$\mathcal{A}$*;*
(vi) definable subcategories of $\mathcal{A}$-Mod.

Write $\mathcal{D}^{\text{d}}$ for the definable subcategory dual in the sense of (v)$\leftrightarrow$(vi) to $\mathcal{D}$.

Note that the bijections within the groups (i)-(iv) and (v),(vi) are order-preserving, those between these two groups order-reversing. Here are some of the direct connections, described explicitly (further details can be found *via* [84, 12.4.1]).

Given $\mathcal{D}$, a definable subcategory of Mod-$\mathcal{A}$, let $\mathcal{S}_\mathcal{D}$ be the Serre subcategory of fun-$\mathcal{A} = (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\text{fp}}$ consisting of those finitely presented functors F which annihilate $\mathcal{D}$, i.e. such that $\overrightarrow{F}\mathcal{D} = 0$. To get the torsion class for the corresponding torsion theory in (ii), just remove the restriction that F be finitely presented: this torsion class $\mathcal{T}_\mathcal{D}$ is, as said before, the closure of $\mathcal{S}_\mathcal{D}$ under direct limits. Let $\mathcal{S}^{\text{d}}_\mathcal{D}$ be the dual (in the sense of 4.5) Serre subcategory of $\text{fun}^{\text{d}}\text{-}\mathcal{A} = (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\text{fp}}$: $\mathcal{S}^{\text{d}}_\mathcal{D} = \{dF : F \in \mathcal{S}_\mathcal{D}\}$. By 4.6 this can be characterised directly as $\mathcal{S}^{\text{d}}_\mathcal{D} = \{F \in \text{fun}^{\text{d}}\text{-}\mathcal{A} : (F, D \otimes_\mathcal{A} -) = 0 \ \forall D \in \mathcal{D}\}$. The hereditary torsion theory, $\tau^{\text{d}}_\mathcal{D}$, of finite type generated by $\mathcal{S}^{\text{d}}_\mathcal{D}$ is also characterised as that cogenerated by the set of indecomposable injectives (5.12), $D \otimes_{\mathcal{A}} -$ as D ranges over the indecomposable pure-injective objects in $\mathcal{D}$. The definable category $\mathcal{D}^{\mathrm{d}}$ dual in the sense of (v)↔(vi) to $\mathcal{D}$ is $\mathcal{D}^{\mathrm{d}} = \{L \in \mathcal{A}\text{-Mod} : \overrightarrow{dF}(L) = 0\ \forall F \in \mathcal{S}_{\mathcal{D}}\} = \{L \in \mathcal{A}\text{-Mod} : \overrightarrow{F}L = 0 \text{ for every } F \in \mathcal{S}^{\mathrm{d}}_{\mathcal{D}}\}$.

Starting with a finite type hereditary torsion theory τ on Fun-$\mathcal{A}$, the Serre subcategory in (i) is the intersection, $\mathcal{T}_\tau \cap \text{fun-}\mathcal{A}$, of the torsion class with the class of finitely presented functors. The torsion theory, τ^{d}, in (iv) is defined *via* duality applied to this Serre subcategory. The corresponding definable subcategory consists of those $D \in \text{Mod-}\mathcal{A}$ such that the functor $D \otimes_{\mathcal{A}} -$ is τ^{d}-torsionfree, that is, see 5.12, the category of absolutely pure τ^{d}-torsionfree objects in Fun$^{\mathrm{d}}$-$\mathcal{A}$.

A particular case for modules, going back to Eklof and Sabbagh [23] [96] (and [40, 9.3] for the formal duality), is the following (see 5.8 and 6.1).

flatabsdual

Example 8.2. Suppose that $\mathcal{A}$-Mod is locally coherent. Then Flat-$\mathcal{A}$ is a definable subcategory of Mod-$\mathcal{A}$ and $\mathcal{A}$-Abs is a definable subcategory of $\mathcal{A}$-Mod and these correspond under the bijection (v)↔(vi) in 8.1.

To see that left absolutely pure is dual to right flat, let $\mathcal{S}$ denote the Serre subcategory corresponding in the sense of 8.1 to $\mathcal{A}$-Abs. Given a monomorphism $f : B \longrightarrow C$ in $\mathcal{A}$-mod there is the exact sequence $(C,-) \xrightarrow{(f,-)} (B,-) \longrightarrow F_f \longrightarrow 0$ of functors in $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$, where $F_f = \operatorname{coker}(f,-)$. By 5.6, $L \in \mathcal{A}\text{-Mod}$ is absolutely pure iff $F_f L = 0$ for all such f. Thus $\mathcal{S}$ is generated as a Serre subcategory by $\{F_f : f \in \mathcal{A}\text{-mod is monic}\}$. Applying the duality d (4.5) gives the exact sequence $0 \longrightarrow dF_f \longrightarrow d(B,-) = (- \otimes B) \xrightarrow{d(f,-)=(-\otimes f)} d(C,-) = (- \otimes C)$, so the subcategory of Mod-$\mathcal{A}$ corresponding to $\mathcal{S}^{\mathrm{d}}$, which is generated as a Serre subcategory by $\{dF_f : F_f \in \mathcal{S}\}$ contains exactly those $M \in \text{Mod-}\mathcal{A}$ such that $\overrightarrow{dF_f}M = 0$, that is, such that for every monomorphism $f : B \longrightarrow C$ in $\mathcal{A}$-mod the morphism $M \otimes_{\mathcal{A}} B \xrightarrow{1_M \otimes f} M \otimes_{\mathcal{A}} C$ is monic, that is, such that M is flat.

exloccohdef

Example 8.3. If $\mathcal{C}$ is a locally coherent category and τ is a hereditary torsion theory of finite type on $\mathcal{C}$ then the localised category $\mathcal{C}_\tau$, regarded as a subcategory of $\mathcal{C}$, is definable. For $\mathcal{C}$ a module category this is [73, 2.20] and the result for functor categories is [75, 0.1]. The general case then follows by 10.1 for instance.

Using 6.1, which implies that every finitely accessible additive category with products is a definable subcategory of a locally coherent functor category, together with 7.3, one extends 8.1 as follows.

flatabsdual1

Corollary 8.4. *Suppose that $\mathcal{C}$ is finitely accessible with products. Then there is an order-preserving bijection between the definable subcategories of $\mathcal{C}$ and those of the (elementary) dual category $\mathcal{C}^{\mathrm{d}} = \mathcal{C}^{\mathrm{fp}}$-Abs (for which see the next section).*

This bijection will allow us to define, in the next section, the elementary dual of any definable category.

duallocfp

Proposition 8.5. *Let τ be a torsion theory of finite type on* Fun-$\mathcal{A}$. *Then the categories of finitely presented objects of the localisations* $(\text{Fun-}\mathcal{A})_\tau$ *and* $(\text{Fun}^{\text{d}}\text{-}\mathcal{A})_{\tau^{\text{d}}}$ *are opposite:* $(\text{fun}^{\text{d}}\text{-}\mathcal{A})/\mathcal{S}^{\text{d}} \simeq ((\text{fun-}\mathcal{A})/\mathcal{S})^{\text{op}}$, *where $\mathcal{S}$ is the Serre subcategory of finitely presented τ-torsion objects.*

This follows directly from the fact that d is a duality, $\text{fun}^{\text{d}}\text{-}\mathcal{A} \simeq (\text{fun-}\mathcal{A})^{\text{op}}$ (4.5), and 7.3.

9 Conjugate and dual categories

secconjdual

Given $\mathcal{C}$ finitely accessible, so (3.4) $\mathcal{C} \simeq \text{Flat-}\mathcal{C}^{\text{fp}}$, there is the **conjugate category** $\widetilde{\mathcal{C}} = \mathcal{C}^{\text{fp}}\text{-Flat}$ (Roos [93, p. 204], at least for $\mathcal{C}$ locally coherent). By 3.4 $\widetilde{\mathcal{C}}$ is again finitely accessible and, by 3.5, it is locally coherent iff $\mathcal{C}$ is. Since, by 3.4, $\left(\widetilde{\mathcal{C}}\right)^{\text{fp}} \simeq (\mathcal{C}^{\text{fp}})^{\text{op}}$ it follows that $\widetilde{\widetilde{\mathcal{C}}} \simeq \mathcal{C}$.

For example, if $\mathcal{C} = \text{Mod-}R$ then $\widetilde{\mathcal{C}} = \text{Flat}(\text{mod-}R, \mathbf{Ab})$. Every flat functor in $(\text{mod-}R, \mathbf{Ab})$ is a direct limit of representables and these form the image of the restriction functor $(\text{mod-}R)^{\text{op}} \longrightarrow (\text{mod-}R, \mathbf{Ab})$ taking M to $(M, -)$, so the objects of $\widetilde{\mathcal{C}}$, being direct limits of representables, correspond, by 3.3, to inverse limits of objects of mod-R. Therefore $\widetilde{\mathcal{C}}$ is the opposite of the **pro-completion** (the opposite of the ind-completion of the opposite category, see [3, p. 119] or [47, p. 233]) of mod-R. Compare with $\mathcal{C}$, which is the ind-completion of mod-R.

A second category which we associate to $\mathcal{C}$ is the (**elementary**) **dual** category $\mathcal{C}^{\text{d}} = \mathcal{C}^{\text{fp}}\text{-Abs}$. This definition will be extended in Section 10 with an arbitrary definable category in place of $\mathcal{C}$. There it will be seen that the notation is in agreement with that introduced after 8.1 and, in particular, $(\mathcal{C}^{\text{d}})^{\text{d}} \simeq \mathcal{C}$. This dual first appeared for definable subcategories of module categories in the model theory of modules, in [40], where Herzog showed that there is a natural duality (see 8.1) between definable subcategories of Mod-R and R-Mod. The extension to definable subcategories of arbitrary functor categories is discussed in [79]. The same concept, for finitely accessible additive categories (not necessarily with products) appears in [22], where it is called the "pseudodual". The definition there is completely different from the model-theoretic one just referred to. This, but defined as above, is extended in [20] to definable categories and there it is called the "symmetric" category. It is a consequence of the results of [53] that the dual does not depend on the particular representation of a definable category.

For example if $\mathcal{C} = \text{Mod-}\mathcal{A}$ then $\mathcal{C}^{\text{d}} = \text{Abs}(\text{mod-}\mathcal{A}, \mathbf{Ab}) \simeq \mathcal{A}\text{-Mod}$, the equivalence by 5.12 (also cf. [99, p. 355]).

Example 9.1. The elementary dual, $\mathcal{C}^{\text{d}}$, of a finitely accessible category $\mathcal{C}$ need not be finitely accessible. Let $\mathcal{A} = \mathbb{Z}$, so $\mathcal{A}^{++}$ is the category of free abelian groups of finite rank. Set $\mathcal{C} = \text{Flat-}\mathcal{A}$. By 3.3 this is the category of flat abelian groups: a finitely accessible category with products. Then $\mathcal{C}^{\text{d}} = \text{Abs}(\mathbb{Z}^{++}, \mathbf{Ab})$ which is the category of injective abelian groups and that, as is shown in Example 10.3 below, is not finitely accessible (indeed, there is no non-zero finitely presented object).

We compare these categories - conjugate and dual - both of which (but especially the dual) will be useful to us. We have already noted that $\widetilde{\mathcal{C}}$ is finitely accessible and that $\mathcal{C}^{\mathrm{d}}$ need not be. The next result is just a re-phrasing of .

Note that condition (b)(vi) of 6.1 is that if $\mathcal{C}$ is finitely accessible then $\mathcal{C}^{\mathrm{d}}$ is a definable subcategory of $\mathcal{C}^{\mathrm{fp}}$-Mod iff $\mathcal{C}$ has products. Since (5.8) the subcategory of absolutely pure objects of a locally finitely presented category (such as $\mathcal{C}^{\mathrm{fp}}$-Mod) always is closed under products and pure subobjects, this condition is that $\mathcal{C}^{\mathrm{d}}$ be closed under direct limits in $\mathcal{C}^{\mathrm{fp}}$-Mod. In contrast, even if $\mathcal{C}$ has products, $\widetilde{\mathcal{C}}$ need not, as is shown by the next example.

Example 9.2. Let R be left coherent. Then $\widetilde{\text{Mod-}R}$ has products iff R is right coherent.

For by 3.10 $\widetilde{\text{Mod-}R}$ = (mod-R)-Flat = Flat-(mod-R)$^{\mathrm{op}}$ has products iff (mod-R)$^{\mathrm{op}}$ has pseudocokernels iff mod-R has pseudokernels iff (by 6.3) mod-R has kernels iff R is right coherent.

Thus if $\mathcal{C} = \text{Mod-}\mathcal{A}$ then, whilst $\mathcal{C}^{\mathrm{d}} = \mathcal{A}\text{-Mod}$, the conjugate $\widetilde{\mathcal{C}} = \text{Flat-(mod-}\mathcal{A})^{\mathrm{op}}$ need not have products. If, however, there is a duality, (mod-$\mathcal{A}$)$^{\mathrm{op}} \simeq \mathcal{A}$-mod (e.g. if $\mathcal{A} = R$ is an artin algebra) then $\widetilde{\mathcal{C}} = \text{Flat-}(\mathcal{A}\text{-mod}) \simeq \mathcal{A}\text{-Mod} = \mathcal{C}^{\mathrm{d}}$. Note that the existence of a duality (mod-$\mathcal{A}$)$^{\mathrm{op}} \simeq \mathcal{A}$-mod implies that mod-$\mathcal{A}$ and $\mathcal{A}$-mod are abelian and hence that $\mathcal{A}$ is left and right coherent.

Taking the preadditive category $\mathcal{A}$ in the above paragraph to be itself of the form (mod-$\mathcal{A}$)$^{\mathrm{op}}$, we have that if $\mathcal{C} = (\text{mod-}\mathcal{A}, \mathbf{Ab}) = \text{Fun-}\mathcal{A}$ then $\mathcal{C}^{\mathrm{d}} = ((\text{mod-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})$ and $\widetilde{\mathcal{C}} = (\text{fun-}\mathcal{A})\text{-Flat} = \text{Flat-}(\text{fun-}\mathcal{A})^{\mathrm{op}}$ which, by the duality, 4.5, between fun-$\mathcal{A}$ and fun$^{\mathrm{d}}$-$\mathcal{A}$, is equivalent to Flat-(fun$^{\mathrm{d}}$-$\mathcal{A}$) which, by 3.4, is equivalent to Fun$^{\mathrm{d}}$-$\mathcal{A}$. That is, $\widetilde{\text{Fun-}\mathcal{A}} = \text{Fun}^{\mathrm{d}}\text{-}\mathcal{A}$. In particular, if there is a duality $(\text{mod-}\mathcal{A})^{\mathrm{op}} \simeq \mathcal{A}\text{-mod}$ then $(\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{d}} \simeq (\mathcal{A}\text{-mod}, \mathbf{Ab}) \simeq \widetilde{(\text{mod-}\mathcal{A}, \mathbf{Ab})}$.

Proposition 9.3. *Let $\mathcal{A}$ be a small preadditive category. If there is a duality* (mod-$\mathcal{A}$)$^{\mathrm{op}} \simeq \mathcal{A}$-mod *between the categories of right and left finitely presented $\mathcal{A}$-modules then elementary dual, $\mathcal{C}^{\mathrm{d}}$, and conjugate, $\widetilde{\mathcal{C}}$, coincide for both $\mathcal{C} =$* Mod-$\mathcal{A}$ *and $\mathcal{C} =$* Fun-$\mathcal{A}$.

Recall the full embedding, 5.12, of Mod-$\mathcal{A}$ into $(\mathcal{A}\text{-mod}, \mathbf{Ab})$.

inclpurdef

Theorem 9.4. *Suppose that $\mathcal{D}$ is a definable subcategory of* Mod-$\mathcal{A}$. *Then the image of $\mathcal{D}$ under the full embedding $\epsilon : \text{Mod-}\mathcal{A} \longrightarrow (\mathcal{A}\text{-mod}, \mathbf{Ab})$, which takes M to $M \otimes_{\mathcal{A}} -$, is the class of absolutely pure $\tau_{\mathcal{D}}^{\mathrm{d}}$-torsionfree objects. This category is, moreover, equivalent to the category of absolutely pure objects of the localised category* $(\mathcal{A}\text{-mod}, \mathbf{Ab})_{\tau_{\mathcal{D}}^{\mathrm{d}}}$.

Proof. By definition of $\tau_{\mathcal{D}}^{\mathrm{d}}$, objects of the form $D \otimes -$ with $D \in \mathcal{D}$ are $\tau_{\mathcal{D}}^{\mathrm{d}}$-torsionfree. The converse follows from (the details of) 8.1. The second statement, which may be found at [56, A7] for example, follows from 8.3 and the comments after 5.3. The details also may be found at [84, §12.3]). □

10 Definable subcategories

secdefsub

Given a finitely accessible additive category with products a full subcategory (or the corresponding class of objects) $\mathcal{D}$ is **definable** if it is closed under products, direct limits and pure subobjects (the last we take to include closure under isomorphism). We will refer to such a category $\mathcal{D}$ as a "definable subcategory" or simply as a "definable category", though the latter term begs some questions: whether the "definable structure" (see §12) may be recovered from $\mathcal{D}$ *qua* category and whether such categories have an intrinsic, independent of representation, characterisation.

Let $\mathcal{C}$ be a finitely accessible additive category with products and consider the inclusion $\mathcal{C} \simeq \text{Flat-}\mathcal{C}^{\text{fp}} \subseteq \text{Mod-}\mathcal{C}^{\text{fp}}$. By 5.11 an exact sequence in $\mathcal{C}$ is pure-exact iff its image in Mod-$\mathcal{C}^{\text{fp}}$ is pure-exact. By 3.10 the inclusion of $\mathcal{C}$ in Mod-$\mathcal{C}^{\text{fp}}$ preserves products, and it preserves direct limits (since a direct limit of flat functors is flat). Therefore, extending 6.1, we have the following.

finaccdef

Proposition 10.1. *Let $\mathcal{C}$ be a finitely accessible category with products. Then $\mathcal{C}$ is a definable subcategory of* Mod-$\mathcal{C}^{\text{fp}}$*. Therefore every definable subcategory of a finitely accessible category is a definable subcategory of a functor category.*

In fact, by the remark after 5.12, every definable category is a definable subcategory of a locally coherent functor category.

The terminology "definable" refers to the natural model-theoretic language for $\mathcal{C}$ based on $\mathcal{C}^{\text{fp}}$ (§18) and one can obtain the above fact starting from the model-theoretic definition of "definable subclass" (see 19.4).

defex1

Example 10.2. Let $\mathcal{T}$ be the category of torsion abelian groups. This is not a definable subcategory of Mod-$\mathbb{Z}$ (it is not closed under products) but it is finitely accessible with products (hence a definable category): the finitely presented objects are the finitely generated = finitely presented torsion groups and product is given by taking the torsion subgroup of the product in Mod-$\mathbb{Z}$. It is a definable subcategory of Mod-$(\mathcal{T}^{\text{fp}})$.

It is not the case that a definable subcategory need be finitely accessible.

defex2

Example 10.3. Let Inj-$\mathbb{Z}$ be the category of injective abelian groups - a definable subcategory of Mod-$\mathbb{Z}$. Any $\varinjlim$-generating set for Inj-$\mathbb{Z}$ must contain a module with $\mathbb{Q}$ as a direct summand (since $(\mathbb{Z}_{p^\infty}, \mathbb{Q}) = 0$) but $\mathbb{Q}$ is not a finitely presented object. To see this, regard $M = \mathbb{Z}_{p^\infty}^{(\aleph_0)}$ as a direct limit $\mathbb{Z}_{p^\infty} \longrightarrow \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty} \longrightarrow \dots \longrightarrow \mathbb{Z}_{p^\infty}^n \longrightarrow \dots$. Define a morphism from $\mathbb{Q}$ to M by sending 1 to $(a_{11}, 0, \dots)$ where $a_{11} \neq 0$ and $a_{11}p = 0$, sending $1/p$ to $(a_{12}, a_{21}, 0, \dots)$ where $a_{12}p = a_{11}$, $a_{21} \neq 0$ and $a_{21}p = 0$, etc. (then extending to all of $\mathbb{Q}$). Clearly any such morphism factors through no finite subsum. This argument applies to any indecomposable injective, showing that $(\text{Inj-}\mathbb{Z})^{\text{fp}} = 0$.

A subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is **covariantly finite** if for every $C \in \mathcal{C}$ there is $f : C \to B \in \mathcal{B}$ such that every morphism from C to an object of $\mathcal{B}$ factors (though not necessarily uniquely) through f.

limclosdef

Proposition 10.4. *([19, 4.1]) Suppose that $\mathcal{C}$ is finitely accessible with products and let $\mathcal{D}_0$ be a covariantly finite subcategory of $\mathcal{C}^{\mathrm{fp}}$. Then the closure, $\overrightarrow{\mathcal{D}_0}$, of $\mathcal{D}_0$ under direct limits is a definable subcategory of $\mathcal{C}$ and is finitely accessible, with $(\overrightarrow{\mathcal{D}_0})^{\mathrm{fp}}$ being the closure of $\mathcal{D}_0$ under direct summands of finite direct sums.*

Also see [61, §2] for related results.

covfinex

Example 10.5. The category Flat-$\mathbb{Z}$ of flat abelian groups is the $\varinjlim$-closure of the category, proj-$\mathbb{Z}$, of finitely generated projectives which, since flat=torsionfree for $\mathbb{Z}$-modules, clearly is covariantly finite in Mod-$\mathbb{Z}$. Hence the finitely accessible category Flat-$\mathbb{Z}$ is a definable subcategory (of course, we know this already since $\mathbb{Z}$ is coherent). Observe that this category is not abelian (e.g. the monomorphism $2 \times - : \mathbb{Z} \to \mathbb{Z}$ is not a kernel) but, since it has cokernels, it is locally finitely presented by 3.7.

Example 10.6. Let R be a tame hereditary finite-dimensional algebra over a field (for example the path algebra of an extended Dynkin quiver). The set $\mathcal{P}_0$ of finite-dimensional preprojective right R-modules is (by the description of homomorphisms between indecomposable finite-dimensional modules) covariantly finite, so the closure of this class under direct limits is, by 10.4, a finitely accessible category.

The modules in $\overrightarrow{\mathcal{P}_0}$ are exactly those M such that $(N, M) = 0$ for every indecomposable regular or preinjective finite-dimensional module N, because these conditions together define $\mathcal{P}_0$ within mod-R, that is, the Serre subcategory $\mathcal{S}$ of (mod-$R, \mathbf{Ab}$) corresponding in the sense of 8.1 to $\overrightarrow{\mathcal{P}_0}$ is generated as such by these functors $(N, -)$. The dual category $(\overrightarrow{\mathcal{P}_0})^{\mathrm{d}}$ is the definable category generated by the preinjective left R-modules. There are various ways to see this, for example, given the above description of $\mathcal{S}$ it follows that $\mathcal{S}^{\mathrm{d}}$ is generated as a Serre subcategory by the functors $d(N, -) = (N \otimes -)$ with N as before. Now, if $L \in R$-mod then $N \otimes L = 0$ iff $(L, N^*) = 0$ (* being the duality $\mathrm{Hom}_k(-, k)$ between mod-R and R-mod). Since N^* ranges over the indecomposable preprojective and regular left R-modules we deduce that the $L \in R$-mod satisfying these conditions are the finite-dimensional preinjective left R-modules, as claimed.

In model-theoretic terms, the conditions of the form $(N, -) = 0$, with N as above, form a set of axioms, of the kind in 19.4, for $\overrightarrow{\mathcal{P}_0}$ and the conditions $(N \otimes -) = 0$ are obtained from them by applying elementary duality (§23).

Various related examples of definable subcategories may be found in [92], [61] and [91].

Now we extend the definition of the associated functor category from categories of the form Mod-$\mathcal{A}$ to definable categories. If $\mathcal{D}$ is a definable subcategory, of Mod-$\mathcal{A}$ say, then the **functor category**, $\mathrm{Fun}(\mathcal{D})$, of $\mathcal{D}$ is $(\text{Fun-}\mathcal{A})_{\tau_{\mathcal{D}}} = (\text{mod-}\mathcal{A}, \mathbf{Ab})_{\tau_{\mathcal{D}}}$ where $\tau_{\mathcal{D}}$ is the finite type hereditary torsion theory on (mod-$\mathcal{A}, \mathbf{Ab}$) which corresponds, in the sense of 8.1, to $\mathcal{D}$. Set $\mathrm{fun}(\mathcal{D}) = \mathrm{Fun}(\mathcal{D})^{\mathrm{fp}}$ which, by 7.3, equals $(\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}/\mathcal{S}_{\mathcal{D}}$, that is $\mathrm{fun}(\mathcal{D}) = \text{fun-}\mathcal{A}/\mathcal{S}_{\mathcal{D}}$ (we also refer to this as the **functor category** of $\mathcal{D}$, adding a prefix "finitely presented" if we need to make the distinction). It will be seen in 22.2 that this category is equivalent to the category $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ of (model-theoretic) imaginaries of $\mathcal{D}$. Since, by 7.3, $\mathrm{Fun}(\mathcal{D})$ is locally coherent the category $\mathrm{fun}(\mathcal{D})$ is abelian. It will be shown later, 12.2, that this is a good definition in the sense that if $\mathcal{D}$ is a definable subcategory of Mod-$\mathcal{A}$ and if $\mathcal{D}'$ is a definable subcategory of Mod-$\mathcal{A}'$ such that $\mathcal{D} \simeq \mathcal{D}'$ then $\mathrm{fun}(\mathcal{D}) \simeq \mathrm{fun}(\mathcal{D})'$. We also extend our earlier notation by writing $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$ for $(\text{Fun-}\mathcal{A}_{\tau^{\mathrm{d}}_{\mathcal{D}}}$ and $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$ for $\big(\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})\big)^{\mathrm{fp}} \simeq \text{fun-}\mathcal{A}/\mathcal{S}^{\mathrm{d}}_{\mathcal{D}}$. By 8.5, $\mathrm{fun}^{\mathrm{d}}(\mathcal{D}) \simeq \big(\mathrm{fun}(\mathcal{D})\big)^{\mathrm{op}}$.

Example 10.7. Let $\mathcal{D} \subseteq$ Mod-$\mathbb{Z}$ be the set of torsionfree abelian groups, considered earlier at 3.16 and 10.5. Then $\mathrm{fun}(\mathcal{D})$ is the localisation of $(\text{mod-}\mathbb{Z}, \mathbf{Ab})^{\mathrm{fp}}$ at the Serre subcategory, $\mathcal{S}$, generated by the functors $(\mathbb{Z}_{p^n}, -)$ (p prime, $n \geq 1$). It follows by 7.3 that $\mathrm{fun}(\mathcal{D}) = (\text{mod-}\mathbb{Z}, \mathbf{Ab})^{\mathrm{fp}}/\mathcal{S}$ is generated by the image of $(\mathbb{Z}, -)$ under this localisation. One may check directly that $\mathrm{fun}(\mathcal{D})$ is equivalent to mod-$\mathbb{Z}$ (it also follows, since flat =torsionfree over $\mathbb{Z}$, by 12.4).

The next, key, result, identifying the objects of a definable category with the exact functors on its associated functor category, is due to Herzog for definable subcategories of module categories and to Krause [53] in the general case.

defsubexfun1

Theorem 10.8. *Let $\mathcal{D}$ be a definable subcategory. Then* $\mathcal{D} \simeq \mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathbf{Ab})$.

Proof. Without loss of generality (10.1) $\mathcal{D}$ is a definable subcategory of a functor category Mod-$\mathcal{A}$. By 9.4, $\mathcal{D}$ is equivalent to the full subcategory of absolutely pure objects of the localised category $(\mathcal{A}\text{-mod}, \mathbf{Ab})_{\tau^{\mathrm{d}}_{\mathcal{D}}}$. This category is locally coherent (7.3) so, by 5.7, $\mathcal{D} \simeq \mathrm{Ex}\big((((\mathcal{A}\text{-mod}, \mathbf{Ab})_{\tau^{\mathrm{d}}_{\mathcal{D}}})^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab}\big)$. Also, by 7.3, $((\mathcal{A}\text{-mod}, \mathbf{Ab})_{\tau^{\mathrm{d}}_{\mathcal{D}}})^{\mathrm{fp}} \simeq ((\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}})_{\tau^{\mathrm{d}}_{\mathcal{D}}}$ and this is the category we have denoted $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$. Since also $(\mathrm{fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}} \simeq \mathrm{fun}(\mathcal{D})$ (8.5), the proof is complete. □

The proof of 10.8 goes *via* the opposite functor category, that is, *via* the representation $\mathcal{D} \simeq \mathrm{Ex}((\mathrm{fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}}, \mathbf{Ab})$, given by $D \mapsto (-, (D \otimes -)_{\tau^{\mathrm{d}}_{\mathcal{D}}})$. From this a direct description of the equivalence may be extracted. Since $D \otimes -$ is $\tau^{\mathrm{d}}_{\mathcal{D}}$-torsionfree and $\tau^{\mathrm{d}}_{\mathcal{D}}$-divisible (the latter since $D \otimes -$ is absolutely pure = fp-injective and since $\tau^{\mathrm{d}}_{\mathcal{D}}$ is of finite type so it is enough to check that $\mathrm{Ext}^1(\mathcal{S}^{\mathrm{d}}_{\mathcal{D}}, D \otimes -) = 0$) we have $((dF)_{\tau^{\mathrm{d}}_{\mathcal{D}}}, (D \otimes -)_{\tau^{\mathrm{d}}_{\mathcal{D}}}) \simeq (dF, D \otimes -)$ by 7.2 and the latter is isomorphic to $\overrightarrow{F}D$ by 4.6.

defsubexfun2

Corollary 10.9. *The equivalence of 10.8 is, from left to right, $D \mapsto \mathrm{ev}_D$, evaluation at D (cf. 4.7), given by $\mathrm{ev}_D(F) = \overrightarrow{F}D$ and, in the other direction, is given by taking an exact functor E from* $\mathrm{fun}(\mathcal{D})$ *to* $\mathbf{Ab}$ *to the object, D_E, of* Mod-$\mathcal{A}$ *which takes an object $A \in \mathcal{A}$ to $E\big(((-, A), -)_{\tau_{\mathcal{D}}}\big)$, which is indeed an object of $\mathcal{D}$.*

For the second statement, use that since both E and evaluation at D_E are exact on $\mathrm{Fun}(\mathcal{D})$ it is enough to check that they agree on (localisations of) representable functors $(M, -)$ and, again by exactness, it is enough to check that they agree on $\mathcal{A}^{++}$ (if $M \in$ mod-$\mathcal{A}$, say $(-, A) \longrightarrow (-, B) \longrightarrow M \longrightarrow 0$ is exact where $A \longrightarrow B$ is in $\mathcal{A}^{++}$ then $0 \longrightarrow (M,-) \longrightarrow ((-,B),-) \longrightarrow ((-,A),-)$ is exact), equivalently on $\mathcal{A}$. But that is direct from the definition of D_E (and the Yoneda Lemma).

Suppose that $\mathcal{D}$ is a definable subcategory of Mod-$\mathcal{A}$. According to 8.1 there is a definable subcategory, $\mathcal{D}^{\mathrm{d}}$, of $\mathcal{A}$-Mod corresponding to $\mathcal{D}$. Extending earlier terminology call this the (**elementary**) **dual** of $\mathcal{D}$. Note that, corresponding to $\mathcal{D} \simeq \mathrm{Ex}(\mathrm{fun}(\mathcal{D}))$ we have $\mathcal{D}^{\mathrm{d}} \simeq \mathrm{Ex}(\mathrm{fun}^{\mathrm{d}}(\mathcal{D}))$, that is, $\mathrm{fun}(\mathcal{D}^{\mathrm{d}}) = \mathrm{fun}^{\mathrm{d}}(\mathcal{D})$ (model-theoretically this says that $\mathcal{D}$ and $\mathcal{D}^{\mathrm{d}}$ have opposite categories of imaginaries: $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+} \simeq (\mathbb{L}(\mathcal{D}^{\mathrm{d}})^{\mathrm{eq}+})^{\mathrm{op}}$). It will follow from 11.4 below that this dual is well-defined (is independent of representation of $\mathcal{D}$ as a definable subcategory). Also, by 8.2, this definition agrees on finitely accessible categories which have products with the definition given in Section 9 (this also shows that $\mathcal{C}^{\mathrm{dd}} \simeq \mathcal{C}$).

Thus we have the following result which was proved by Herzog, [40, 2.9], for $\mathcal{D}$ being of the form Mod-R (R a ring) and, more generally [40, 6.2], for definable subcategories of module categories.

fundualdef

Theorem 10.10. *Let $\mathcal{D}$ be a definable category. Then* $\mathrm{fun}^{\mathrm{d}}(\mathcal{D}) = \mathrm{fun}(\mathcal{D}^{\mathrm{d}}) \simeq (\mathrm{fun}(\mathcal{D}))^{\mathrm{op}}$.

defsubexfun3

Corollary 10.11. *Let $\mathcal{D}$ be a definable category. Then* $\mathcal{D} \simeq \mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathbf{Ab}) \simeq \mathrm{Ex}\big((\mathrm{fun}(\mathcal{D}^{\mathrm{d}}))^{\mathrm{op}}, \mathbf{Ab}\big)$ *where $\mathcal{D}^{\mathrm{d}}$ is the elementary dual of $\mathcal{D}$.*

We refer to $\mathrm{fun}^{\mathrm{d}}(\mathcal{D}) = \mathrm{fun}(\mathcal{D}^{\mathrm{d}})$ (as well as $\mathrm{Fun}(\mathcal{D}^{\mathrm{d}})$) as the **dual functor category** of $\mathcal{D}$.

(Note that $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$ is in general different from $(\mathrm{Fun}(\mathcal{D}))^{\mathrm{d}}$: take $\mathcal{D} = $ Mod-$\mathcal{A}$, so $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}) = (\mathcal{A}\text{-mod}, \mathbf{Ab})$, whereas $(\mathrm{Fun}(\mathcal{D}))^{\mathrm{d}} = (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{d}} = ((\text{mod-}\mathcal{A})^{\mathrm{op}}, \mathbf{Ab})$ and, in the absence of a duality $\mathcal{A}\text{-mod} \simeq (\text{mod-}\mathcal{A})^{\mathrm{op}}$, these are different.)

From this we also have a canonical tensor product $\mathcal{D} \times \mathcal{D}^{\mathrm{d}} \xrightarrow{\otimes} \mathbf{Ab}$ where the tensor is over $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$, i.e. with objects of $\mathcal{D}$ and $\mathcal{D}^{\mathrm{d}}$ being seen as right, respectively left, $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$-modules.

defsubcatdual1

Corollary 10.12. *For any definable category $\mathcal{D}$ there is an inclusion-preserving bijection between definable subcategories of $\mathcal{D}$ and definable subcategories of $\mathcal{D}^{\mathrm{d}}$.*

This is immediate from 8.1 because a definable subcategory of $\mathcal{D}$ is simply a definable subcategory of Mod-$\mathcal{A}$ for some suitable $\mathcal{A}$ (we can take $\mathcal{A} = \mathrm{fun}^{\mathrm{d}}(\mathcal{D})$) which is contained in $\mathcal{D}$.

The next point, though labelled as a remark, is a key one in our approach to purity since it provides a definition of pure embedding which applies in categories that satisfy quite weak hypotheses.

Remark 10.13. If $C \longrightarrow D$ is a pure embedding in a definable subcategory, $\mathcal{D}$ of, say, a functor category Mod-$\mathcal{A}$ then, as remarked after 5.3, this is also a pure embedding in Mod-$\mathcal{A}$, say $0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$ is pure-exact in Mod-$\mathcal{A}$. Then $E \in \mathcal{D}$: for, by 5.3, some ultrapower, E^*, of E is a direct summand of an ultrapower of D. The latter is in $\mathcal{D}$, hence so is E^*. But, 20.2, E is pure in E^*, hence $E \in \mathcal{D}$.

11 Exactly definable categories

secexdef

A category is **exactly definable** (Krause [56]) if $\mathcal{D} \simeq \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$ for some (skeletally) small abelian category $\mathcal{A}$. (Since the opposite of an abelian category is abelian, the "op" is a choice rather than a necessity.) In this case Mod-$\mathcal{A}$ will, by 6.3, be locally coherent and so, by 5.7, $\mathcal{D} \simeq$ Abs-$\mathcal{A}$, where the latter denotes the category of absolutely pure = fp-injective objects of Mod-$\mathcal{A}$. Therefore, by 6.1, $\mathcal{D}$ is a definable subcategory of the locally coherent category Mod-$\mathcal{A}$.

Conversely, if $\mathcal{D}$ is a definable subcategory of some finitely accessible category with products then, by 10.11, $\mathcal{D} \simeq \mathrm{Ex}((\mathrm{fun}(\mathcal{D}^{\mathrm{d}}))^{\mathrm{op}}, \mathbf{Ab})$ so $\mathcal{D}$ is exactly definable. Thus the concepts are equivalent. Note that both, however, depend on a representation of the category being given, so fail to be "intrinsic" definitions.

exdefeqdefsub

Proposition 11.1. *A category is a definable subcategory of a finitely accessible category with products iff it is exactly definable iff it is equivalent to the full subcategory of absolutely pure objects of a locally coherent category.*

If $\mathcal{D} = \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$, with $\mathcal{A}$ abelian, is exactly definable, hence a definable subcategory of Mod-$\mathcal{A}$, then, as noted above, $\mathcal{D} \simeq \mathrm{Ex}((\mathrm{fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}}, \mathbf{Ab})$ where $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$ is computed from this representation of $\mathcal{D}$ as a definable subcategory. It will follow from 12.2 that $\mathcal{A} \simeq \mathrm{fun}^{\mathrm{d}}(\mathcal{D})$ but this can be seen now.

Aisfun

Proposition 11.2. *If* $\mathcal{D} = \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$ *where* $\mathcal{A}$ *is small abelian then,* $\mathcal{A} \simeq \mathrm{fun}^{\mathrm{d}}(\mathcal{D})$*, where* $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$ *is defined with respect to the inclusion of* $\mathcal{D}$ *as a definable subcategory of* Mod-$\mathcal{A}$.

Proof. (sketch) We show that $\mathcal{A}^{\mathrm{op}} \simeq \mathrm{fun}(\mathcal{D})$ and then take opposite categories. Recall, 4.3, that $\mathrm{fun}^{\mathrm{d}}$-$\mathcal{A}$ is the free abelian category, $\mathrm{Ab}(\mathcal{A})$, of $\mathcal{A}$ and hence fun-$\mathcal{A} \simeq (\mathrm{Ab}(\mathcal{A}))^{\mathrm{op}} \simeq \mathrm{Ab}(\mathcal{A}^{\mathrm{op}})$. So, by 4.2 the exact functors on $\mathcal{A}^{\mathrm{op}}$, that is the objects of $\mathcal{D}$, may be identified with the exact functors on fun-$\mathcal{A}$. By considering the construction of fun-$\mathcal{A}$ from $\mathcal{A}$ (in the proof of 4.3) we see that the functors added are those which express non-exactness, hence on which exact functors are 0. More precisely, applying the functor from $(\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$ to $(\text{mod-}\mathcal{A})^{\mathrm{op}}$ seen in 4.11 followed by that in 3.6 from $(\text{mod-}\mathcal{A})^{\mathrm{op}}$ to $\mathcal{A}^{\mathrm{op}}$ gives an exact functor from fun-$\mathcal{A}$ to $\mathcal{A}^{\mathrm{op}}$ which, note, contains $\mathcal{S}_{\mathcal{D}} = \{F \in \text{fun-}\mathcal{A} : \overrightarrow{F}\mathcal{D} = 0 \text{ i.e. for all } D \in \mathcal{D},\ \mathrm{ev}_D F = 0\}$ in its kernel and hence which factors through the quotient map fun-$\mathcal{A} \longrightarrow \mathrm{fun}(\mathcal{D}) = \text{fun-}\mathcal{A}/\mathcal{S}_{\mathcal{D}}$. Then one checks that this is an equivalence (note that already we had $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathrm{fun}(\mathcal{D})$ *via* $A \mapsto ((-, A), -)_{\mathcal{D}}$). $\square$

$$\begin{tikzcd}
\text{Mod-}\mathcal{A} \\
\text{Flat-}\mathcal{A} \simeq \mathrm{Lex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}) \arrow[u] \\
\mathcal{D} \simeq \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}) \simeq \text{Abs-}\mathcal{A} \arrow[u]
\end{tikzcd}$$

If $\mathcal{D} = \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$ with $\mathcal{A}$ abelian let $\mathcal{C} = \text{Flat-}\mathcal{A} \simeq \mathrm{Lex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$ (3.8) denote the locally coherent category corresponding, in the sense of 3.5, to $\mathcal{A}$, so $\mathcal{A} \simeq \mathcal{C}^{\mathrm{fp}}$. Then, since $\mathrm{Abs}(\mathcal{C}) \simeq \mathrm{Ex}((\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab})$, the above diagram may be re-expressed as follows.

$$\begin{tikzcd}
\text{Mod-}\mathcal{C}^{\mathrm{fp}} \\
\mathcal{C} \simeq \text{Flat-}\mathcal{C}^{\mathrm{fp}} \simeq \mathrm{Lex}((\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab}) \arrow[u] \\
\mathcal{D} \simeq \mathrm{Ex}((\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab}) = \mathrm{Abs}(\mathcal{C}) \arrow[u]
\end{tikzcd}$$

We remark, parenthetically, that, as observed already, since $\text{Mod-}\mathcal{C}^{\mathrm{fp}}$ is locally coherent $\text{Abs-}\mathcal{C}^{\mathrm{fp}}$ also is a definable subcategory of $\text{Mod-}\mathcal{C}^{\mathrm{fp}}$ (5.8), but there is no particular relation between this category, $\mathrm{Abs}(\text{Mod-}\mathcal{C}^{\mathrm{fp}}) \simeq \mathrm{Ex}((\text{mod-}\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab})$, and $\mathcal{D} = \mathrm{Abs}(\mathcal{C}) \simeq \mathrm{Ex}((\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab})$, because the inclusion of $\mathcal{C}^{\mathrm{fp}}$ in $\text{mod-}\mathcal{C}^{\mathrm{fp}}$ is not exact.

dualfuncat

Corollary 11.3. *In the above diagram $\mathcal{C} \simeq \mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$, the dual functor category of $\mathcal{D}$. That is, if $\mathcal{D} \simeq \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}) \simeq \text{Abs-}\mathcal{A}$ with $\mathcal{A}$ abelian then $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}) \simeq \mathrm{Lex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}) \simeq \text{Flat-}\mathcal{A}$ and $\mathrm{fun}^{\mathrm{d}}(\mathcal{D}) \simeq \text{flat-}\mathcal{A} \simeq \mathcal{A}$.*

Proof. (Since we have not yet shown independence of $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$ from the representation of $\mathcal{D}$ as a definable subcategory, $\mathrm{fun}(\mathcal{D})$ should be understood as being defined with respect to the above representation of $\mathcal{D}$ as a definable subcategory.) By 11.2, $\mathcal{C}^{\mathrm{fp}} \simeq \mathrm{fun}^{\mathrm{d}}(\mathcal{D})$, therefore $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}) \simeq \mathrm{Ind}(\mathrm{fun}^{\mathrm{d}}(\mathcal{D})) \simeq \mathrm{Ind}(\mathcal{C}^{\mathrm{fp}}) \simeq \mathcal{C}$. $\square$

The next result therefore follows from 10.10 and 10.11.

tensordualcat

Corollary 11.4. *If $\mathcal{D} = \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$ then $\mathcal{D}^{\mathrm{d}} \simeq \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$.*

As noted after 10.11 there is a tensor functor $\mathcal{D}^{\mathrm{d}} \times \mathcal{D} \longrightarrow \mathbf{Ab}$ given by $(D, D') \mapsto D \otimes_{\mathcal{A}} D'$. The next statement follows from 5.12 and the proof of 10.8.

abseqtens

Corollary 11.5. *In the second diagram after 11.2 the equivalence $\mathcal{D} \simeq \mathrm{Abs}(\mathrm{Fun}(\mathcal{D}^{\mathrm{d}}))$ is given by $M (\in \mathcal{D}) \mapsto (M \otimes_{\mathcal{A}} -)_{\mathcal{D}}$: here $M \otimes_{\mathcal{A}} -$ is the functor in $\mathcal{C} = \mathrm{Fun}^{\mathrm{d}}(\mathcal{D}) = \mathrm{Fun}(\mathcal{D}^{\mathrm{d}})$ given by $L \mapsto M \otimes_{\mathcal{A}} L$ for $L \in (\mathcal{A} = \mathcal{C}^{\mathrm{fp}})\text{-mod}$, and the subscript denotes localisation at the torsion theory on $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$ corresponding to $\mathcal{D}^{(\mathrm{d})}$.*

ppevalpur

Corollary 11.6. *Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence in the definable category $\mathcal{D}$. Then the sequence is pure-exact iff for every $F \in \mathrm{fun}(\mathcal{D})$ the sequence $0 \to FL \to FM \to FN \to 0$ (more precisely, $0 \to \overrightarrow{F}L \to \overrightarrow{F}M \to \overrightarrow{F}N \to 0$) of abelian groups is exact.*

Proof. Evaluation of F at an object $M \in \mathcal{D}$ is equivalent to computing $(dF, (M \otimes -)_{\mathcal{D}}) \simeq (G, M \otimes -)$ where $G \in \text{fun-}\mathcal{A}$ is a pre-image of dF in fun-$\mathcal{A}$ (see after 10.8), so this follows from the above since, by 9.4, $(M \otimes -)_{\mathcal{D}}$ is absolutely pure=fp-injective (5.6). □

We also note the following results on the relation between $\mathcal{D}$ and $\text{fun}(\mathcal{D})$. The first stated as [53, 2.3] (the proof is sketched at [84, 20.1.10]).

injeqfpfun

Proposition 11.7. *If $\mathcal{A}$ is skeletally small abelian and $\mathcal{D} = \text{Ex}(\mathcal{A}^{\text{op}}, \mathbf{Ab})$ then the Yoneda embedding $\mathcal{A} \longrightarrow \text{Mod-}\mathcal{A}$, given by $A \mapsto (-, A)$, induces an equivalence $\text{Inj}(\mathcal{A}) \simeq \mathcal{D}^{\text{fp}}$.*

modexfun

Example 11.8. ([38, 5.5]) Take $\mathcal{D} = \text{Mod-}R \simeq \text{Ex}\big(((R\text{-mod}, \mathbf{Ab})^{\text{fp}})^{\text{op}}, \mathbf{Ab}\big) \leq \text{Mod-}((R\text{-mod}, \mathbf{Ab})^{\text{fp}})^{\text{op}}$. Under the Yoneda correspondence $\text{mod-}R \simeq \text{Inj}((R\text{-mod}, \mathbf{Ab})^{\text{fp}})$ and this is explicitly given by $M \mapsto M \otimes -$ (although $(M \otimes -)$ is only absolutely pure = fp-injective in the full functor category it is injective in the category of finitely presented functors).

finaccdef1

Corollary 11.9. *([53, 2.3]) A definable category $\mathcal{D}$ is finitely accessible iff $\text{fun}^{\text{d}}(\mathcal{D})$ has enough injectives, that is,*
a definable category $\mathcal{D} = \text{Ex}(\mathcal{A}, \mathbf{Ab})$ is finitely accessible iff the abelian category $\mathcal{A}$ has enough projectives.

12 Recovering the definable structure

secrecdef

Is the definable structure on a definable category unique? That is, given a definable category $\mathcal{D}$ is it possible to construct $\text{fun}(\mathcal{D})$ purely from the category-theoretic structure on $\mathcal{D}$, and, if so, how? Without a positive answer to that question it would not even make sense to talk about $\text{fun}(\mathcal{D})$ for definable *categories* as opposed to subcategories since, *a priori*, there is no reason to exclude the possibility of definable subcategories $\mathcal{D}$ and $\mathcal{D}'$ which are equivalent as categories but with $\text{fun}(\mathcal{D})$ not equivalent to $\text{fun}(\mathcal{D}')$. A positive solution is provided by Krause [53]. The further question as to what is an intrinsic definition of "definable category" is, however, not yet answered.

Let $\mathcal{D}$ be a definable category. First recall (from after 5.3) that there is an *intrinsic* theory of purity on $\mathcal{D}$ (which must, therefore, coincide with that induced by any representation of $\mathcal{D}$ as a definable subcategory). Namely an exact sequence in $\mathcal{D}$ is pure-exact iff some ultrapower (which, 21.3, may be chosen uniformly over $\mathcal{D}$) of it is split exact (for ultrapowers see Section 20). Therefore the class, $\text{Pinj}(\mathcal{D})$, of pure-injective objects of $\mathcal{D}$ may be defined intrinsically as consisting of those $D \in \mathcal{D}$ such that every pure monomorphism in $\mathcal{D}$ with domain D is split. This class also may be defined directly, as in [53, §2], using the Jensen-Lenzing test 5.4(iii) for pure-injectivity.

Now we make use of the fact (see the second diagram in §11) that if $\mathcal{D}$ is exactly definable then $\mathcal{D} \simeq \text{Ex}((\mathcal{C}^{\text{fp}})^{\text{op}}, \mathbf{Ab})$ for a locally coherent category $\mathcal{C}$ and, *via* the embedding of $\text{Ex}((\mathcal{C}^{\text{fp}})^{\text{op}}, \mathbf{Ab})$ into $\text{Lex}((\mathcal{C}^{\text{fp}})^{\text{op}}, \mathbf{Ab}) \simeq \mathcal{C}$, one has the identification (11.1) of $\mathcal{D}$ with $\mathrm{Abs}(\mathcal{C})$ and hence of $\mathrm{Pinj}(\mathcal{D})$ with $\mathrm{Inj}(\mathcal{C})$ which is given explicitly (see 11.5) by $D \mapsto (D \otimes_{\mathcal{C}^{\mathrm{fp}}} -)_{\mathcal{D}}$. We consider the Category $(\mathrm{Inj}(\mathcal{C}), \mathbf{Ab})$, but need only a part in which the morphisms between any pair objects form a set rather than a proper class.

funopinj

Lemma 12.1. *[53, 2.7] Suppose that $\mathcal{C}$ is abelian with enough injectives. Then the Yoneda map $\mathcal{C}^{\mathrm{op}} \longrightarrow (\mathrm{Inj}(\mathcal{C}), \mathbf{Ab})^{\mathrm{fp}}$ given by $C \mapsto (C, -) \upharpoonright \mathrm{Inj}(\mathcal{C})$ is an equivalence.*

Proof. If $F \in (\mathrm{Inj}(\mathcal{C}), \mathbf{Ab})^{\mathrm{fp}}$ then there is $f : E \to E'$ in $\mathrm{Inj}(\mathcal{C})$ such that $(E', -) \xrightarrow{(f,-)} (E, -) \to F \to 0$ is exact. Let $K = \ker(f)$ so $0 \to K \to E \xrightarrow{f} E'$ is exact. Since we are evaluating at injectives the induced sequence $(E', -) \to (E, -) \to (K, -) \to 0$ also is exact, hence $F \simeq (K, -)$.

Conversely, if $K \in \mathcal{C}$ then there is an exact sequence $0 \to K \to E \xrightarrow{f} E'$ where E and E' are injective so, as above, $(K, -) = \mathrm{coker}(f, -)$ and $(K, -) \upharpoonright \mathrm{Inj}(\mathcal{C})$ is finitely presented, as required. □

Putting this together with the equivalence $\mathrm{Pinj}(\mathcal{D}) \simeq \mathrm{Inj}(\mathcal{C})$ one obtains a description of $\mathrm{fun}(\mathcal{D})$ from $\mathcal{D}$.

charfundef

Theorem 12.2. *Given a definable category $\mathcal{D} = \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$, where $\mathcal{A}$ is abelian, set $\mathcal{C} = \text{Mod-}\mathcal{A}$. Then $(\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\mathrm{fp}} \simeq (\mathrm{Inj}(\mathcal{C}), \mathbf{Ab})^{\mathrm{fp}} \simeq \mathcal{C}^{\mathrm{op}}$, hence, by 11.3, $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}) \simeq ((\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}}$ and so $\mathrm{fun}^{\mathrm{d}}(\mathcal{D}) \simeq \big(((\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}}\big)^{\mathrm{fp}}$ and $\mathrm{fun}(\mathcal{D}) \simeq \Big(\big(((\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}}\big)^{\mathrm{fp}}\Big)^{\mathrm{op}}$.*

This shows that $\mathrm{fun}(\mathcal{D})$ may, indeed, be recovered, though *via* a few twists and turns, from the category $\mathcal{D}$.

abbijdef

Theorem 12.3. *([53, 2.9]) There is a bijection between small abelian categories $\mathcal{A}$ and definable categories $\mathcal{D}$ (both up to natural equivalence), given by $\mathcal{A} \mapsto \mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$, with inverse $\mathcal{D} \mapsto \big(((\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}}\big)^{\mathrm{fp}}$.*

Indeed, there is an equivalence of suitable 2-categories, see 13.1 and, for complete details, [86].

modRfun

Example 12.4. If R is a right coherent ring then mod-R is abelian, hence is the functor category for some definable category $\mathcal{D}$. By 10.11, $\mathcal{D} \simeq \mathrm{Ex}((\text{mod-}R)^{\mathrm{op}}, \mathbf{Ab})$ which, by 3.9 and 5.7, is Abs-R. Therefore, by 8.2, R-Flat and Abs-R form the dual pair of definable categories with mod-R as associated functor category. Of course one may replace Mod-R here by any locally coherent category.

In view of 22.2 this has a model-theoretic interpretation. Namely, if R is right coherent then the sorts for $(\text{Abs-}R)^{\mathrm{eq}+}$ correspond to finitely presented R-modules, with the elements of $M \in \text{Abs-}R$ of sort $A \in \text{mod-}R$ being the morphisms (A, M). The well-known fact that the theory of Abs-R over a right coherent ring R has complete elimination of quantifiers is immediate from this since if $\overline{a}$ is a finite generating tuple for A and $\theta(\overline{a})$ is a finite generating system of linear equations for the relations on $\overline{a}$, then $(A, M) \simeq \theta(M)$, hence is quantifier-free-definable.

Example 12.5. If R is a ring then (e.g. [84, 10.2.38]) the embedding 5.12 of Mod-R into Fun$^{\mathrm{d}}$-R is an equivalence iff R is von Neumann regular. So this is exactly the case in which fun$^{\mathrm{d}}$-$R \simeq$ mod-R.

These results leave us with the questions: is there a simpler characterisation of fun($\mathcal{D}$)? and is there a category-theoretic characterisation of definable categories? We answer the first but do not have an answer for the second.

For a positive answer to the first question we take another, model-theoretic, route to defining fun($\mathcal{D}$) (equivalently, by 22.1, $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$) directly from $\mathcal{D}$.

We begin by assuming that $\mathcal{D}$ is given as a definable subcategory of a functor category Mod-$\mathcal{A}$. We will work in $(\mathcal{D}, \mathbf{Ab})$: a category in which the morphisms between two objects may, because $\mathcal{D}$ is not skeletally small, form a proper class, so some care will be necessary. For instance, a functor might not be determined by its action on any particular *set* of objects of $\mathcal{D}$ and hence there might not be an epimorphism from a direct sum of a set of representable functors to it. We will say that a functor $G \in (\mathcal{D}, \mathbf{Ab})$ **has a presentation** if there are index sets I, J and an exact sequence $\bigoplus_{j\in J}(E_j, -) \longrightarrow \bigoplus_{i\in I}(D_i, -) \longrightarrow G \to 0$ in $(\mathcal{D}, \mathbf{Ab})$ with the $E_j, D_i \in \mathcal{D}$.

dc1.8

Proposition 12.6. *Suppose that $\mathcal{D}$ is an additive category with products and let $G \in (\mathcal{D}, \mathbf{Ab})$ be a functor with a presentation. Then G is finitely presented (as an object of $(\mathcal{D}, \mathbf{Ab})$) iff G commutes with products.*

Proof. First note that, just by definition of product, each representable functor $(D, -)$ with $D \in \mathcal{D}$ commutes with products. If G is finitely presented then there is a morphism $C \to D$ in $\mathcal{D}$, such that $(D, -) \to (C, -) \to G \to 0$ is exact in $(\mathcal{D}, \mathbf{Ab})$. Given $M = \prod_k M_k$ in $\mathcal{D}$, the sequence $(D, M) \to (C, M) \to GM \to 0$, which is $\prod(D, M_k) \to \prod(C, M_k) \to GM \to 0$, is exact. Also, for each k, the sequence $(D, M_k) \to (C, M_k) \to GM_k \to 0$ is exact so the sequence $\prod(D, M_k) \to \prod(C, M_k) \to \prod GM_k \to 0$ is exact, from which we deduce $GM = \prod GM_k$ (more accurately, $GM \simeq \prod GM_k$ canonically), as required.

For the converse, we are assuming that G commutes with products and that there is a presentation $\bigoplus_j(E_j, -) \longrightarrow \bigoplus_i(D_i, -) \xrightarrow{\pi} G \to 0$ with the E_j, D_i in $\mathcal{D}$. First we show that G is finitely generated.

Let α run over the finite subsets of the set indexing the D_i and set $F_\alpha = \bigoplus_{i\in\alpha}(D_i, -)$ with $i \in \alpha$. So G is the directed sum of the subfunctors $\pi_\alpha F_\alpha$, where π_α is the restriction of π to F_α. Assume, for a contradiction that each inclusion $\pi_\alpha F_\alpha \le G$ is proper, say $D_\alpha \in \mathcal{D}$ is such that $\pi_\alpha F_\alpha \cdot D_\alpha < GD_\alpha$, and choose $a_\alpha \in GD_\alpha \setminus \pi_\alpha F_\alpha D_\alpha$.

Set $a = (a_\alpha)_\alpha \in \prod_\alpha GD_\alpha, = G\prod_\alpha D_\alpha$ since G commutes with products. Now $G = \sum_{\overrightarrow{}}{}_\beta \pi_\beta F_\beta$, so $a \in \sum_{\overrightarrow{}}{}_\beta \pi_\beta F_\beta \cdot \prod_\alpha D_\alpha$ and hence, since the sum is directed, $a \in \pi_\beta F_\beta \cdot \prod_\alpha D_\alpha$ for some β. Although F_β, being representable, preserves products its image $\pi_\beta F_\beta$ might not. However, the obvious maps $(\pi_\beta F_\beta \prod_\alpha D_\alpha \to \pi_\beta F_\beta D_\gamma)_\gamma$ induce a canonical map $\pi_\beta F_\beta \prod_\alpha D_\alpha \to \prod_\alpha \pi_\beta F_\beta D_\alpha$ and the latter group is contained in $\prod_\alpha GD_\alpha = G\prod_\alpha D_\alpha$, so this canonical map is monic (since $\pi_\beta F_\beta \prod_\alpha D_\alpha \to G\prod_\alpha D_\alpha$ is monic) and is just the canonical inclusion of part of the product in the full product. Therefore we may regard a as being in

$\prod_\alpha \pi_\beta F_\beta D_\alpha \leq \prod_\alpha GD_\alpha$. In particular, $a_\beta \in \pi_\beta F_\beta D_\beta$ - contrary to choice of a_β.

Therefore finitely many D_i will do, so G has a presentation of the form $\bigoplus_j (E_j, -) \to (D, -) \to G \to 0$. Now we show that G is finitely related.

Consider $H = \mathrm{im}\big(\bigoplus_j (E_j, -) \to (D, -)\big)$. The sequence $0 \to H \to (D, -) \to G \to 0$ is exact. Given objects M_k of $\mathcal{D}$ there is, for each k, an exact sequence $0 \to HM_k \to (D, M_k) \to GM_k \to 0$, and hence an exact sequence $0 \to \prod_k HM_k \to \prod_k (D, M_k) \to \prod_k GM_k \to 0$. There is also the exact sequence $0 \to H\prod_k M_k \to (D, \prod_k M_k) \to G\prod_k M_k \to 0$. Since both $(D, -)$ and G commute with direct products, we deduce that so does H. The above argument showing that G is finitely generated (and using only that there is an epimorphism from a direct sum of representables to G) therefore applies equally well to show that H is finitely generated and hence that G is finitely presented, as required. □

(The slightly awkward point in the proof above, where we have to take account of the possibility that $\pi_\beta F_\beta$ might not commute with products, is quite well illustrated by considering a product of torsion abelian groups (of unbounded order) and considering the action of the functor which takes a group to its torsion subgroup.)

comprodfinsum

Corollary 12.7. *(of the above proof) Suppose that $\mathcal{D}$ is an additive category with products and that $F : \mathcal{D} \longrightarrow \mathbf{Ab}$ commutes with products and is an image of a direct sum of a set of representables in $(\mathcal{D}, \mathbf{Ab})$. Then F is finitely generated.*

For a category $\mathcal{C}$ denote by $(\mathcal{C}, \mathbf{Ab})^{\prod}$ the category of functors $\mathcal{C} \longrightarrow \mathbf{Ab}$ which commute with direct products.

comprodeqfp2

Corollary 12.8. *If $\mathcal{D}$ is a definable category then $(\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\prod} \simeq (\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}}$.*

Proof. This follows from 12.6 and 12.2 once, in order to apply 12.6, it has been shown that each functor G on $\mathrm{Pinj}(\mathcal{D})$ is determined by its action on a set of pure-injective objects (and hence is an epimorphic image of the direct sum of the corresponding representable functors, and the same for the kernel of this epimorphism). By 21.7 (existence of an elementary cogenerator for $\mathcal{D}$) there is $N \in \mathrm{Pinj}(\mathcal{D})$ such that every $N' \in \mathrm{Pinj}(\mathcal{D})$ is a direct summand of a product of copies of N. Therefore G is determined by its restriction to N, as required. □

Since a product of pure-injectives is pure-injective we have the restriction map from $(\mathcal{D}, \mathbf{Ab})^{\prod}$ to $(\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\prod}$ which, by 12.8, is equivalent to $(\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}}$.

The next lemma summarises, for use in the proof that follows, the actions of the dual pair of functor categories on a definable category (the connection between these actions follows immediately from 4.6).

fpcomprod

Lemma 12.9. *Let $\mathcal{D}$ be represented as a definable subcategory of $\mathrm{Mod}\text{-}\mathcal{A}$; so $\mathcal{D} \simeq \mathrm{Ex}\big((\mathrm{fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}}, \mathbf{Ab}\big)$, where $\mathrm{fun}^{\mathrm{d}}(\mathcal{D}) \simeq (\mathcal{A}\text{-mod}, \mathbf{Ab})_{\mathcal{D}}$, the action being given by $(G, D \otimes -) \simeq \big(G_{\mathcal{D}}, (D \otimes -)_{\mathcal{D}}\big)$ for $G \in \mathrm{fun}^{\mathrm{d}}(\mathcal{D})$ and $D \in \mathcal{D}$. Then the action of $\mathrm{fun}(\mathcal{D})$ on $\mathcal{D}$ induced by this action and by the duality $\mathrm{fun}(\mathcal{D}) \simeq (\mathrm{fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}}$ is given for $F \in \mathrm{fun}(\mathcal{D})$ and $D \in \mathcal{D}$ by $\overrightarrow{F}D$.*

The next result was obtained by Krause [53, 7.2] in the case where $\mathcal{D}$ is a finitely accessible category. In that case one has a generating set of finitely presented objects and a considerably simpler proof may be given.

finaccfun

Theorem 12.10. *Let $\mathcal{D}$ be a definable category. Then* $\mathrm{fun}(\mathcal{D}) \simeq (\mathcal{D}, \mathbf{Ab})^{\to\prod}$*: the category of those functors from $\mathcal{D}$ to* $\mathbf{Ab}$ *which commute with direct limits (equivalently filtered colimits) and products.*

Proof. The action of $F \in \mathrm{fun}(\mathcal{D})$ on $\mathcal{D}$ is as described in 12.9. Just from its definition, the functor $\overrightarrow{F}$ commutes with direct limits and it commutes with products, cf. 12.6, since if $(B,-) \longrightarrow (A,-) \longrightarrow F \longrightarrow 0$ is a presentation of F then this, read in $(\text{Mod-}\mathcal{A}, \mathbf{Ab})$, is a presentation of $\overrightarrow{F}$. Note that localisation at the torsion theory, $\tau_{\mathcal{D}}$, corresponding to $\mathcal{D}$ just corresponds to restriction of the action to $\mathcal{D}$. (Model-theoretically this direction is clear by 22.2 and since evaluation of pp formulas is easily seen to commute with products and direct limits.)

Thus every functor in $\mathrm{fun}(\mathcal{D})$ commutes with products and direct limits in its action on $\mathcal{D}$. We must prove the converse.

First, we consider the equivalence $(\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\prod} \simeq (\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}}$ from 12.8. From 11.5 the equivalence of the first with $(\mathrm{Inj}(\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})), \mathbf{Ab})^{\prod}$ takes a representable functor $(N,-)$ $(N \in \mathrm{Pinj}(\mathcal{D}))$ to the functor $((N \otimes -)_{\mathcal{D}}, -)$ which, under the equivalence, 12.1, of $(\mathrm{Inj}(\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})), \mathbf{Ab})$ with $(\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}}$, is mapped to $((N \otimes -)_{\mathcal{D}})^{\mathrm{o}} \in (\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{o}}$ (superscript $^{\mathrm{o}}$ signals the opposite category). Any functor $G \in (\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\prod} = (\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\mathrm{fp}}$ has, see the proof of 12.1, a presentation of the form

$$(N',-) \xrightarrow{(f,-)} (N,-) \to G \to 0$$

for some $f : N \to N'$ in $\mathrm{Pinj}(\mathcal{D})$. This maps under the equivalence (and the replacement of $(\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}))^{\mathrm{op}}$ by $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$) to the exact sequence

$$0 \to G' = \ker((f \otimes -)_{\mathcal{D}}) \longrightarrow (N \otimes -)_{\mathcal{D}} \xrightarrow{(f\otimes -)_{\mathcal{D}}} (N' \otimes -)_{\mathcal{D}}.$$

That is, if $G = \mathrm{coker}(f,-)$ then the corresponding functor G' in $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$ is $(\ker(f \otimes -))_{\mathcal{D}}$ $(= \ker((f \otimes -)_{\mathcal{D}})$ since localisation is exact).

Conversely, starting with a functor H in $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$, take an exact sequence

$$0 \to H \to E(H) \to E\big(E(H)/H\big),$$

where E denotes injective hull, that is (by 9.4),

$$0 \to H \to (N \otimes -)_{\mathcal{D}} \xrightarrow{(f\otimes -)_{\mathcal{D}}} (N' \otimes -)_{\mathcal{D}}$$

for some $f : N \to N'$ in $\mathrm{Pinj}(\mathcal{D})$, so the corresponding functor in $(\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})$ is $\mathrm{coker}(f,-)$.

Each functor $H \in \mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$ acquires an action on objects of $\mathrm{Pinj}(\mathcal{D})$ *via* this equivalence together with the natural action of functors in $(\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})$ on

$\mathrm{Pinj}(\mathcal{D})$. We check that this is just the action in 12.9. Given H and an exact sequence

$$0 \to H \to (N \otimes -)_{\mathcal{D}} \xrightarrow{(f\otimes -)_{\mathcal{D}}} (N' \otimes -)_{\mathcal{D}}$$

as above, one has the corresponding exact sequence $(N', -) \xrightarrow{(f,-)} (N, -) \to H' \to 0$ of functors on $\mathrm{Pinj}(\mathcal{D})$ so, if $N'' \in \mathrm{Pinj}(\mathcal{D})$, then one has the exact sequence

$$(N', N'') \xrightarrow{(f,N'')} (N, N'') \to H'N'' \to 0.$$

Compare this with the exact sequence induced by $(-, (N''\otimes -)_{\mathcal{D}})$ on the original sequence in $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$, that is

$$((N'\otimes -)_{\mathcal{D}}, (N''\otimes -)_{\mathcal{D}}) \xrightarrow{((f\otimes -)_{\mathcal{D}},1)} ((N\otimes -)_{\mathcal{D}}, (N''\otimes -)_{\mathcal{D}}) \to (H, (N''\otimes -)_{\mathcal{D}}) \to 0$$

(this is exact since $(N'' \otimes -)_{\mathcal{D}}$ is injective in $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$), equivalently (since $\mathrm{Pinj}(\mathcal{D}) \longrightarrow \mathrm{Inj}(\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}))$ is an equivalence, hence full)

$$(N', N'') \xrightarrow{(f,N'')} (N, N'') \to (H, (N'' \otimes -)_{\mathcal{D}}) \to 0.$$

It follows that the action of H on $\mathrm{Pinj}(\mathcal{D})$ is indeed that of $(H, -) \upharpoonright \mathrm{Inj}(\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}))$.

Suppose now that $G \in (\mathcal{D}, \mathbf{Ab})^{\prod\rightarrow}$, so $G \upharpoonright \mathrm{Pinj}(\mathcal{D}) \in (\mathrm{Pinj}(\mathcal{D}), \mathbf{Ab})^{\prod}$. Let $G' = \ker(f \otimes -)_{\mathcal{D}}$ where $G \upharpoonright \mathrm{Pinj}(\mathcal{D}) = \mathrm{coker}(f, -)$, be, with notation as above, the corresponding object of $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$. Now, $G' = \sum\limits_{\rightarrow} H_\lambda$ for some finitely generated subobjects H_λ. The action restricted to pure-injectives commutes with products so we may apply (the argument of) 12.6 to obtain that, for some λ, $G' = H_\lambda$ on $\mathrm{Pinj}(\mathcal{D})$. In detail: otherwise choose, for each λ, some $N_\lambda \in \mathrm{Pinj}(\mathcal{D})$ and some $a_\lambda \in GN_\lambda \setminus H_\lambda N_\lambda$. Set $N = \prod_\lambda N_\lambda$, so $a = (a_\lambda)_\lambda \in G'N = \sum\limits_{\rightarrow}{}_\lambda H_\lambda N$ is in $H_\mu N \leq \prod_\lambda H_\mu N_\lambda$ for some μ. But then $a_\mu \in H_\mu N_\mu$, contradiction.

The functor $G' = H_\lambda$, being a finitely generated object of $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$, has, by 22.3, the form $(F_{D\psi}/F_{Dp})_{\mathcal{D}}$ for some pp formula ψ and pp-type p (for pp-types and formulas see §18). Therefore, by the description of the actions above and 23.5, the action on pure-injectives of the original functor G back in $(\mathcal{D}, \mathbf{Ab})^{\rightarrow\prod}$ is $GN = p(N)/\psi(N)$. If this functor were not finitely generated then this would contradict 21.11 since $G \upharpoonright \mathrm{Pinj}(\mathcal{D})$, that is $(p/\psi) \upharpoonright \mathrm{Pinj}(\mathcal{D})$, commutes with direct limits (where defined) and products, hence with those ultraproducts of pure-injectives where the ultraproduct itself is pure-injective.

Therefore, restricted to $\mathrm{Pinj}(\mathcal{D})$, $G = F_{\phi/\psi}$ for some pp formula $\phi \geq \psi$. It remains to show that G and $F_{\phi/\psi}$ agree, not just on $\mathrm{Pinj}(\mathcal{D})$ but on all $\mathcal{D}$.

Suppose, therefore, that $G \in (\mathcal{D}, \mathbf{Ab})^{\rightarrow\prod}$ is such that $G \upharpoonright \mathrm{Pinj}(\mathcal{D}) = \overrightarrow{F} \upharpoonright \mathrm{Pinj}(\mathcal{D})$ for some $F \in \mathrm{fun}(\mathcal{D})$ (we will write F in place of $\overrightarrow{F}$ for simplicity). We show that G and F agree on all of $\mathcal{D}$. Let us be more precise: what we have is a natural isomorphism $\eta : G \upharpoonright \mathrm{Pinj}(\mathcal{D}) \longrightarrow F \upharpoonright \mathrm{Pinj}(\mathcal{D})$. Since both functors commute with products and direct limits on $\mathcal{D}$, they commute with ultraproducts. Let $M \in \mathcal{D}$. By 21.3, for some index set J and ultrafilter $\mathcal{F}$ on J, the ultrapower $M^J/\mathcal{F}$ is pure-injective, so $G(M^J/\mathcal{F}) \simeq F(M^J/\mathcal{F}) =$

$(FM)^J/\mathcal{F}$, the isomorphism being the component of η at $G(M^J/\mathcal{F})$. Choose a pure embedding $i : M \to N$ with $N \in \mathrm{Pinj}(\mathcal{D})$. We can suppose that J, $\mathcal{F}$ are such that $N^J/\mathcal{F}$ also is pure-injective. Consider the diagram shown, where Δ is the diagonal embedding of a structure into an ultraproduct (see §20) and where the vertical arrows are canonical inclusions, respectively induced by i.

$$\begin{tikzcd}
 & G(N^J/\mathcal{F}) = (GN)^J/\mathcal{F} \arrow[r, "\eta"] & (FN)^J/\mathcal{F} & \\
GN \arrow[ur, "\Delta_{GN}"] \arrow[rrr, "\eta"] & & & FN \arrow[ul, "\Delta_{FN}"'] \\
 & G(M^J/\mathcal{F}) \arrow[uu] \arrow[r, "\eta"] & F(M^J)/\mathcal{F} \arrow[uu] & \\
GM \arrow[uu] \arrow[ur, "\Delta_{GM}"'] & & & FM \arrow[uu] \arrow[ul, "\Delta_{FM}"]
\end{tikzcd}$$

By construction of ultraproducts, we have $\mathrm{im}(\Delta_{FN}) \cap \big((FM)^J/\mathcal{F}\big) = \mathrm{im}(\Delta_{FM})$, therefore the image of $\Delta_{GM}(GM)$ under η lies in the image of $\Delta(FM)$, which is naturally isomorphic to FM. This allows us to define an isomorphism $GM \simeq FM$ and thus η extends to a natural isomorphism from G to F on all of $\mathcal{D}$, as required. $\square$

Thus, if $\mathcal{D}$ is a definable category and $\mathrm{fun}(\mathcal{D})$ is the corresponding finitely presented functor category then evaluation gives the action of each of the other: fixing $D \in \mathcal{D}$, evaluation at D, $F \mapsto \mathrm{ev}_D F$ gives an exact functor from $\mathrm{fun}(\mathcal{D})$ to $\mathbf{Ab}$ (10.9) and, given $F \in \mathrm{fun}(\mathcal{D})$, application, $D \mapsto \overrightarrow{F} D$, commutes with direct limits and products (12.9). From now on we will usually write FD instead of $\overrightarrow{F} D$ for this action.

13 Functors between definable categories

secfun

There are three "natural" types of functors between definable categories: those which preserve products and direct limits (and hence also pure-exact sequences, see 13.3); those which are induced by exact functors between the respective finitely presented functor categories, and the interpretation functors from model theory (for these see Section 25). Using the equivalence between the finitely presented functor category and the model-theoretic imaginaries category (22.2), the latter two are seen to be in natural bijection (25.1) and, from the characterisation of the functor category in 12.10, it follows that all three types of functor coincide (we state this as 13.2).

Let $\mathcal{C}$ and $\mathcal{D}$ be definable categories. If $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{C})$ is an exact functor then I_0 induces, by composition, a functor $I = (I_0)^* : \mathcal{C} = \mathrm{Ex}(\mathrm{fun}(\mathcal{C}), \mathbf{Ab}) \longrightarrow \mathcal{D} = \mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathbf{Ab})$. So if $G \in \mathrm{fun}(\mathcal{D})$ and $C \in \mathcal{C}$ then we have (by 10.9 and 12.9) $G(I_0^* C) = \mathrm{ev}_{I_0^* C} G = \mathrm{ev}_C(I_0 G) = I_0 G \cdot C$ $(*)$.

exfundef1

Theorem 13.1. *(the first part in [53, 7.2]) Let* $\mathcal{C}$, $\mathcal{D}$ *be definable categories. If* $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{C})$ *is an exact functor then* $(I_0)^* : \mathcal{C} \longrightarrow \mathcal{D}$ *commutes with direct limits and products.*

Conversely, if $I : \mathcal{C} \longrightarrow \mathcal{D}$ *commutes with direct limits and products then it induces an exact functor* $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{C})$.

These processes are inverse: $(I_0)^*$ *is naturally equivalent to* I *and* $((I_0)^*)_0$ *is naturally equivalent to* I_0.

Proof. Given I_0 let $\{C_\lambda\}_\lambda$ be a set of objects in $\mathcal{C}$. Then $I_0^*(\prod_\lambda C_\lambda)$ is the functor from $\mathcal{D}$ to **Ab** which takes G to $G(I_0^* \prod_\lambda C_\lambda) = I_0G(\prod_\lambda C_\lambda) = \prod_\lambda I_0GC_\lambda$ (since all functors in $\mathrm{fun}(\mathcal{C})$ commute with products, 12.9) and this is the value of $\prod_\lambda I_0^*C_\lambda$ on G. Similarly for direct limits.

The second statement is immediate from 12.10.

For the last statement we have $((I_0)^*)_0 : G \mapsto (C \mapsto G(I_0^*C) = I_0G \cdot C)$, which is the action of I_0 and $(I_0)^* : C \mapsto (G \mapsto I_0G \cdot C = G \cdot IC)$ which is the action of I. □

exfundual2

Corollary 13.2. *([53, 7.2] for the case where* $\mathcal{C}$ *is locally finitely presented) Let* $\mathcal{C}$, $\mathcal{D}$ *be definable categories. Then a functor from* $\mathcal{C}$ *to* $\mathcal{D}$ *commutes with direct limits and products iff it has the form* I_0^* *for some exact functor* I_0 *from* $\mathrm{fun}(\mathcal{D})$ *to* $\mathrm{fun}(\mathcal{C})$.

A functor $I : \mathcal{C} \longrightarrow \mathcal{D}$, between definable categories, which is of the form I_0^* we will refer to as a **definable** functor. This terminology is in agreement with that used in [53] and, as mentioned above, it does turn out to be equivalent to being "definable" (more precisely, to being an interpretation functor) in a model-theoretic sense (see Section 25, 25.3 in particular).

The next couple of results can be found variously in [53, §7] (in full generality) in [46, esp. 7.35] and, said in terms of interpretation functors, in [81, §3].

exfundef2

Proposition 13.3. *Let* $\mathcal{C}$, $\mathcal{D}$ *be definable categories and let* $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{C})$ *be exact. Then* $I = I_0^* : \mathcal{C} \longrightarrow \mathcal{D}$ *preserves pure exact sequences and pure-injectivity. Also* $\ker(I) = \{C \in \mathcal{C} : IC = 0\}$ *is a definable subcategory of* $\mathcal{C}$*. Indeed, for any definable subcategory,* $\mathcal{D}'$*, of* $\mathcal{D}$ *the inverse image,* $I^{-1}\mathcal{D}'$*, is a definable subcategory of* $\mathcal{C}$.

Proof. If $0 \to A \to B \to C \to 0$ is a pure exact sequence in $\mathcal{C}$ then, by 11.6, for any $F \in \mathrm{fun}(\mathcal{C})$ the sequence $0 \to FA \to FB \to FC \to 0$ is exact. Apply this to functors of the form $F = I_0G$ for $G \in \mathrm{fun}(\mathcal{D})$. By $(*)$ and 13.1 we have $G(IC') = I_0G.C'$ for $C' \in \mathcal{C}$ so it follows, using 11.6 again, that the image under I of the sequence is pure exact.

The fact that I preserves pure-injectivity is direct from the Jensen-Lenzing criterion, 5.4(iii), and 13.1. Also by 13.1 the last statement will follow once it has been shown that $I^{-1}(\mathcal{D}')$ is closed under pure subobjects. So suppose that B is a pure subobject of $C \in \mathcal{C}$ where $IC \in \mathcal{D}'$. Some ultrapower, $B^J/\mathcal{U} \to C^J/\mathcal{U}$, of the inclusion is split (21.3) and, since I commutes with ultrapowers (for it commutes with products and direct limits), it follows that $(IB)^J/\mathcal{U}$ is a direct summand of $(IC)^J/\mathcal{U}$, which is in $\mathcal{D}'$. Therefore $(IB)^J/\mathcal{U} \in \mathcal{D}'$. But every object is a pure subobject of each of its ultrapowers (20.2), so $IB \in \mathcal{D}'$, as required.

(An alternative argument for the first part uses the fact that a sequence is pure-exact iff some ultrapower of it is split.) □

prespurpi

Corollary 13.4. *Let $\mathcal{C}$, $\mathcal{D}$ be definable categories and let $I : \mathcal{C} \longrightarrow \mathcal{D}$ commute with products and direct limits. Then I preserves pure exact sequences and pure-injectivity.*

The image of a definable category $\mathcal{C}$ under a functor which preserves products and direct limits need not be a definable subcategory of $\mathcal{D}$ but it is easily checked that the closure, $\mathcal{D}'$ say, of the image under pure subobjects in $\mathcal{D}$ will be definable and then the corresponding morphism $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{C})$ factors through the canonical projection from $\mathrm{fun}(\mathcal{D})$ to $\mathrm{fun}(\mathcal{D}')$.

imnotdef

Example 13.5. Let R be the path algebra over a field k of the quiver A_2 (see 3.1). Let $\mathcal{D}$ be the (definable) subcategory of Mod-$\mathcal{A}$ consisting of those R-modules M in which the map corresponding to the arrow of the quiver is an isomorphism. Note that such a module must be even-dimensional over k. Let I : Mod-$R \longrightarrow$ Mod-k be the forgetful functor. Clearly this commutes with direct limits and products but its image is not definable; to obtain a definable subcategory of Mod-k one must close under direct summands (and then one obtains all of Mod-k).

Assuming that I is full, at least on $\mathrm{Pinj}(\mathcal{C})$, gives stronger results. Say that a functor $I : \mathcal{C} \longrightarrow \mathcal{D}$ between definable categories is **full on pure-injectives** if, restricted to $\mathrm{Pinj}(\mathcal{C})$, it is full (this is equivalent, see 25.9, to the model-theoretic notion of preserving all induced structure). Clearly it is enough to assume that for every $N \in \mathrm{Pinj}(\mathcal{C})$ the induced map $\mathrm{End}(N) \to \mathrm{End}(IN)$ is surjective.

It will be shown at 25.3 that $I : \mathcal{C} \to \mathcal{D}$ is a functor between definable categories which commutes with products and direct limits iff I is an interpretation functor. This notion from model theory will be defined in Section 25. The condition that I be full on pure-injectives is equivalent (25.9) to I preserving all induced structure (this is defined in §25) and, in this case, if $I_0 : \mathrm{fun}(\mathcal{D}) \to \mathrm{fun}(\mathcal{C})$ is the corresponding exact functor between the associated functor categories, then I_0 will be full (but that is not enough to give full on pure-injectives, see 25.11).

Note that, given an exact functor, $I_0 : \mathrm{fun}(\mathcal{D}) \to \mathrm{fun}(\mathcal{C})$, if we take the opposite functor between the opposite categories then, since $(\mathrm{fun}(\mathcal{D}))^{\mathrm{op}} \simeq \mathrm{fun}(\mathcal{D}^{\mathrm{d}})$, we obtain what we will write as $I_0^{\mathrm{d}} : \mathrm{fun}(\mathcal{D}^{\mathrm{d}}) \longrightarrow \mathrm{fun}(\mathcal{C}^{\mathrm{d}})$. This also will be exact, hence will induce a functor $I : \mathcal{C}^{\mathrm{d}} \longrightarrow \mathcal{D}^{\mathrm{d}}$ which commutes with direct limits and products, and I_0^{d} will be full iff I_0 is.

Note also that any functor $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{C})$ has a unique extension to a functor $\overrightarrow{I_0} : \mathrm{Fun}(\mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C})$ by, for example, [54, 5.6]. If I_0 is full then so is $\overrightarrow{I_0}$. For let $H = \varinjlim_\lambda H_\lambda$ and $H' = \varinjlim_\mu H'_\mu$ be functors in $\mathrm{Fun}(\mathcal{D})$ represented as direct limits of finitely presented functors. Denote by $f_{\lambda\infty}, f'_{\mu\infty}$

the canonical maps to the respective direct limits. Let $g : I_0H \longrightarrow I_0H'$. For each λ, μ let $(H_\lambda, H'_\mu)_{g'}$ be the set of morphisms g' from H_λ to H'_μ such that $gI_0(f_{\lambda\infty}) = I_0(f'_{\mu\infty})Ig'$. Note that, since the I_0H_λ are finitely presented and since I_0 is full, for each λ there is some μ such that $(H_\lambda, H'_\mu)_{g'}$ is non-empty. The original directed structures induce an obvious filtered structure on these sets and, by the observation just made, there is an induced morphism $g : H \longrightarrow H'$ which is such that $\overrightarrow{I_0}g = g'$, showing fullness.

Corollary 13.6. *[81, 3.16, 3.17] Suppose that the functor $I : \mathcal{C} \longrightarrow \mathcal{D}$ between definable categories commutes with products and direct limits and is full on pure-injectives. Then I preserves indecomposability of pure-injectives in the weak sense that if $N \in \mathcal{C}$ is indecomposable pure-injective then either $IN = 0$ or IN is indecomposable pure-injective. Also, for every $C \in \mathcal{C}$, one has $IH(C) = H(IC)$ where H denotes pure-injective hull.*

Proof. That I (weakly) preserves indecomposability is immediate from that fact that it is full. For the statement about pure-injective hulls note, by 13.4, that IC is pure in $IH(C)$, which is pure-injective, so $H(IC)$ is a direct summand of $IH(C)$. If it were a proper direct summand then there would, by fullness, be a proper direct summand of $H(C)$ strictly between C and $H(C)$, contradicting the definition of pure-injective hull. $\square$

For further consequences of this fullness condition see Section 15.

The proof of the next result also uses model theory, perhaps in a more essential way, in the sense that it uses a result of model theory (existence of locally atomic models, see, for example, the references given at [81, p. 200]) which appears not to translate in any natural way to something algebraic.

Theorem 13.7. *[81, 3.8] Suppose that the functor $I : \mathcal{C} \longrightarrow \mathcal{D}$ between definable categories commutes with products and direct limits and is full on pure-injectives. Suppose that* $\mathrm{fun}(\mathcal{C})$ *has just countably many objects up to isomorphism. Then for every definable subcategory $\mathcal{C}'$ of $\mathcal{C}$ the image $I\mathcal{C}'$ is a definable subcategory of $\mathcal{D}$.*

An example of Herzog (see [81, 3.9]) shows that this result is not true without some restriction. His example uses a commutative valuation domain which has a non-standard uniserial module (existence of such rings is a result of Fuchs and Shelah [27]).

14 Spectra of definable categories

secspec

Let $\mathcal{D}$ be a definable category. The **Ziegler spectrum**, $\mathrm{Zg}(\mathcal{D})$, of $\mathcal{D}$ is the topological space whose points are the isomorphism classes of indecomposable pure-injectives objects of $\mathcal{D}$ and with a basis of open sets consisting of those sets of the form $(F) = \{N \in \mathrm{Zg}(\mathcal{D}) : \overrightarrow{F}N \neq 0\}$ where F ranges over $\mathrm{fun}(\mathcal{D})$.

The basic results on this space were proved by Ziegler [105] in the context of definable subcategories of module categories. These were extended by various people, see Burke [11], Herzog [41], Krause [52], Prest [79], to more general contexts. The model-theoretic techniques used by Ziegler generally work just as well in the wider contexts (the main difference being that one has to use multi-sorted languages) so model-theoretic proofs were usually not written down, at least, not typed up, again (being "really just the same"). More work was required to produce proofs which use the language of functor categories, either by translating Ziegler's proofs or by working from scratch (and taking a fresh view often produced new results as well as new, and sometimes better, proofs). The "functorial translation" uses in an essential way the fundamental results of Gruson and Jensen [36], [37] and, for extending beyond the abelian case, Crawley-Boevey [19].

The Ziegler spectrum is the main focus of the book [84] but there I do not say much about model theory. In contrast, one of my main aims in these notes is an exposition of model theory in the additive context (after enough, actually rather a lot of, "algebraic/categorical" material has been developed).

zgfunbasis

Theorem 14.1. *([105, 4.9] in the context of modules) The sets of the form* (F), $F \in \mathrm{fun}(\mathcal{D})$*, form a basis for a topology. Each basic open set* (F) *is compact and these are exactly the compact open subsets of* $\mathrm{Zg}(\mathcal{D})$*.*

In contrast with the case where $\mathcal{D}$ is the category of R-modules, the whole space need not be basic open, hence need not be compact. For example if k is a field and $\mathcal{A}$ is a preadditive category with infinitely many objects A_i, $i \in \mathbb{N}$ such that $\mathrm{End}(A_i) = k$ for all k and $(A_i, A_j) = 0$ for all $i \neq j$ then it is clear that $\mathrm{Zg}(\text{Mod-}\mathcal{A})$ is not compact.

corresp

Theorem 14.2. *(based on [105, 4.10] but extended and developed quite a bit since) Given a definable category* $\mathcal{D}$ *there are natural bijections between:*
(i) definable subcategories $\mathcal{D}'$ *of* $\mathcal{D}$*;*
(i)′ definable subcategories of $\mathcal{D}^{\mathrm{d}}$*;*
(ii) closed subsets X *of* $\mathrm{Zg}(\mathcal{D})$*;*
(ii)′ closed subsets of $\mathrm{Zg}(\mathcal{D}^{\mathrm{d}})$*;*
(iii) Serre subcategories $\mathcal{S}$ *of* $\mathrm{fun}(\mathcal{D})$*;*
(iii)′ Serre subcategories $\mathcal{S}'$ *of* $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$*.*

Many of the direct correspondences already have been described after 8.1; here are some more.
(i)→(ii) $X = \mathcal{D}' \cap \mathrm{Zg}(\mathcal{D})$ - the set of (isomorphism classes of) indecomposable pure-injectives in $\mathcal{D}'$;
(ii)→(i) $\mathcal{D}'$ is the closure of X under products, direct limits and pure subobjects;
(ii)→(iii) $\mathcal{S} = \{F \in \mathrm{fun}(\mathcal{D}) : \overrightarrow{F} N = 0 \text{ for all } N \in X\}$;
(ii)→(iii)′ $\mathcal{S}' = \{G \in \mathrm{fun}^{\mathrm{d}}(\mathcal{D}) : (G, N \otimes -) = 0 \text{ for all } N \in X\}$ (since the corresponding hereditary torsion theory on $\mathrm{Fun}(\mathcal{D})$ is of finite type it is determined by the indecomposable torsionfree injective objects $N \otimes -$);
(iii)→(ii) X is the set of $N \in \mathrm{Zg}(\mathcal{D})$ such that $\overrightarrow{F} N = 0$ for all $F \in \mathcal{S}$;
(iii)′→(ii) X is the set of $N \in \mathrm{Zg}(\mathcal{D})$ such that $N \otimes -$ is an indecomposable injective which is torsionfree for the hereditary torsion theory (of finite type) generated by $\mathcal{S}'$.

There is another natural topology on the same set of points, namely the **Gabriel-Zariski spectrum**, $\mathrm{Zar}(\mathcal{D})$, of $\mathcal{D}$. This has, for a basis of open sets, the complements of the compact open sets of $\mathrm{Zar}(\mathcal{D})$; we use the notation $[F] = \{N \in \mathrm{Zg}(\mathcal{D}) : \overrightarrow{F}N = 0\}$ where $F \in \mathrm{fun}(\mathcal{D})$ for these basic Gabriel-Zariski-open sets. The terminology is explained in [78, p. 200 ff.] or, in more detail, in [82] (also see [84, Chpt. 14]): for example, $\mathrm{Zar}(\text{Mod-}R)$ is a direct generalisation of the Zariski spectrum of a commutative noetherian ring but applied with (the "ring with many objects") mod-R in place of R.

There is a natural presheaf, $\mathrm{Def}(\mathcal{D})$, of small abelian categories over $\mathrm{Zar}(\mathcal{D})$, which, given $F \in \mathrm{fun}(\mathcal{D})$, assigns the ("finite") localisation $\mathrm{fun}(\mathcal{D})/\langle F\rangle$ to the basic Gabriel-Zariski-open set $[F]$. Here $\langle F\rangle$ denotes the Serre subcategory generated by F. This is seldom a sheaf and we denote its sheafification $\mathrm{LDef}(\mathcal{D})$, referring to it as the sheaf of **locally definable scalars** (**in all sorts**). The terminology arose originally from the case where $\mathcal{D}$ is Mod-R and where we take only a small part of this (pre-)sheaf of categories, namely the endomorphism ring (*viz.* R) of the forgetful functor $(R_R, -)$ and of its various localisations at Serre subcategories of fun-R. For then the ring $\mathrm{End}((R,-)_{\langle F\rangle})$ has a natural interpretation as the ring of definable functions ("definable scalars") on the modules in the definable subcategory $\{M \in \text{Mod-}R : \overrightarrow{F}M = 0\}$ of modules annihilated by F (see Section 22, [82], [84, §12.8] for more on this).

In the next section we consider the effect of definable functors on these spectra.

15 Definable functors and spectra

secdeffunspec

Proposition 15.1. *([80, Prop. 2] for a special case) Suppose that $I : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor between definable categories which commutes with products and direct limits. Let $\mathcal{C}'$ be a definable subcategory of $\mathcal{C}$ and suppose that N is an indecomposable pure-injective direct summand of IC for some $C \in \mathcal{C}'$. Then there is $N' \in \mathrm{Zg}(\mathcal{C}')$ such that N is a direct summand of IN'.* inddirsum

Proof. The model-theoretic proof in [80, Prop. 2] works just as well in the general case, alternatively follow the functorial version of this given in [53, 7.6]. Here we adapt the proof of [80, Prop. 2] to this context. We get away with the weaker (than in the original result) hypothesis because of 13.2.

Let a be a non-zero element of N, say $a \in ((-,B), N)$ is of sort $(-,B)$ (where $\mathcal{D}$ is a definable subcategory of Mod-$\mathcal{B}$ and $B \in \mathcal{B}$). Set $p = \mathrm{pp}^N(a)$ to be the pp-type of a in N (see Section 18). Consider the map $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{C})$ such that $I = I_0^*$ (existence by 13.2). Identifying these functor categories with the imaginaries categories (22.2), the image under I_0 of the sort $(-,B)$ is a sort, σ say, of $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ and the image of the pp-type p under the induced map (see after 25.3) from $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$ to $\mathcal{L}(\mathcal{C})^{\mathrm{eq}+}$ is a pp-type, p_1 say, of sort σ. Also set Ψ to be the image under I_0 of $p^- = \{\psi \in \mathcal{L}(\mathcal{D})^{\mathrm{eq}+} : \psi$ is pp of sort $(-,B)$ and $\psi \notin p\}$.

Consider the lattice of pp formulas of $\mathcal{L}(\mathcal{C})^{\mathrm{eq}+}$ of sort σ. The filter generated by p_1 and the ideal generated by Ψ have empty intersection since, regarding a

as an element, a', of C of sort σ (see after 25.6), a' satisfies each formula in the filter and none in the ideal (note that, by 18.3, $\mathrm{pp}^{IC}(a) = \mathrm{pp}^N(a)$ since N is pure in IC).

The necessary adaptations having been made, the proof now continues as in [80, Prop. 2]. Let q be a pp-type of sort σ maximal with respect to containing p_1 and missing Ψ. We check Ziegler's criterion (24.3) to show that q is irreducible.

So let η_1, η_2 be pp formulas of sort σ which are not in q. By maximality of q there are $\phi_1, \phi_2 \in p_1$ and $\psi_1, \psi_2 \in \Psi$ such that $\eta_i \wedge \phi_i \leq \psi_i$ $(i = 1, 2)$. Without loss of generality $\phi_1 = \phi_2 = \phi$ say (replace each by $\phi_1 \wedge \phi_2$). Let $\phi' \in p$ and $\psi_1', \psi_2' \in p^-$ be such that $\phi = I_0\phi'$ and $\psi_i = I_0\psi_i'$ $(i = 1, 2)$ (notation as after 25.4). Since p is irreducible, by Ziegler's criterion there is $\phi_0 \in p$ with $\phi_0 \leq \phi'$ such that $\phi_0 \wedge \psi_1' + \phi_0 \wedge \psi_2' \notin p$ and hence such that $(I_0\phi_0) \wedge \psi_1 + (I_0\phi_0) \wedge \psi_2 \in \Psi$ (since the action of I_0 on formulas commutes with $\wedge$ and $+$). Since $I_0\phi_0 \in p_1 \subseteq q$ this is as required for 24.3, so q is irreducible.

Therefore, see 24.1, there is an indecomposable pure-injective object $N' \in \mathcal{C}$ and an element b' of $\sigma(N')$ such that $\mathrm{pp}^{N'}(b') = q$. Consider $IN' \in \mathcal{D}$ and the element, b say, of IN' of sort $(-, B)$ such that b in IN' corresponds, in the sense of 25.6, to b' in N'. Clearly $\mathrm{pp}^{IN'}(b) = p$. Since N is indecomposable it is the hull, $H(p)$, of p and hence, by 24.2, N is isomorphic to a direct summand of IN', as required. $\square$

interpctscl

Proposition 15.2. *Suppose that $I : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor between definable categories which commutes with products and direct limits. Then I induces a closed and continuous map from $\mathrm{Zg}(\mathcal{C})$ to $\mathrm{Zg}(\mathcal{D})$ at the level of topology: that is, it induces a map I_*, from the lattice of closed sets of $\mathrm{Zg}(\mathcal{C})$ to that of $\mathrm{Zg}(\mathcal{D})$ which preserves finite union and arbitrary intersection and it induces a map, I^{-1}, from the lattice (in fact, the complete Heyting algebra) of open subsets of $\mathrm{Zg}(\mathcal{D})$ to that of $\mathrm{Zg}(\mathcal{C})$ which preserves arbitrary union and finite intersection.*

Proof. Let $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{C})$ be the corresponding (in the sense of 13.1) exact functor.

Given a closed subset X of $\mathrm{Zg}(\mathcal{C})$, let $\mathcal{X}$ be the corresponding definable subcategory of $\mathcal{C}$. Then the closure of $I\mathcal{X}$ under pure subobjects is a definable subcategory of $\mathcal{D}$ (see after 13.4) and so determines, by 14.2, a closed subset, let us denote it I_*X, of $\mathrm{Zg}(\mathcal{D})$.

If Y is another closed subset of $\mathrm{Zg}(\mathcal{C})$ and $\mathcal{Y}$ is the corresponding definable subcategory then the definable subcategory corresponding to the closed set $X \cup Y$ is (see, e.g. [77, 4.67] or [84, 3.4.9]) $\{L : L | M' \oplus M'', M' \in \mathcal{X}, M'' \in \mathcal{Y}\}$ and hence the definable subcategory of $\mathcal{D}$ generated by the image of $X \cup Y$ under I is $\{L : L | IM' \oplus IM'', M' \in \mathcal{X}, M'' \in \mathcal{Y}\}$. By 15.1 and this description of $X \cup Y$ any indecomposable pure-injective in that set must already be in I_*X or I_*Y so I_* does commute with finite unions.

Next, suppose that $(X_\lambda)_\lambda$ are closed subsets of $\mathrm{Zg}(\mathcal{C})$ and let X be their intersection. Denote the corresponding definable subcategories of $\mathcal{C}$ by $\mathcal{X}_\lambda$, respectively $\mathcal{X}$. Since, clearly, I_* preserves inclusions, $I_*X \subseteq \bigcap_\lambda I_*X_\lambda$. Suppose that $N_0 \notin I_*X$. Then there is $F \in \mathrm{fun}(\mathcal{D})$ such that $FN_0 \neq 0$ but $FIC = 0$ for every $C \in X$ (since, Section 14, the open sets (G) for $G \in \mathrm{fun}(\mathcal{D})$ form a basis of open sets and I_*X is closed). Therefore, by $(*)$ in Section 13, $I_0F \cdot C = 0$ for every $C \in X$, that is, the intersection of the compact (14.1) open set (I_0F) with X is empty. Therefore, for some λ, $(I_0F) \cap X_\lambda = \emptyset$. That is, $F(IC) = I_0F \cdot C = 0$ for every $C \in X_\lambda$ and hence, by 15.1, $FN = 0$ for every $N \in I_*X_\lambda$. Therefore, $N_0 \notin I\mathcal{X}_\lambda$, as required.

Regarding continuity, the $(G) \cap \mathrm{im}(\mathrm{Zg}(\mathcal{C}))$ for $G \in \mathrm{fun}(\mathcal{D})$ form a basis of open sets for the induced topology on $\mathrm{im}(\mathrm{Zg}(\mathcal{C}))$ and the formula $G(IN) = I_0G \cdot N$ from Section 13 shows that the inverse image of this set under I is $(I_0G) \subseteq \mathrm{Zg}(\mathcal{C})$, which is open. So I is continuous (at the level of open sets) hence I^{-1} is as described. □

Let us extract the restatement of 15.1 used in the proof above.

inddirsum1

Corollary 15.3. *Suppose that $I : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor between definable categories which commutes with products and direct limits. Let X be a closed subset of $\mathrm{Zg}(\mathcal{C})$ and let $N \in I_*X$ (notation as in 15.2). Then N is a direct summand of IN' for some $N' \in X$.*

Corollary 15.4. *(see [81, §3], [53, 7.8]) Suppose that $I : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor between definable categories which commutes with products and direct limits. If I is full on pure-injectives then I induces a map $\mathrm{Zg}(\mathcal{C}) \to \mathrm{Zg}(\mathcal{D})$. This map is continuous and closed.*

Proof. Since I is full on pure-injectives $N \in \mathrm{Zg}(\mathcal{C})$ implies IN also is indecomposable hence, if non-zero, is in $\mathrm{Zg}(\mathcal{D})$. Also it follows from 15.3 and the observation just made that the image under I of a closed set is already closed, so I_* (as in 15.2) is just the usual direct image map induced by I on sets. □

Actually, we should be more careful in the statement since we do not usually allow the zero module to be a point of the spectrum and it could well be that $IN = 0$ for some $N \in \mathrm{Zg}(\mathcal{C})$. So, if we denote by K the closed (by 13.3) subset of $\mathrm{Zg}(\mathcal{C})$ corresponding to the kernel of I, then the correct statement is that I induces a map from $\mathrm{Zg}(\mathcal{C}) \setminus K$ to $\mathrm{Zg}(\mathcal{D})$.

Theorem 15.5. *(mainly from [81, 3.19]) Suppose that $I : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor between definable categories which commutes with products and direct limits and which is full on pure-injectives. Set $K = \{N \in \mathrm{Zg}(\mathcal{C}) : IN = 0\}$. Then the induced map $I_* : \mathrm{Zg}(\mathcal{C}) \setminus K \longrightarrow \mathrm{Zg}(\mathcal{D})$ is a homeomorphism of its domain with its image, which is the closed subset of $\mathrm{Zg}(\mathcal{D})$ corresponding to the definable subcategory of $\mathcal{D}$ generated (under taking pure subobjects) by the image of I.*

Proof. If $N, N' \in \mathrm{Zg}(\mathcal{C}) \setminus K$ then any isomorphism $IN \simeq IN'$ would, by fullness, be the image of an isomorphism $N \simeq N'$ so I_* is an embedding of $\mathrm{Zg}(\mathcal{C}) \setminus K$. It remains to see that the topology on $\mathrm{Zg}(\mathcal{C}) \setminus K$ is no finer than that induced by $\mathrm{Zg}(\mathcal{D})$ on its image. But, by the discussion before 25.9, any $\mathcal{C}$-definable structure is $\mathcal{D}$-definable, so this follows since the topology is defined in each case in terms of pairs of pp formulas. □

16 Triangulated Categories

sectriang

The requirement that our categories have direct limits does exclude some important examples of additive categories, triangulated categories in particular. What one can do, following Krause [55], is to map a triangulated category $\mathcal{C}$ to an associated functor category. We assume that our triangulated category is **compactly generated**, meaning that the subcategory, $\mathcal{C}^{\mathrm{c}}$, of compact objects is skeletally small and that if $C \in \mathcal{C}$ is such that $(A, C) = 0$ for every compact object A then $C = 0$. An object A is **compact** if $(A, -)$ commutes with arbitrary direct sums. Thus, in a compactly generated triangulated category, the compact objects see every object: it is not usually the case that they see every morphism and one says that $f : C \to D$ is **phantom** if for every $A \in \mathcal{C}^{\mathrm{c}}$ the induced map $(A, f) : (A, C) \to (A, D)$ is zero. Benson and Gnacadja showed [9, 4.2.5], in the context of stable module categories, and Krause showed in general [55, 1.4], that an object of a compactly generated triangulated category is pure-injective iff there are no non-zero phantom maps with it as codomain.

Consider the functor $\mathcal{C} \longrightarrow ((\mathcal{C}^{\mathrm{c}})^{\mathrm{op}}, \mathbf{Ab})$ given by $C \mapsto (-, C) \restriction \mathcal{C}^{\mathrm{c}}$. Because $\mathcal{C}$ is compactly generated this is faithful on objects (but not on morphisms: it is exactly the phantom maps which disappear). This is now treated just like the embedding, 9.4, of a definable category $\mathcal{D}$ into its dual functor category. So one defines a distinguished triangle $C \to D \to E \to C[1]$ of $\mathcal{C}$ to be pure-exact iff the sequence of functors $0 \to (-, C) \to (-, D) \to (-, E) \to 0$ on $\mathcal{C}^{\mathrm{c}}$ is exact, and an object $N \in \mathcal{C}$ is defined to be pure-injective if its image in $((\mathcal{C}^{\mathrm{c}})^{\mathrm{op}}, \mathbf{Ab})$ is injective ("internal" definitions also may be given, see [55, §1]). One shows that every injective functor is isomorphic to the image of a (pure-injective) object of $\mathcal{C}$ *et cetera et cetera*: in short, everything that could reasonably work does work. Including the model theory, which is described in [32].

17 Some Open Questions

What is an "intrinsic" definition of the notion of definable category? (so that the current two equivalent definitions, of definable subcategory and of exactly definable category, become representation theorems for a certain kind of category).

When is a category of the form $\sigma[M]$ (for M a module) definable? By $\sigma[M]$ we mean the subcategory of Mod-R consisting of those modules which are submodules of factor modules of direct sums of copies of M. It is a Grothendieck category.

More generally, when is a Grothendieck category definable?

When are categories of the form $\sigma[M]$, respectively Mod-$\mathcal{O}_X$ (where $\mathcal{O}_X$ is a ringed space), locally finitely presented? (Some partial answers are in [88] and [87] respectively).

18 Model theory in finitely accessible categories

seclangfinacc

From now on we assume at least passing acquaintance with model theory. Actually rather little is needed: what is covered in the introductory article [83] is easily enough background and anyone acquainted with the content of [84] should, with occasional recourse to standard references, have no problem understanding the model theory that is here.

Many model theory texts introduce concepts in the context of single-sorted languages. This is rather at odds with current practice in model theory, where it is normal to introduce extra sorts and/or to set results within the context of (Shelah's) imaginaries (these are extra definable sorts, use of which simplifies statements and proofs and they form part of the conceptual framework of current model theory). This limitation is also unnatural from the viewpoint of the more category-theoretic approaches to model theory, see [1], [64], [66] for example.

Each finitely accessible (additive, though the initial comments apply without this restriction) category $\mathcal{C}$ has an associated **canonical language** $\mathcal{L}(\mathcal{C})$ (see the above references). This language has a sort s_A for every object A belonging to some fixed small version of the subcategory, $\mathcal{C}^{\mathrm{fp}}$, of finitely presented objects (if there are only κ isomorphism types of objects of $\mathcal{C}^{\mathrm{fp}}$ then it would be perverse to use more than κ sorts). The language also has a function symbol for each morphism of $\mathcal{C}^{\mathrm{fp}}$: if $f : A \to B$ is in $\mathcal{C}^{\mathrm{fp}}$ then the corresponding symbol (we will usually just write "f" for this symbol) has domain s_B and codomain s_A. The reason for this contravariance will be seen now.

Every object $M \in \mathcal{C}$ becomes an $\mathcal{L}(\mathcal{C})$-structure as follows. The **elements of** M **of sort** $A \in \mathcal{C}^{\mathrm{fp}}$ are the morphisms from A to M: $s_A M = (A, M)$. Every morphism $f : A \to B$ in $\mathcal{C}^{\mathrm{fp}}$ induces, by composition, the morphism $(f, M) : (B, M) \to (A, M)$: this morphism is the interpretation of the corresponding function symbol in M. In other words we have embedded $\mathcal{C}$ into $((\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab}) = \mathrm{Mod}\text{-}\mathcal{C}^{\mathrm{fp}}$ via the Yoneda embedding $M \mapsto (-, M) \restriction \mathcal{C}^{\mathrm{fp}}$ and then used the language for the functor category $((\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab})$ which is based on the copy of $\mathcal{C}^{\mathrm{fp}}$ sitting inside it as a generating set of projective functors.

We know, by 3.4, that $\mathcal{C}$ is equivalent to the subcategory, $\mathrm{Flat}\text{-}\mathcal{C}^{\mathrm{fp}}$, of flat functors and, assuming (as we always will) that $\mathcal{C}$ has products hence, 3.8, $\mathrm{Flat}\text{-}\mathcal{C}^{\mathrm{fp}} = \mathrm{Lex}((\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}, \mathbf{Ab})$, it follows, by 6.1, that (the image of) $\mathcal{C}$ is a definable subcategory of $\mathrm{Mod}\text{-}\mathcal{C}^{\mathrm{fp}}$.

It is worth emphasising that our choice of language forces our view of objects of $\mathcal{C}$ as functors and *vice versa*. This view is the natural extension of the identification of a right module M over a ring R with (R_R, M) (recall that the projective module R_R is really the representable functor $(-, R)$, where we take R to be a category with one object).

If we take $\mathcal{C} = \mathrm{Mod}\text{-}R$, the category of right modules over a ring R, then the language described above has a sort for every finitely presented module and this language fits with the view of $\mathrm{Mod}\text{-}R$ as equivalent to the definable subcategory $\mathrm{Lex}((\mathrm{mod}\text{-}R)^{\mathrm{op}}, \mathbf{Ab}) \subseteq ((\mathrm{mod}\text{-}R)^{\mathrm{op}}, \mathbf{Ab}) = \mathrm{Mod}\text{-}(\mathrm{mod}\text{-}R)$ of left exact functors. This language is, note, intermediate between the usual 1-sorted language for R-

modules (i.e. that corresponding to the view of Mod-R as a category of sets with structure) and the richest expansion, that based on the full pp-imaginaries $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ language (for which see Section 22), which corresponds to the view of R-modules as exact functors on fun-$R \simeq \mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$. These comments apply equally well with any skeletally small preadditive category in place of R.

This kind of language is used for locally finitely presented additive categories in, for example, [74, Chpt. II] (there the definition certainly was modelled on that used for toposes) or, for functor categories Mod-$\mathcal{A}$, in [49].

There is the usual notion of **formula** in the language $\mathcal{L}(\mathcal{C})$: especially in [84] we have used the term **condition** interchangably with "formula", partly because this throws attention somewhat away from the specific syntactic form of the formula and more towards the solution sets it defines in objects - that is towards the corresponding solution-set functor (the difference is like that between a presentation of a group and the group so defined: presentations are very useful, especially for calculations, but normally it is the group we actually care about). Our notation is simplified by the observation that since the set of sorts in the canonical language $\mathcal{L}(\mathcal{C})$ is closed under finite products (because $\mathcal{C}^{\mathrm{fp}}$ has finite direct sums) every formula is equivalent, see below, to one with a single free variable.

A **quantifier-free** formula is one which is built up without using quantifiers: one which is equivalent to a disjunction of conjunctions of equations and inequations. Note that any conjunction of equations (i.e. system of homogeneous linear equations) defines a group: when evaluated at any $M \in \mathcal{C}$ the resulting solution set carries an abelian group structure induced by that on M (literally, if M is a set with structure, and induced by the morphism groups of $\mathcal{C}$ otherwise). A **pp formula** is one which is obtained by existentially quantifying out some variables from a system of linear equations. Thus such a formula ϕ defines, on any M, a projection of the solution set to a system of linear equations, so the solution set, written $\phi(M)$, carries a natural abelian group structure. Each such formula $\phi(x)$ (as commented above, there is, without loss of generality, just one free variable) is equivalent to one of the form $\exists y(xf = yg)$ where, say, x has sort A, y has sort B and $f : C \to A$, $g : C \to B$ are morphisms in $\mathcal{C}^{\mathrm{fp}}$.

$$\begin{tikzcd}
A \arrow[dr, "x"'] & C \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, phantom, "xf \,\|\, yg \,\Downarrow"] & B \arrow[dl, "y"] \\
& M &
\end{tikzcd}$$

Note that if $\phi(x)$ is pp and $M \in \mathcal{C}$ then the solution set $\phi(M)$ (a subset of (A, M) if x is a variable of sort A) is actually a subgroup of (A, M). The first reason for the central role of pp conditions is the following.

ppchar1

Lemma 18.1. *(cf. proof of [77, 16.5(1)]) Suppose that $\zeta(x)$ is a formula of the language $\mathcal{L}(\mathcal{C})$ where $\mathcal{C}$ is a finitely accessible additive category with products. If for every $M \in \mathcal{C}$ the solution set $\zeta(M)$ is a group then ζ is pp.*

Proof. By pp-elimination of quantifiers (19.1) ζ is equivalent to a boolean combination of pp formulas (the sentence part, "σ" in 19.1, is empty since there is at least the zero solution in each module). Say ζ is equivalent to $\bigvee_i(\phi_i \wedge \bigwedge_j \neg\psi_{ij})$ where each ϕ_i and ψ_{ij} is pp. For each i, $\phi_i \wedge \bigwedge_j \neg\psi_{ij}$ implies ζ so, for each $M \in \mathcal{C}$, $\phi_i(M) \subseteq \zeta(M) \cup \bigcup_j \psi_{ij}(M)$. Since this is also true for $M^{\aleph_0}$ (in which all indices of one subgroup in another are 1 or infinite) Neumann's Lemma implies that either $\phi_i(M) \subseteq \zeta(M)$ or $\phi_i(M) \subseteq \psi_{ij}(M)$ for some j. In fact this must be the case uniformly for all $M \in \mathcal{C}$ (because $\phi(M_1 \oplus \cdots \oplus M_n) = \phi(M_1) \oplus \cdots \oplus \phi(M_n)$ etc.). In the case that $\zeta(M) \subseteq \psi_{ij}(M)$ for all M, the disjunct $\phi_i \wedge \bigwedge_j \neg\psi_{ij}$ is never satisfied so may be dropped. Therefore it may be assumed that each ϕ_i implies ζ. So ζ is equivalent to $\bigvee_i \phi_i$. Another application of Neumann's Lemma yields that ζ is equivalent to ϕ_i for some i, as required. $\square$

Neumann's Lemma (see e.g. [77, 2.12]) says that if a coset aH is contained in a finite union, $\bigcup_j a_j H_j$, of cosets then it is contained in the union of only those cosets where the index of $H_j \cap H$ in H is finite. In particular, if all indices are 1 or infinite then the containment is for trivial reasons.

Similarly (see [79, A1.0], [14, 2.1]) if a formula defines an additive function when evaluated at every object of $\mathcal{C}$ then it is equivalent to a pp formula. These two results explain why our definition of the imaginaries category in Section 22 uses pp, rather than general, formulas.

A second reason for the importance of pp formulas is the next result, which says that pp conditions are exactly those whose solution sets are preserved by morphisms (it is quite straightforward to prove it from 19.1).

ppchar2

Lemma 18.2. *Suppose that $\phi(x)$ is a formula of the language $\mathcal{L}(\mathcal{C})$ where $\mathcal{C}$ is a finitely accessible additive category with products. If for every morphism $g : M \to N$ of $\mathcal{C}$ one has $g\phi(M) \subseteq \phi(N)$ then ϕ is pp. It is enough to check just for g in $\mathcal{C}^{\mathrm{fp}}$.*

Therefore pp formulas are exactly those formulas which define additive functors from $\mathcal{C}$ to **Ab**. Given a pp formula ϕ the functor from $\mathcal{C}$ to **Ab** given on objects by $M \mapsto \phi(M)$ and in the obvious (in view of 18.2) way on morphisms, is denoted F_ϕ. There is a deeper reason for the model-theoretic importance of pp formulas in this additive situation, which we discuss in the next section.

Next, note the connection with purity.

purviapp

Lemma 18.3. *Let $\mathcal{C}$ be a finitely accessible additive category (with products). Then a monomorphism $L \to M$ in $\mathcal{C}$ is pure iff for every pp formula ϕ in $\mathcal{L}(\mathcal{C})$, say with free variable of sort $A \in \mathcal{C}^{\mathrm{fp}}$, one has $\phi(L) = \phi(M) \cap (A, L)$ (with respect to the induced inclusion $(A, L) \leq (A, M)$).*

Proof. More precisely, since the various definitions of purity diverge in this generality (see 5.2), this definition of purity using pp formulas coincides with that seen in condition (iii) of 5.2.

To show this, suppose first that one has a morphism $h : A \longrightarrow B$ in $\mathcal{C}^{\mathrm{fp}}$ and a commutative diagram as shown.

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k'} \\ L & \longrightarrow & M \end{array}$$

Since A and B are finitely presented they correspond to sorts of the canonical language of $\mathcal{C}$ so we have the pp formula (with a constant, hence a projection of a non-homogeneous system of equations) $\exists y_B(k = y_B h)$, where y_B is a variable of sort B and k is being regarded as an element of L of sort A, which is satisfied by M. From the condition on pp formulas one deduces that this is true in L and that is precisely the statement that there is $l : B \longrightarrow L$ such that $k = lh$.

For the converse, suppose that $\phi = \phi(x_A)$ is a pp formula, say ϕ is $\exists y_B(x_A h' = y_B h)$ where $h : C \longrightarrow B$ and $h' : C \longrightarrow A$ are morphisms in $\mathcal{C}^{\mathrm{fp}}$. As commented before 18.1, this is the typical form of a pp formula. Suppose also that $k : A \longrightarrow B$ is such that the composition of this with the inclusion, f say, of L in M is a solution to ϕ in M, say $k' : B \longrightarrow M$ is such that $(fk)h' = k'g$. By the assumed condition (iii) of 5.2 (with $C \xrightarrow{h} B$ and $C \xrightarrow{h'} A$ at the top of the diagram) there is $l : B \longrightarrow L$ with $kh' = lh$, as required. $\square$

As usual, if $\overline{a}$ is a finite tuple of elements (of various sorts) of an object M then the **pp-type of $\overline{a}$ in** M is $\mathrm{pp}^M(\overline{a}) = \{\phi : \overline{a} \in \phi(M)\}$. (We recall, but only occasionally use, the model-theoretic notation $M \models \phi(\overline{a})$ which means $\overline{a} \in \phi(M)$.) If the list of free variables of ϕ is $\overline{x} = (x_1, \ldots, x_n)$ where x_i is a variable of sort A_i then the solution set $\phi(M)$ will be a subset of the product $(A_1, M) \times \cdots \times (A_n, M)$. Since $\mathcal{C}^{\mathrm{fp}}$ is closed under direct sum we can replace this tuple, using the canonical isomorphism $(A_1 \oplus \cdots \oplus A_n, M) \simeq (A_1, M) \times \cdots \times (A_n, M)$, by a single variable of sort $A_1 \oplus \cdots \oplus A_n$. Of course this does not apply to all other choices of language (in particular not for the usual 1-sorted language for R-modules).

A **pp-type** is any collection of pp conditions which is closed under implication and conjunction. It is the case that every pp-type is the pp-type of some tuple of elements of some object: the proof is as for modules (because there are infinitely many sorts involved in the quantified variables of a pp-type we cannot reduce, as in the proof of 19.1 below, to the 1-sorted case so we do have to choose a proof for modules and actually check that it applies in this generality. In fact, this is an easy consequence of Neumann's Lemma (for which see after 18.1) since it follows from that result that, if p is a pp-type with free variable x_A say, then $p \cup \{\neg\psi(x_A) : \psi \notin p\}$ is finitely satisfied in some object M hence, by the compactness theorem, is realised in an elementary extension of M.

If p is a pp-type then we denote by F_p the functor from $\mathcal{C}$ to $\mathbf{Ab}$ which takes C to the solution set $p(C) = \bigcap\{\phi(C) : \phi \in p\}$.

We now give an extended example which illustrates some of these ideas. Let X be any topological space and let $\mathcal{O}_X$ be a sheaf of rings (not necessarily commutative) with 1 on X. The category, PreMod-$\mathcal{O}_X$, of presheaves of $\mathcal{O}_X$-modules is locally finitely presented abelian (a proof of local finite presentation can be found at [87, §2] for example). Let τ be the hereditary torsion theory on PreMod-$\mathcal{O}_X$ (e.g. [72, §4.7]) such that the corresponding localisation functor Q_τ (7.2) is the sheafification functor. We denote by Mod-$\mathcal{O}_X$ the category of sheaves of $\mathcal{O}_X$-modules, that is, the localisation (PreMod-$\mathcal{O}_X)_\tau$. The space X is **noetherian** if every open subset of X is compact, equivalently if X has the descending chain condition on closed subsets.

shffintp

Proposition 18.4. *([87, 3.8]) The sheafification torsion theory is of finite type iff X is a noetherian space.*

tfpreshf

Corollary 18.5. *([87, 3.9]) The class of monopresheaves is a definable subset of* PreMod-$\mathcal{O}_X$ *iff X is noetherian.*

A **monopresheaf**, also called a "separated presheaf", is one which is τ-torsionfree, that is, if a section is locally zero then it is zero. The corollary follows from [73, 2.4] which says that a hereditary torsion theory is of finite type iff the torsionfree objects form a definable subclass (proved there for modules but generalised in [74, 3.3] to locally finitely presented abelian categories).

A finite type hereditary torsion theory τ on a locally finitely presented category $\mathcal{C}$ is said to be **elementary** ([75]) if, for every morphism $f : G \longrightarrow F$ in the category, $\mathcal{C}^{\mathrm{fp}}$, of finitely presented objects of $\mathcal{C}$, if $\mathrm{im}(f)$ is τ**-dense** in F (i.e. $F/\mathrm{im}(f)$ is torsion) then $\ker(f)$ is τ-finitely generated (i.e. it contains a finitely generated τ-dense subobject). It is proved in [75, 0.1] ([73, 2.20] for modules) that a hereditary torsion theory τ on the locally finitely presented $\mathcal{C}$ is elementary iff the localised category $\mathcal{C}_\tau$ (regarded as a subcategory of $\mathcal{C}$, see 7.2) is a definable subcategory of $\mathcal{C}$ and, furthermore, [75, 2.1], in this case $\mathcal{C}_\tau$ is locally finitely presented, with $(\mathcal{C}^{\mathrm{fp}})_\tau = (\mathcal{C}_\tau)^{\mathrm{fp}}$ (cf. 7.3).

Theorem 18.6. *([87, 3.10, 3.11, 3.12]) The category* Mod-$\mathcal{O}_X$ *of sheaves of modules over $\mathcal{O}_X$ is a definable subcategory of* PreMod-$\mathcal{O}_X$ *iff X is noetherian and, in this case,* Mod-$\mathcal{O}_X$ *is locally finitely presented.*

The last part is not an equivalence, indeed in [87, 5.7] it is shown that if X has a basis of compact open sets then Mod-$\mathcal{O}_X$ is locally finitely presented (in that paper some necessary and some sufficient conditions for Mod-$\mathcal{O}_X$ to be locally finitely presented are found but there remains quite a wide gap between these two sets of conditions). In this case the extensions by zero, $j_!\mathcal{O}_U$, of the restrictions, $\mathcal{O}_U$, of the structure sheaf to compact open sets U, form a generating set of finitely presented objects. So a suitable language (see §18) for the model theory of $\mathcal{O}_X$-modules, where X has a basis of compact open sets, has a sort for each compact open set U and, if $M \in$ Mod-$\mathcal{O}_X$, then the elements of M of sort corresponding to U are the elements of $(j_!\mathcal{O}_U, M) \simeq MU$, that is, the sections of M over U. Some of the model theory of sheaves based on this is developed in [85].

If X is a noetherian space then it is easy to write down axioms which define, within the category of $\mathcal{O}_X$-presheaves the property of being a sheaf. Namely, for each open set U and finite cover $U = U_1 \cup \cdots \cup U_n$, closure of the pp-pair $\big(\bigwedge_{i=1}^n x_U r_{UU_i} = 0\big) \,/\, (x_U = 0)$ expresses the monopresheaf property for this cover and closure of the pair $\exists x_U (x_U r_{UU_i} = x_{U_i}) \,/\, \big(\bigwedge_{ij} x_{U_i} r_{U_i U_{ij}} = x_{U_j} r_{U_j U_{ij}}\big)$

expresses the glueing property for this cover, where suffixes on variables denote their sorts, $U_{ij} = U_i \cap U_j$ and where the r are restriction maps (between sets denoted by their suffices) - these are indeed function symbols of the language since $r_{UV} \in (j_!\mathcal{O}_U, j_!\mathcal{O}_V)$ (e.g. see [85, p. 1190]).

We also remark that ([87, 2.18]) if for each open $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is right coherent then PreMod-$\mathcal{O}_X$ is locally coherent so, if X is also noetherian then, by 7.3 and [87, 3.10], Mod-$\mathcal{O}_X$ is locally coherent.

Example 18.7. As stated in 3.17, the category of comodules over a coalgebra is an example of a locally finitely presented category. Comodules are considered as structures for the canonical language of that category in [21].

19 pp-Elimination of quantifiers

secppeq

The basic result in the model theory of modules is that every formula is equivalent to a finite boolean combination of pp formulas: a result referred to as **pp-elimination of quantifiers**. There were theorems in the subject before this result was proved but, with this result, their proofs became easier and much more became possible. This result holds in complete generality: that is, it holds for finitely accessible additive categories with products. To prove this we could simply repeat the usual proof for modules (e.g. [77, §2.4] or [105, 1.1]), adapting it as necessary but that seems rather heavy-handed. Instead we will reduce to the 1-sorted case.

Write $\phi \geq \psi$ if $\phi(M) \geq \psi(M)$ for every $M \in \mathcal{C}$, that is, if F_ψ is a subfunctor of F_ϕ: we refer to these together as a **pp pair** and write ϕ/ψ to refer to this pair. By an **invariants condition** we mean a sentence of $\mathcal{L}(\mathcal{C})$ which says that the index of the solution set to ψ in the solution set of ϕ is at least n, where ψ and ϕ are pp conditions (hence define groups) with $\phi \geq \psi$ and n is a positive integer. (It is easy to write down a sentence expressing this condition.) By an **invariants statement** I will mean a finite boolean combination of invariants conditions.

ppeq

Theorem 19.1. *(see [77, p. 36] for references in the modules case) Let $\xi(x)$ be a formula of the canonical language, based on $\mathcal{C}^{\mathrm{fp}}$, of a finitely accessible additive category $\mathcal{C}$ with products. Then there is an invariants statement, σ, and a finite boolean combination, $\eta(x)$, of pp formulas such that $\xi(x)$ is equivalent to (on every object has the same solution set as) $\sigma \wedge \eta(x)$ on $\mathcal{C}$.*

Proof. (Sketch) The formula ξ involves variables (free or quantified) coming from only finitely many sorts. Let $\mathcal{A}$ be a finite subcategory of $\mathcal{C}^{\mathrm{fp}}$ containing, for each of these sorts, at least one corresponding object. Then ξ may be regarded as a formula in the language of right $\mathcal{A}$-modules and the functor from $\mathcal{C} \simeq$ Flat-$\mathcal{C}^{\mathrm{fp}}$ to Mod-$\mathcal{A}$ given by restriction to $\mathcal{A}$ clearly has the property that if $\zeta(x)$ is any formula with all variables corresponding to objects of $\mathcal{A}$ then the solution set of ζ on each object of $\mathcal{C}$ is unchanged whether that object is regarded as a (contravariant) functor on $\mathcal{C}^{\mathrm{fp}}$ or on $\mathcal{A}$. Thus we reduce to the case where we are dealing with formulas in a language (not the canonical one, but that makes no difference) for the category of functors on a *finite* preadditive category. But such a functor category is equivalent to a module category over a ring and so we can pull back pp-elimination of quantifiers of that module category to the original category $\mathcal{C}$. $\square$

An observation which was used in the proof above is that the various languages of a functor category are equivalent in such a way that pp formulas of the one language can be translated to pp formulas of any other (the only restriction is that any language should include enough sorts that the corresponding objects form a generating (under coproducts and cokernels) set of finitely presented objects for the category). That is easy to see directly but it also follows from the lemmas (18.1, 18.2) characterising pp conditions. Usually, given a finitely accessible additive category $\mathcal{C}$ with products, we do not specify (since we do not need to) which language we are dealing with: a subset of, say, (A, M) is pp-definable in one such language iff it is pp-definable in every such language.

ppinvars

Corollary 19.2. *Let $\mathcal{C}$ be a finitely accessible additive category with products and let M, N be objects of $\mathcal{C}$. Then M is elementarily equivalent to N, we write $M \equiv N$, iff for every pair ϕ/ψ of pp formulas the quotients $|\phi(M)/\psi(M)|$ and $|\phi(N)/\psi(N)|$ are both infinite or both are finite and have the same number of elements.*

pursubelem

Corollary 19.3. *Let $\mathcal{C}$ be a finitely accessible additive category with products and let $M \leq N$ be objects of $\mathcal{C}$ with $M \equiv N$. If M is pure in N then M is an elementary substructure of N.*

The technique, of reducing to the 1-sorted case, that we used in the proof of 19.1 allows us to avoid having to re-prove many of the results from the model theory of modules over a ring. For example, all pp formulas have free realisations, where a **free realisation** of a pp formula $\phi(x)$, where the variable x has sort A, is a pair (C, c) consisting of a finitely presented object C and an element c of C of sort A (i.e. $c \in (A, C)$) such that the pp-type of c in C is generated by ϕ. By saying that ϕ **generates** a pp-type p we mean that $p = \langle\phi\rangle$ where $\langle\phi\rangle = \{\psi \text{ pp} : \psi \geq \phi\}$. More generally, a subset of a pp-type p **generates** p if it does so under conjunction and implication.

The next result shows how definable subcategories arise model-theoretically. Given a pp pair ϕ/ψ there is the corresponding functor F_ϕ/F_ψ, defined on objects by $C \mapsto \phi(C)/\psi(C)$. By 22.1 this is a typical finitely presented functor on $\mathcal{C}$. Therefore, by 14.2, definable subcategories of $\mathcal{C}$ are defined (hence the name) by the coincidence of certain pairs of pp formulas.

defisdef

Theorem 19.4. *Let $\mathcal{C}$ be a finitely accessible additive category with products. Then a subcategory $\mathcal{D}$ of $\mathcal{C}$ is a definable subcategory iff there is a set $T = \{\phi_\lambda/\psi_\lambda\}_\lambda$ of pp-pairs such that the class $\mathcal{D}$ of objects is exactly the set of objects on which all these pairs are closed, $\mathcal{D} = \{C \in \mathcal{C} : \phi_\lambda(C) = \psi_\lambda(C) \text{ for every } \lambda\}$, that is iff $\mathcal{D}$ is the set of models of the sentences $\forall x\, (\phi_\lambda(x) \to \psi_\lambda(x))$ where $\phi_\lambda/\psi_\lambda \in T$.*

Proof. (Sketch) It is trivial to check that the class of models of such a set of sentences is closed under products, pure submodules and, with a little more work, direct limits (the point here is that witnesses to solvability of a finite system of equations in the direct limit have preimages somewhere back in the directed system and, moving forward in the directed system if necessary, one may find preimages which also solve a preimage of the system of equations).

For the converse, any definable subcategory is axiomatisable: it is closed under ultraproducts and pure (hence, by 19.3, under elementary) subobjects. By 19.1 the sentences axiomatising the class are boolean combinations of invariants conditions and it is straightforward to check that the closure properties of the class imply that one may take these axioms all to be of the form saying that some pp-pair is closed. An "algebraic" proof which avoids pp-elimination of quantifiers is given (for modules, but it generalises) in [84, 3.4.7]. □

20 Ultraproducts

secup

The ultraproduct construction is an algebraic one which is well-known in model theory. Since it plays an important part in some arguments in this paper I include a brief summary here.

Let $\mathcal{D}$ be a category of (algebraic/relational) structures of some kind and let $(D_i)_{i\in I}$ be a family of objects of $\mathcal{D}$. A **filter** on the index set I is a non-empty collection, $\mathcal{F}$, of non-empty subsets of I which is closed upwards ($J \subseteq J' \subseteq I$ and $J \in \mathcal{F}$ implies $J' \in \mathcal{F}$) and under finite intersection. An **ultrafilter** is a maximal filter on I. By Zorn's Lemma every filter is contained in an ultrafilter. If a set of subsets of I has the **finite intersection property** (the intersection of any finitely many is non-empty) then it extends to at least one (ultra)filter on I. Given a filter $\mathcal{F}$ on I, define an equivalence relation on the product (we assume that $\mathcal{D}$ contains all products of objects in it) $\prod_i D_i$ by $(a_i)_i \sim (b_i)_i$ iff $\{i \in I : a_i = b_i\} \in \mathcal{F}$. The quotient $\prod_i D_i/\sim$ is denoted $\prod_i D_i/\mathcal{F}$ and carries a natural structure induced from that on the D_i: this is the corresponding **reduced product** of the D_i. It is an **ultraproduct** if $\mathcal{F}$ is an ultrafilter and an **ultrapower** if, furthermore, the D_i all are isomorphic to some single object D, in which case the notation $D^I/\mathcal{F}$ is used. We write $(a_i)_i/\sim$ or $(a_i)_i/\mathcal{F}$ for the typical element of $\prod_i D_i/\mathcal{F}$.

This structure may alternatively be described as a direct limit of certain products, as follows. For each $J \in \mathcal{F}$ consider the product $\prod_J D_i = \prod_{i\in J} D_i$. If $J \supseteq K \in \mathcal{F}$ there is the canonical projection $\pi_{JK} : \prod_J D_i \longrightarrow \prod_K D_i$. So we have a diagram with objects $(\prod_J D_i)_{J\in\mathcal{F}}$ and morphisms the π_{JK} with $J \supseteq K \in \mathcal{F}$. Since $\mathcal{F}$ is closed under intersection this diagram is directed by $\mathcal{F}$ under reverse inclusion. Then one may check that the direct limit of this system (let us assume that $\mathcal{D}$ has direct limits) is exactly the ultraproduct defined above (or one may prefer to take this as the definition).

The basic result on ultraproducts is Łos' Theorem (for which see, for instance, [15, 4.1.9], [44, 9.5.1]).

Los

Theorem 20.1. *(Los' Theorem) If D_i $(i \in I)$ are structures for a language $\mathcal{L}$, if $\mathcal{F}$ is an ultrafilter on I, if ϕ is a formula of $\mathcal{L}$ and if $a = (a_i)_i/\sim$ is an element in $D^* = \prod_I D_i/\mathcal{F}$, then $a \in \phi(D^*)$ iff $\{i \in I : a_i \in \phi(D_i)\} \in \mathcal{F}$.*

(There is also a version for reduced products: the same form of statement but with ϕ restricted to being a pp formula.)

elemsubup

Theorem 20.2. *(see, e.g., [15, 4.1.13], [44, 9.5.2]) Every object is an elementary substructure of, in particular a pure subobject of, each of its ultrapowers.*

Note that, given $D \in \mathcal{D}$, there is a natural, **diagonal**, embedding of D into each of its ultrapowers, $\Delta : D \longrightarrow D^I/\mathcal{F}$, defined, in terms of elements, by sending $a \in D$ to the $\sim$-equivalence class of the constant tuple a (and in terms of the category-theoretic definition, *via* the diagonal embeddings of D into the partial products D^J, $J \in \mathcal{F}$).

21 Pure-injectives and elementary equivalence

secpiee

As in categories of modules, it turns out that the concepts of pure-injective and algebraically compact coincide for objects of finitely accessible additive categories $\mathcal{C}$ with products, moreover each object is elementarily equivalent to its pure-injective hull (and these exist 5.5).

An object $M \in \mathcal{C}$ is **algebraically compact** if every set of pp-definable cosets (i.e. cosets of pp-definable subgroups) with the finite intersection property has non-empty intersection. More precisely, if x is a variable of sort A, $\phi_i(x)$ is, for each i, a pp formula and a_i is an element of M of sort A, such that the set of subsets $(a_i + \phi_i(M))_i$ of (A, M) has the finite intersection property, then the whole set has non-empty intersection. In other words, every pp-type *with parameters* from M (i.e. possibly with some free variables replaced with non-zero constants from M) has a solution in M. The next result is, therefore, immediate.

satpi

Proposition 21.1. *If κ is an infinite cardinal and $\mathcal{C}^{\mathrm{fp}}$ has no more than κ objects and morphisms up to isomorphism then every κ^+-saturated object of $\mathcal{C}$ is algebraically compact.*

We draw a distinction between a *solution* of a pp-type p (i.e. an element which satifies all the formulas in p) and a **realisation** of p, meaning an element whose pp-type is exactly p (and no more). For example, if p is a pp-type with no extra (i.e. non-zero) parameters then 0 always is a solution but not a realisation unless p contains the formula $x = 0$. Algebraic compactness refers only to solutions: an algebraically compact object may well not realise every pp-type. A simple example is the 2-adic integers $\overline{\mathbb{Z}_{(2)}}$, regarded as an abelian group (or as a module over itself or over the localisation of $\mathbb{Z}$ at 2; it makes no difference). This is algebraically compact but the pp-type which describes an element divisible by every power of 2 (and which is realised in the elementary extension obtained by adding on a copy of $\mathbb{Q}$ as a direct summand) is not realised in $\overline{\mathbb{Z}_{(2)}}$ (though 0 is a solution, it is not a realisation).

Theorem 21.2. *Let $\mathcal{C}$ be a locally finitely presented additive category with products. An object of $\mathcal{C}$ is algebraically compact iff it is pure-injective.* pieqac

The usual proofs (for which see just about any of the basic references for the model theory of modules or, indeed, the early works on algebraic compactness in general systems) work in this more general context.

Corollary 21.3. *If $\mathcal{D}$ is a definable category (say a definable subcategory of* Mod-$\mathcal{A}$*) then there is an index set I (which may be taken arbitrarily large) and an ultrafilter $\mathcal{F}$ on I such that for every object $D \in \mathcal{D}$ the ultrapower $D^I/\mathcal{F}$ is pure-injective. In particular an embedding $f : C \longrightarrow D$ in $\mathcal{D}$ is pure iff some/this particular ultrapower $f^I/\mathcal{U} : C^I/\mathcal{U} \longrightarrow D^I/\mathcal{U}$ of it is split.* lgupsplit

It follows that if $\mathcal{D}$ is a definable subcategory of Mod-$\mathcal{A}$ then an object $D \in \mathcal{D}$ is pure-injective regarded as an object of $\mathcal{D}$, meaning that every pure embedding $D \longrightarrow D' \in \mathcal{D}$ is split iff it is so in Mod-$\mathcal{A}$, so we have an unambiguous notion of pure-injectivity/algebraic compactness for objects of definable categories.

The above corollary follows from a standard result, e.g. [15, 6.1.4, 6.1.8], of model theory which says that, given a cardinal κ, there is I, $\mathcal{F}$ as above such that every ultrapower is κ^+-saturated and that property (for large enough κ) implies, in particular, saturation for pp formulas, that is algebraic compactness = pure-injectivity.

A key property of pure-injectives is 21.5 below. There is an analogue, 21.4, for finitely presented objects. Both can be proved as in the case where $\mathcal{C}$ is a module category (e.g. [77, 8.5, 2.8]) but really they follow from that case (at least from the functor category case) by using 6.1b(v).

Proposition 21.4. *Suppose that $\mathcal{C}$ is a finitely accessible additive category with products, let $A, C \in \mathcal{C}^{\mathrm{fp}}$ and let $a \in (A, C)$ be an element of C of sort A. Then there is a pp formula ϕ such that $\mathrm{pp}^C(a) = \langle \phi \rangle$.* morsfp

Furthermore, if $M \in \mathcal{C}$ and $b \in (A, M)$ is an element of M of sort A such that $b \in \phi(M)$ (i.e. such that $\phi \in \mathrm{pp}^M(b)$) then there is a morphism $f : C \to M$ such that $fa = b$.

$$\begin{tikzcd} A \arrow[rr, "a"] \arrow[dr, "b"'] & & C \arrow[dl, dashed, "f"] \\ & M & \end{tikzcd}$$

Compare this with the next result.

Proposition 21.5. *Suppose that $\mathcal{C}$ is a finitely accessible additive category with products, let $M \in \mathcal{C}$ and let $a \in (A, M)$ be an element of M of sort $A \in \mathcal{C}^{\mathrm{fp}}$. Set $p = \mathrm{pp}^M(a)$. Suppose that $N \in \mathcal{C}$ is pure-injective and that $b \in (A, N)$ is such that $\mathrm{pp}^N(b) \supseteq p$ (i.e. for every ϕ pp, $a \in \phi(M)$ implies $b \in \phi(N)$). Then there is a morphism $f : M \to N$ such that $fa = b$.* morspi

More generally, the result holds if $\mathcal{C}$ is a definable category and we use any suitable language for this category.

endpi

Corollary 21.6. *Suppose that $\mathcal{D}$ is a definable category, that $N \in \mathcal{D}$ is pure-injective and that a is an element of N. Let $p = \mathrm{pp}^N(a)$. Then* $\mathrm{End}(N) \cdot a = F_p(N)$*, the group of solutions of p in N.*

If $\mathcal{D}$ is a definable category then an **elementary cogenerator** for $\mathcal{D}$ is a pure-injective object, N, of $\mathcal{D}$ such that every $M \in \mathcal{D}$ is a pure subobject of some product of copies of N. In particular, every pure-injective object of $\mathcal{D}$ is then a direct summand of some power of N.

elemcog

Theorem 21.7. *[77, 9.36] Every definable subcategory has an elementary cogenerator.*

Proof. It is enough, by 21.5, that $N \in \mathrm{Pinj}(\mathcal{D})$ realise every pp-type for $\mathcal{D}$ (i.e. realise every pp-type realised by some element/tuple in some object of $\mathcal{D}$) for then by 21.5 any $M \in \mathcal{D}$ embeds *via* the natural map $(a \mapsto (fa)_f)$ into $M^{(M,N)}$ and this map will preserve pp-types hence (18.3) will be pure. So just take N to be weakly saturated (i.e. to realise every (pp-)type without parameters). □

Given pp-types $p_1, \dots, p_n$, all with free variable of some fixed sort, define $p_1 + \dots + p_n = \{\exists x_1, \dots, x_n (x = x_1 + \dots + x_n) \wedge \phi_1(x_1) \wedge \dots \wedge \phi_n(x_n)) : \phi_i \in p_i\}$. If N_i is a pure-injective object of $\mathcal{D}$ containing a realisation, a_i, of p_i (recall that this means $\mathrm{pp}^{N_i}(a_i) = p_i$) then it is straightforward to check that $\mathrm{pp}^N(a) = p_1 + \dots + p_n$, where $N = N_1 \oplus \dots \oplus N_n$ and $a = (a_1, \dots, a_n)$: in particular $p_1 + \dots + p_n$ is a pp-type. It also follows that $p_1 + \dots + p_n$ is, as a set of pp formulas, exactly $p_1 \cap \dots \cap p_n$.

Lemma 21.8. *Suppose that $N \in \mathrm{Mod}\text{-}\mathcal{A}$ is pure-injective and that L is an* $\mathrm{End}(N)$*-submodule of N. Then $L = \sum_\lambda p_\lambda(N)$ for some set, $\{p_\lambda\}_\lambda$, of pp-types.*

Proof. For $a \in L$ set $p_a = \mathrm{pp}^N(a)$. If $\mathcal{A}$ is a ring then "$\in L$" may be read naively. Otherwise it can be taken to mean $a \in (C, N)$ for some $C \in \mathrm{mod}\text{-}\mathcal{A}$, If one prefers, the sort C can be restricted to be of the form $(-, A)$ for some $A \in \mathcal{A}$, so $a \in NA$ (thinking of N as a functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathbf{Ab}$)). In any case, "$a \in L$" and the sum "$\underrightarrow{\sum}_\lambda$" in the statement make perfectly good sense if read appropriately. Then, by 21.6, $\mathrm{End}(N) \cdot a = p_a(N)$ so the result follows. □

tpsum

Lemma 21.9. *Let $p_1, \dots, p_n$ be pp-types all with free variable of the same sort, set $p = p_1 + \dots + p_n$ and suppose that N is pure-injective. Then $p(N) = p_1(N) + \dots + p_n(N)$*

Proof. Clearly $p(N) \geq \sum_1^n p_i(N)$. For the converse let $a \in p(N)$. Then, by definition of p, every formula in the set $\{a = x_1 + \dots + x_n\} \cup \{\phi_1(x_1) \wedge \dots \wedge \phi_n(x_n) : \phi_i \in p_i\}$ has a solution in N. Note that this set is closed under finite conjunction, since the p_i are. So this is a set of pp formulas which is finitely satisfied in N hence, since N is pure-injective, with a solution in N, as required. □

The next result is related to those in Section 12.

Proposition 21.10. *Suppose that $\mathcal{D}$ is a definable subcategory of* Mod-$\mathcal{A}$ *(without loss of generality suppose that $\mathcal{A}$ is additive) and that $F : \mathrm{Pinj}(\mathcal{D}) \longrightarrow \mathbf{Ab}$ is a subfunctor of some functor $((-, A), -) \restriction \mathrm{Pinj}(\mathcal{D})$ with $A \in \mathcal{A}$. Suppose that F commutes with products. Then $F = F_p \restriction \mathrm{Pinj}(\mathcal{D})$ for some pp-type p.*

Proof. Let $\mathcal{P}$ be the set of pp-types p with free variable of type $(-, A)$ such that there exists $N \in \mathrm{Pinj}(\mathcal{D})$ and $a \in FN$ with $\mathrm{pp}^N(a) = p$. Then, it is claimed, $F = \sum_{p\in\mathcal{P}} F_p$. For, continuing with this notation, if $b \in N' \in \mathrm{Pinj}(\mathcal{D})$ with $b \in F_p(N')$ then there is, by 21.5, $f : N \to N'$ taking a to b, hence $b \in FN'$. Thus $F \geq F_p$ for each $p \in \mathcal{P}$ and the equality of the statement is then immediate from the definition of $\mathcal{P}$ (and the fact that F is a subfunctor of $(-, A)$).

By 12.6, F is finitely generated (it is a consequence of what we have just shown that F has a presentation), so $F = F_{p_1} + \cdots + F_{p_n}$ (on $\mathrm{Pinj}(\mathcal{D})$) for some $p_1, \ldots, p_n \in \mathcal{P}$. Since these are being evaluated on pure-injectives the right-hand side is, by 21.9, F_p where $p = p_1 + \cdots + p_n$. □

Let $\mathcal{D}$ be a definable category and choose a language for $\mathcal{D}$ (for convenience of exposition we assume that the language has product sorts, hence the assumption on $\mathcal{A}$ in the next result). Suppose that p is a pp-type and that ψ is a pp formula in this language, both with free variable of sort $(-, A)$, say. By p/ψ we denote the pp-type in the sort $(-, A)/\psi$ (see Section 22) which consists of the pp-pairs $(\phi + \psi)/\psi$ with $\phi \in p$ (and those equivalent modulo the theory of $\mathcal{D}$ to such pairs). So p could be regarded as $p/(x_A = 0)$ (we will write x_A for $x_{(-,A)}$). The corresponding functor, F_p/F_ψ, is the intersection of the subfunctors $(F_\phi + F_\psi)/F_\psi$ of $((-, A), -)/F_\psi$, all read as functors in $\mathrm{Fun}(\mathcal{D})$ (for which see Section 10), so really localisations of these at $\tau_{\mathcal{D}}$ (for which see after 8.1). Since we are working modulo $\mathcal{D}$ we have, for example, that $\langle\phi\rangle$ will mean the set of all pp formulas θ such that $\theta(M) \geq \phi(M)$ for every $M \in \mathcal{D}$ or, in terms of functors, such that $(F_\theta)_{\mathcal{D}} \geq (F_\phi)_{\mathcal{D}}$, where the subscript denotes localisation at the torsion theory/Serre subcategory corresponding in the sense of 8.1 to $\mathcal{D}$.

funcomup

Proposition 21.11. *Suppose that $\mathcal{D}$ is a definable subcategory of* Mod-$\mathcal{A}$ *($\mathcal{A}$ additive) and suppose that $F : \mathrm{Pinj}(\mathcal{D}) \longrightarrow \mathbf{Ab}$ has the form $F = (F_p/F_\psi) \restriction \mathrm{Pinj}(\mathcal{D})$ for some pp-type p and pp formula ψ. If p/ψ is not a finitely generated pp-type for $\mathcal{D}$ (i.e. if the image of F_p/F_ψ in $\mathrm{Fun}(\mathcal{D})$ is not finitely generated) then there is a pure-injective object $N \in \mathcal{D}$ and a pure-injective ultrapower, $N^* = N^J/\mathcal{F}$, of N such that $F(N^*) > (FN)^J/\mathcal{F}$. In particular, if F commutes with ultraproducts (meaning, commutes with those ultraproducts of pure-injective objects of $\mathcal{D}$ which are themselves pure-injective) then $(F_p/F_\psi)_{\mathcal{D}}$ must be a finitely generated functor.*

Proof. Suppose that p/ψ is not finitely generated. Then for each pp formula $\phi \in p$ there is an object $D = D^{(\phi)}$ in $\mathcal{D}$ and an element $a^{(\phi)}$ of D (of sort $(-, A)$ where that is the sort of the free variable of ψ and p) such that its image, $a^{(\phi)}_\psi = a^{(\phi)}/\psi$, modulo $\psi(D)$ satisfies ϕ/ψ but does not satisfy all of p/ψ. That is, $a^{(\phi)}_\psi \in (\phi(D)/\psi(D))$ but $a^{(\phi)}_\psi \notin F(D) = (p(D)/\psi(D))$. We may replace D

by its pure-injective hull, $N = H(D)$ (5.5), and also make a uniform choice of D over all $\phi \in p$ simply by taking the direct product of all the D obtained from individual ϕ. This pure-injective object we denote by N.

Now, let J be the set of all pp formulas $\phi \in p$ and consider the sets of the form $S_\phi = \{\phi' : \phi' \in p \text{ and } \phi'(N) \le \phi(N)\}$ as ϕ ranges over p; note that, by our assumption, these sets have the finite intersection property. We want an ultrafilter as in 21.3. To obtain that, choose a large enough $\kappa \ge \mathrm{card}(J)$ so that κ-saturation implies pure-injectivity, replace J by $J' = J \times \kappa$ and define S'_ϕ to be $S_\phi \times \kappa$. Each of these sets has cardinality κ and, together, they have the finite intersection property. Also partition κ into countably many disjoint subsets I_n, $n \in \omega$ and define $T_n \subseteq J'$ to consist of all those elements whose second coordinate does not belong to I_n. Then $\bigcap_n T_n = \emptyset$ and the set of the S'_ϕ together with the T_n still has the finite intersection property. By the comment before [15, 6.1.8], this set can be extended to a κ^+-good and, by definition of the T_n, countably incomplete, ultrafilter $\mathcal{F}$ on J'. By [15, 6.1.8], the ultrapower $N^* = N^{J'}/\mathcal{F}$ is κ-saturated hence pure-injective.

Let a^* be the element of N^* which is the image in N^* of the tuple $(a_\psi^{(\phi)})_{\phi\times\alpha}$ formed from the elements chosen above. By Łos' theorem, 20.1, we have $a^*/\psi \in p(N^*)/\psi(N^*)$: for each $\phi \in p$ the set $\{(\phi', \alpha) : a_\psi^{(\phi')} \in \phi(N)/\psi(N)\} \supseteq S_\phi \times \kappa = S'_\phi$ is in $\mathcal{F}$ and hence $a^* \in \phi(N^*)/\psi(N^*)$, as claimed.

If it were the case that $F = (p/\psi)$ commutes with ultraproducts, hence $p(N^*)/\psi(N^*) = (p(N)/\psi(N))^J/\mathcal{F}$, it would follow that $a^* = (b_\psi^{(\phi,\alpha)})_{\phi,\alpha}/\mathcal{F}$ for some elements $b^{(\phi,\alpha)} \in N$ with $b_\psi^{(\phi,\alpha)} \in p(N)/\psi(N)$ for each ϕ. By construction of ultraproducts, it must be that $a_\psi^{(\phi)} = b_\psi^{(\phi,\alpha)}$ for some (in fact, many) indices (ϕ, α). But that is contrary to choice of the $a^{(\phi)}$, as required. $\square$

The next result, proved by Sabbagh (for modules), is an example of a result which was first obtained before pp-elimination of quantifiers had been established and whose proof became much easier after that result. The theorem after that is Ziegler's (again, for modules) and begins to explain why the Ziegler spectrum (§14), whose points are the indecomposable pure-injectives, is so central to the model theory of modules (and, more generally, the model theory of objects of definable additive categories).

Theorem 21.12. *([95, Cor. 4 to Thm. 4] for modules) Let $\mathcal{D}$ be a definable additive category. Every object of $\mathcal{D}$ is an elementary subobject of its pure-injective hull.*

This follows directly from 21.3, 20.2, definition of pure-injective hull (see 5.5) and 19.3. The original proof for the next result works in the general context.

Theorem 21.13. *([105, 6.9] for modules) Let $\mathcal{D}$ be a definable additive category. Every object of $\mathcal{D}$ is elementarily equivalent to a direct sum of indecomposable pure-injectives.*

Let $\mathcal{D}$ be a definable category (for most statements and proofs there is no loss in generality in taking $\mathcal{D}$ to be a functor category Mod-$\mathcal{A}$). The **support**, supp(M), of $M \in \mathcal{D}$ is $\{N \in \mathrm{Zg}(\mathcal{D}) : N|M' \text{ for some } M' \equiv M\}$ (by $N|M'$ we mean that N is a isomorphic to a direct summand of M'). Just as for modules, this is exactly the set of (isomorphism classes of) indecomposable pure-injectives in $\langle M \rangle$, the definable subcategory of $\mathcal{D}$ generated by M.

infee

Corollary 21.14. *Let $M, M_1 \in \mathcal{D}$ where $\mathcal{D}$ is a finitely accessible additive category with products. Then M and M_1 generate the same definable subcategory of $\mathcal{D}$ iff $M^{\aleph_0} \equiv M_1^{\aleph_0}$ iff* $\mathrm{supp}(M) = \mathrm{supp}(M_1)$.

In particular, if the category $\mathcal{D}$ is such that there are no non-trivial finite pp-definable quotients (for example if $\mathcal{D}$ is a k-linear category, so every pp-definable subgroup is a k-vectorspace, where k is an infinite field) then the conditions above are equivalent to $M \equiv M_1$ (since, under this assumption on $\mathcal{D}$, $M \equiv M^{\aleph_0}$ by 19.2). Even without that assumption it is the relation expressed in 21.14, rather than elementary equivalence, which is important for almost all algebraic applications and even the majority of model-theoretic ones: details about the values of finite indices are something of a refinement in this additive situation, in part because of the emphasis on algebraic applications but also because much of the subtlety and depth of model-theoretic stability theory is lost on modules which are, model-theoretically, rather plain structures.

Indeed, the model theory of modules has a rather different flavour from much of model theory in that it is really the category of structures (rather, significant parts of it) which is the main concern, more than individual or elementary equivalence classes of structures. One can give algebraic expression to stability-theoretic concepts and results in modules, see [77, Chapter 6] and [58] for instance. Perhaps that is the point: what stability theory says can, often rather easily, be said algebraically. It would, however, be interesting to try to lift some of the particular model-theoretic structure that has been found in the context of modules to more general contexts: for instance, the interpretetation of the Ziegler spectrum as a space of equivalence classes of weight one types (see [77, §6.2]) surely has some mileage in it outside the additive context (a start has been made in [70]).

In fact it is some of the more basic ideas from model theory which have turned out to be useful in algebraic applications of the model theory of modules, especially the notion of (pp-)definable set, pp-type, definable category, the category of pp-imaginaries, as well as the notions which are (at least at first sight) peculiar to the model theory of modules, in particular the Ziegler spectrum and the Cantor-Bendixson analysis of that space.

22 Imaginaries and finitely presented functors

secimag

Formulas may have more than one free variable and so one has to deal with n-tuples of elements. Notationally this can be awkward. For example, look at various proofs in the, highly influential, papers [29], [30], [31] of Garavaglia on the model theory of modules. It was a common observation that frequently one could work out an argument for formulas with just one free variable and then "put a bar above everything". In effect, this is treating tuples as elements belonging to another sort. Shelah realised that not only finite tuples but also such tuples modulo definable equivalence relations really should be regarded as new kinds of elements. He formalised this, with his "imaginaries", or "eq", construction, by adding sorts for powers of the structure modulo definable equivalence relations, and adding function symbols for the projection-to-equivalence-classes maps, thus giving a richer, but definably equivalent, multi-sorted structure (see, e.g., [44, p. 151]). In a sense one had been working in a multi-sorted structure already whenever one used tuples of variables of length greater than one and this was still somewhat implicit in Shelah's construction, in that the canonical injections and projections involving tuples were not usually explicitly there in the language. Indeed, it took some time for model theory to work with the *category* of sorts: for instance symbols for definable maps between sorts were not usually added to the enriched language (here I mean "classical" model theory; in category-theoretic model theory one may see something very like this in [67]). In this respect development in the model theory of modules was faster, see [40] and [57], since, much earlier, it was apparent that: (i) one should add sorts for definable subsets (in the usual set-up this was done rather artificially by collapsing the complement of a definable subset to a single point); (ii) one should add, explicitly, symbols for canonical projections and injections and, more important, all definable functions, between sorts. It was also soon apparent that, at least for modules, using only pp-definable sorts and pp-definable functions between sorts gives a structure which fits better with the algebra. Indeed, it was shown by Burke [11, 3.2.5] that the resulting structure is then equivalent to the category of finitely presented functors (22.1 below).

Now we give the precise definition of the imaginary category, at least, the additive, pp, version. Let $\mathcal{C}$ be a finitely accessible additive category with products. Denote by $\mathcal{L}(\mathcal{C})^{\mathrm{eq}+}$ the language with: a sort for every pp-pair ϕ/ψ (we use the same notation for the sort except when ϕ is $x_A = x_A$ when we also write A/ψ for the sort); a function symbol for each pp-definable function from sort ϕ/ψ to ϕ'/ψ', where, if ϕ (hence also ψ) has free variable of sort A and ϕ' has free variable of sort A' then a definable function from ϕ/ψ to ϕ'/ψ' is given by a pp formula $\rho(x, x')$, with x, respectively x', of sort A, resp. A', which satisfies the obvious necessary conditions to define a function from ϕ/ψ to ϕ'/ψ' (note that these are elementary conditions, being expressible by sentences of the language). Formally, a definable function is an equivalence class of such formulas ρ (the equivalence relation being that of defining the same function on all structures). (If the language one started with were not the canonical language of $\mathcal{C}$ then x, x' might have to be tuples rather than single variables.) Clearly this language is a **conservative extension** of the original one, in that everything definable in the new language may be defined, though possibly with more work, in the original one. Also, whatever the language for $\mathcal{C}$ that one begins with (provided it is based on a generating set of finitely presented objects of $\mathcal{C}$) one ends up with "equivalent" enriched languages. More precisely, the corresponding associated categories (below) are naturally equivalent, since 22.1 applies, whatever language one starts with.

It would be difficult not to notice the category that was almost defined in the construction above: its objects are the pp pairs and its morphisms are the pp-definable maps between pp-pairs. We denote this category by $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ (in case $\mathcal{C} = \text{Mod-}\mathcal{A}$ we write $\mathbb{L}_{\mathcal{A}}^{\mathrm{eq}+}$). Burke's proof ([11, 3.2.5], or see [84, §10.2.5]), done for modules, of the fact that this is just the category, $\mathrm{fun}(\mathcal{C})$ (§10, also §4), of finitely presented functors works as well in this more general context.

fpfuneqpp

Theorem 22.1. *Let $\mathcal{C}$ be a finitely accessible additive category with products. Then $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+} \simeq \mathrm{fun}(\mathcal{C})$.*

This result, which is extended to definable categories in 22.2, is the key to relating the model-theoretic approach to more algebraic, particularly functor-category, approaches. In effect, it means that there is a dictionary for translating relevant model-theoretic ideas into purely algebraic ones (and, for a certain circle of algebraic ideas, *vice versa*).

For example, let us consider the notion of the pp-type of an element, a, of sort $A \in \mathcal{C}^{\mathrm{fp}}$ of an object $M \in \mathcal{C}$. In the first instance this may be considered as the set of those subfunctors $F \in \mathrm{fun}(\mathcal{C}) = (\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ of the functor $(A, -)$ such that $a \in \overrightarrow{F}M$ (recall that $\overrightarrow{F}$ is the unique extension of F to a functor on $\mathcal{C}$ which commutes with direct limits). This is a filter of finitely generated (=finitely presented since $\mathrm{fun}(\mathcal{C})$ is locally coherent 7.3) subfunctors of $(A, -)$ and every such filter corresponds to a pp-type: for it is easily derived from 22.1 that every finitely generated subfunctor of the functor $(A, -)$ is of the form F_ϕ for some pp formula ϕ of sort A. Now, a filter of finitely presented functors, though understandable purely algebraically, is perhaps not natural algebraically. We obtain something much more recognisable by moving to the dual category $\mathcal{C}^{\mathrm{d}}$ (§9 and recall, 10.10, that $\mathrm{fun}(\mathcal{C}^{\mathrm{d}}) \simeq (\mathrm{fun}(\mathcal{C}))^{\mathrm{op}}$). Namely, the duals of the subfunctors in the filter form an ideal of finitely generated subfunctors of the dual of $(A, -)$. Regarding $(A, -)$ (on $\mathcal{C}$) as the restriction of $((-, A), -)$ (on $\text{Mod-}\mathcal{C}^{\mathrm{fp}}$) the dual functor is $(A, -) \otimes_{\mathcal{C}^{\mathrm{fp}}} -$ (on $\mathcal{C}^{\mathrm{fp}}\text{-Mod}$) (see before 4.7) which we may regard as $(A \otimes -)$ on $\mathcal{C}^{\mathrm{d}}$ (see 11.4). The resulting ideal of finitely generated subfunctors of $A \otimes -$ may be replaced by the sum of the functors in it. Thus pp-types of sort A correspond to *arbitrary* subfunctors of $A \otimes -$. Furthermore, if $(a \otimes -) : (A \otimes -) \to (M \otimes -)$ is the morphism induced by a then the above ideal is, one may check using 23.3, exactly the kernel of this map. Thus, for instance, the set of pp-types (of sort A) realised in $M \in \mathcal{C}$ "is" exactly the set of "generalised annihilators" of A-elements of M, that is kernels of morphisms from $(A \otimes -)$ to $(M \otimes -)$.

If $\mathcal{D}$ is a definable subcategory of $\mathcal{C}$ then the corresponding category of imaginaries, $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$, is defined to have the same objects as $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ but to have as morphisms all those maps between sorts which are defined by pp formulas that define functions *on objects of* $\mathcal{D}$. For instance, if the pp-pair ϕ/ψ is **closed on** $\mathcal{D}$, meaning that $\phi(D)/\psi(D) = 0$ for every $D \in \mathcal{D}$, then the object ϕ/ψ is isomorphic to the zero object in $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$.

Theorem 22.1 extends to arbitrary definable categories $\mathcal{D}$ as follows. Suppose that $\mathcal{D}$ is a definable subcategory of $\text{Mod-}\mathcal{A}$. Let $\mathcal{S}_{\mathcal{D}}$ be the Serre subcategory of fun-$\mathcal{A}$ corresponding to $\mathcal{D}$ (see 8.1): recall that fun($\mathcal{D}$) is defined to be fun-$\mathcal{A}/\mathcal{S}_{\mathcal{D}}$. Since the functors in $\mathcal{S}_{\mathcal{D}}$ are exactly those which annihilate $\mathcal{D}$ (rather, whose canonical $\varinjlim$-preserving extensions to Mod-$\mathcal{A}$ annihilate $\mathcal{D}$), these exactly correspond to the pp-pairs which are closed on $\mathcal{D}$, and the next result follows easily.

Theorem 22.2. *Let $\mathcal{D}$ be a definable category. Then $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+} \simeq \mathrm{fun}(\mathcal{D})$.* fpfuneqpp1

Thus, associated to any definable category $\mathcal{D}$ we have the above category of "finitely presented functors on $\mathcal{D}$" which may equally be regarded as the category of pp-defined maps between pp-defined sorts of objects of $\mathcal{D}$ - the **category of definable scalars** of $\mathcal{D}$. If $\mathcal{D}$ is a definable subcategory of the category of modules over some ring R then, as a (small) part of this category, we have the endomorphism ring of "the forgetful functor on $\mathcal{D}$", $(R_R, -)_{\mathcal{D}}$ (the image of the forgetful functor on Mod-R under localisation at $\mathcal{S}_{\mathcal{D}}$). This ring, $\mathrm{End}((R_R, -)_{\mathcal{D}})$, is the **ring of definable scalars** of $\mathcal{D}$. It is a rather general kind of localisation of the ring R (see [14] for more on this).

We extend the notations F_ϕ and F_p from Section 18. If p is a pp-type and ϕ is a pp formula with the same free variable write $F_{D\phi/Dp}$ for $(F_{D\phi} + F_{Dp})/F_{Dp}$ (here D denotes elementary duality, see Section 23). Note that this, being isomorphic to $F_{D\phi}/F_{D\phi} \cap F_{Dp}$, is a finitely generated object of $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$. If $\mathcal{D}$ is a definable subcategory then we write $\mathcal{L}(\mathcal{D})$ for any suitable language for $\mathcal{D}$ (if $\mathcal{D}$ is a definable subcategory of the finitely accessible category $\mathcal{C}$ with products then $\mathcal{L}(\mathcal{C})$ will be suitable).

Corollary 22.3. *Let $\mathcal{D}$ be a definable category. Then every finitely generated object of $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$ is isomorphic to one of the form $F_{D\phi}/F_{Dp}$ (more accurately the localisation, $(F_{D\phi}/F_{Dp})_{\mathcal{D}}$) for some pp-type p and pp formula ϕ in $\mathcal{L}(\mathcal{D})$. (Recall that localisation of such a functor at the torsion theory corresponding to $\mathcal{D}$ simply corresponds to restricting its action to $\mathcal{D}$.)* fgobj

Proof. We use subscript $\mathcal{D}$ to denote images of objects of $\mathrm{Fun}^{\mathrm{d}}$-$\mathcal{A}$ under the localisation $\mathrm{Fun}^{\mathrm{d}}\text{-}\mathcal{A} \to \mathrm{Fun}^{\mathrm{d}}(\mathcal{D}) = \mathrm{Fun}^{\mathrm{d}}\text{-}\mathcal{A}/\tau^{\mathrm{d}}_{\mathcal{D}}$ (where $\mathcal{D}$ is a definable subcategory of Mod-$\mathcal{A}$, where $\mathcal{A}$ is additive). The $((-, A) \otimes_{\mathcal{A}} -)_{\mathcal{D}}$ for $A \in \mathcal{A}$ form a generating set of finitely presented objects of $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$, closed under finite direct sum, so the finitely generated objects of $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$ are the quotients of objects of the form $((-, A) \otimes -)_{\mathcal{D}}$ by arbitrary subobjects. Every subobject of $(-, A) \otimes -$ has the form $F_{Dp} = \sum\limits_{\rightarrow}\{F_{D\phi} : \phi \in p\}$ for some pp-type p (see after 22.1). Hence every subobject of $((-, A) \otimes -)_{\mathcal{D}}$ has the form $(F_{Dp})_{\mathcal{D}} = \sum\limits_{\rightarrow}\{(F_{D\phi})_{\mathcal{D}} : \phi \in p\}$ (using that localisation, being a left adjoint, see 7.2, commutes with colimits). In the latter equation we can drop the subscripts $\mathcal{D}$ if we read the formulas and types as applying to $\mathcal{D}$. Hence an arbitrary finitely generated object of $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$ has the form $((-, A) \otimes -)/F_{Dp}$, that is $(x_A = x_A)/F_{Dp}$. Replacing $x = x$ by any pp formula ϕ with free variable of sort A (more accurately $(A, -)$) generalises the representation but not the class of objects being represented (just as every finitely presented object F in Fun-$\mathcal{A}$ is equivalent to one of the form θ/ψ where θ is quantifier-free, corresponding to an epimorphism $((-, A), -) \to F$). $\square$

The richest functional language which one may use for a definable category $\mathcal{D}$ is that with a sort for each object of the category which appears in 22.2 and a function symbol for each morphism of that category. We denote this language by $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$. We describe this, and the way in which each object of $\mathcal{D}$ becomes a stucture for it both model-theoretically and algebraically.

Model-theoretically, we are adding a sort for each pp-pair ϕ/ψ and then, if $D \in \mathcal{D}$ the corresponding sort of D is the factor group $\phi(D)/\psi(D)$. The elements of this sort are simply cosets modulo $\psi(D)$ of elements $a \in \phi(D)$, we write a_ψ or a/ψ for $a+\psi(D)$. The pp formulas with free variable of this quotient sort may simply be expressed as "quotients" of pp formulas modulo ψ, that is, as pp-pairs θ/ψ with $\phi \geq \theta \geq \psi$. One may extend the notation to allow general θ (with free variable of the correct sort) by replacing θ with $\phi\cap(\theta+\psi) = (\phi\cap\theta)+\psi$ (the equality by modularity of the lattice). Then the pp-type, $\mathrm{pp}^D(a_\psi)$, of an element as above may be regarded as $\{\theta/\psi : \theta \in \mathrm{pp}^D(a)\}$. We write p/ψ for $\{\phi/\psi : \phi \in p\}$; so $\mathrm{pp}^D(a_\psi) = \mathrm{pp}^D(a)/\psi$.

Functorially, we are simply regarding $D \in \mathcal{D}$ as an exact functor from $\mathrm{fun}(\mathcal{D})$ to $\mathbf{Ab}$, see 10.8 and comments after that. So D in sort $F \in \mathrm{fun}(\mathcal{D})$ is simply $\mathrm{ev}_D F = \overrightarrow{F} D$ (see 10.9).

In general if σ is a sort of some language for $\mathcal{D}$ we will use the notation $\sigma(D)$ for the group of elements of D of sort σ and we can think of this as the value of σ at D.

The usual notation for an object $D \in \mathcal{D}$ considered as an $\mathcal{L}(\mathcal{D}^{\mathrm{eq}+})$-structure is $D^{\mathrm{eq}+}$. Note that this is essentially the same as what Krause calls the "endocategory" of D, so his [53, 8.4] can be seen as a statement about interpretations.

23 Elementary duality

secelemdual

Let $\mathcal{C}$ be a finitely accessible category with products. Recall (§9) that $\mathcal{C}^{\mathrm{d}}$ denotes its dual category and, 10.10, that $\mathrm{fun}(\mathcal{C}^{\mathrm{d}}) \simeq (\mathrm{fun}(\mathcal{C}))^{\mathrm{op}}$. In view of 22.1 this says that the categories of pp pairs from $\mathcal{L}(\mathcal{C})$ and from $\mathcal{L}(\mathcal{C}^{\mathrm{d}})$ are opposite: $\mathbb{L}(\mathcal{C}^{\mathrm{d}})^{\mathrm{eq}+} \simeq (\mathbb{L}(\mathcal{C})^{\mathrm{eq}+})^{\mathrm{op}}$. We describe this functor. The language for $\mathcal{C}$ coincides with that for Mod-$\mathcal{C}^{\mathrm{fp}}$ based on $\mathcal{C}^{\mathrm{fp}}$ so we will actually describe duality for pp pairs of that language.

Let ϕ be a pp formula of $\mathcal{L}(\mathcal{C})$ of sort $A \in \mathcal{C}^{\mathrm{fp}}$ (identified with $(-,A) \in$ mod-$\mathcal{C}^{\mathrm{fp}}$). The exact sequence $0 \to F_\phi \to ((-,A),-) \to ((-,A),-)/F_\phi \to 0$ dualises (4.5) to an exact sequence $0 \to d\big(((-,A),-)/F_\phi\big) \to d((-,A),-) = ((-,A) \otimes -) = ((A,-),-) \to dF_\phi \to 0$. The functor $d\big(((-,A),-)/F_\phi\big)$ is a finitely generated subfunctor of $((A,-),-)$ which we denote by DF_ϕ. Since every finitely generated subfunctor of $((A,-),-)$ has the form F_ψ for some pp formula ψ in $\mathcal{L}(\mathcal{C}^{\mathrm{d}})$ (which is the language for $(\text{Mod-}\mathcal{C}^{\mathrm{fp}})^{\mathrm{d}} = \text{Mod-}(\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}} = \mathcal{C}^{\mathrm{fp}}\text{-Mod}$ based on $\mathcal{C}^{\mathrm{fp}}$) we may define $D\phi$ to be any pp formula of that language (all choices of ψ are equivalent) such that $DF_\phi = F_{D\phi}$. Any such formula is referred to as the (**elementary**) **dual** of ϕ. Given ϕ explicitly one may write down (a specific) $D\phi$ explicitly, as follows.

Suppose that $\phi(x)$ is the formula $\exists y\,(xf = yg)$ (notation, including A, B,

C, as before 18.1), which we rewrite as $\exists y\,(x\,y)\begin{pmatrix} f \\ g \end{pmatrix} = 0$. (As is usual, we are identifying sorts and the objects of $\mathcal{C}^{\mathrm{fp}}$ which index them, as well as identifying function symbols and corresponding morphisms between sorts: all this is quite harmless and convenient.) Then $D\phi(x)$ is the formula $\exists z \begin{pmatrix} 1 & f \\ 0 & g \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} = 0$ with z of sort C^{o}, which can be rewritten as $\exists z(x = fz \wedge gz = 0)$. Here, if x has sort A then x (really we should use notation such as x^{o} since now we are dealing with $\mathcal{L}(\mathcal{C}^{\mathrm{d}})$) has sort A^{o}. As in the case of ordinary modules this can be seen as a duality between annihilation and divisibility (write ϕ in the equivalent form $g|xf$ and consider $D\phi$ especially in the case $f = 1$).

Note that the languages $\mathcal{L}(\mathcal{C})$ and $\mathcal{L}(\mathcal{C}^{\mathrm{d}})$ have "the same" collection of sorts and function symbols, one indexed by $\mathcal{C}^{\mathrm{fp}}$, the other by $(\mathcal{C}^{\mathrm{fp}})^{\mathrm{op}}$, and it is only in the variance of the function symbols that one really sees a difference. As in the case of modules this is reflected in writing function symbols (in modules, multiplications by ring elements) on the right or on the left.

Also note that, if ϕ has free variable of sort A then $D\phi$ has free variable of sort A^{o}, therefore, for any object $M \in \mathcal{C} \subseteq \text{Mod-}\mathcal{C}^{\mathrm{fp}}$, $\phi(M)$ is a subgroup of $((-,A),M)$, that is, of MA and, for any object $L \in \mathcal{C}^{\mathrm{d}} \subseteq \mathcal{C}^{\mathrm{fp}}\text{-Mod}$, $D\phi(L)$ is a subgroup of $((A,-),L)$, that is, of LA.

This duality is extended to pairs of pp formulas: if $\psi \leq \phi$ is a pp-pair then the corresponding pp-pair for the dual category is $D\phi \leq D\psi$ and the duality $\mathbb{L}(\mathcal{C}^{\mathrm{d}})^{\mathrm{eq}+} \simeq (\mathbb{L}(\mathcal{C})^{\mathrm{eq}+})^{\mathrm{op}}$ takes the sort ϕ/ψ to $D\psi/D\phi$. All this applies, *via* localisation, to arbitrary definable categories $\mathcal{D}$ in place of $\mathcal{C}$.

3.2

Theorem 23.1. *[40, 3.2] (Herzog's Criterion) Let $\mathcal{D}$ be definable additive category, let $M \in \mathcal{D}$ and let a be an element of M of sort A. Let $L \in \mathcal{D}^{\mathrm{d}}$ be an object of the dual category and let b be an element of L of sort A^{o}. Then $a \otimes b = 0$ in $M \otimes L$ iff there is a pp formula ϕ in $\mathcal{L}(\mathcal{D})$ such that $a \in \phi(M)$ and $b \in D\phi(L)$.*

The tensor product above may be taken to be over $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$, see 11.4 and 10.11. The proof, which follows quite formally from properties of $\otimes$, is as in the modules case (e.g. [84, 1.3.7]), indeed reduces to the modules case as in the proof of 19.1.

Of course elements of the tensor product $M \otimes L$ are *sums* of tensors, $\sum_{i=1}^n a_i \otimes b_i$ but our multi-sorted framework immediately converts this into a simple tensor $a \otimes b$ where a is an element of M of sort $A_1 \oplus \cdots \oplus A_n$ (if a_i is of sort A_i) and b is of sort $(A_1 \oplus \cdots \oplus A_n)^{\mathrm{o}} = A_1^{\mathrm{o}} \oplus \cdots \oplus A_n^{\mathrm{o}}$. There is no need to make this conversion but it does simplify statements and notation.

dualhomfmla0

Theorem 23.2. *Let $\mathcal{C}$ be a finitely accessible additive category with products and let $M \in \mathcal{C}$. Suppose that ϕ is a pp formula of $\mathcal{L}(\mathcal{C})$ of sort $A \in \mathcal{C}^{\mathrm{fp}}$ (identified with the sort $(-,A)$ for $\text{Mod-}\mathcal{C}^{\mathrm{fp}}$). Then there is a canonical isomorphism $\phi(M) \simeq \big(((-,A)\otimes -/F_{D\phi}), M \otimes -\big)$ where the right-hand side refers to functors in $\mathrm{fun}^{\mathrm{d}}(\mathcal{C})$.*

Note that the exact sequence $0 \to F_{D\phi} \to ((-,A)\otimes -) \to ((-,A)\otimes -)/F_{D\phi} \to 0$ gives the sequence $0 \to ((-,A)\otimes -)/F_{D\phi}, M\otimes -) \to \big(((-,A)\otimes -), M\otimes -\big) = MA \to (F_{D\phi}, M\otimes -) \to 0$ which is exact since $M\otimes -$ is absolutely pure =fp-injective (5.12).

We give two generalisations of this, to pp pairs and to pp-types (the proofs are as for modules, e.g. [78, p. 193], [12, 5.4], [84, 10.3.8, 12.2.4, 12.3.15]).

dualhomfmla

Theorem 23.3. *Let $\mathcal{C}$ be a finitely accessible additive category with products and let $M \in \mathcal{C}$. Suppose that ϕ/ψ is a pp pair of $\mathcal{L}(\mathcal{C})$. Then there is a canonical isomorphism $\phi(M)/\psi(M) \simeq (F_{D\psi}/F_{D\phi}, M\otimes -)$.*

Localising (and applying 22.2) one has the general case.

dualhomfmla1

Corollary 23.4. *Let $\mathcal{D}$ be a definable category, let $M \in \mathcal{D}$ and let $F \in \mathrm{fun}(\mathcal{D})$. Then $\overrightarrow{F}M \simeq ((dF)_{\mathcal{D}}, (M\otimes -)_{\mathcal{D}}) \simeq (dF, M\otimes -)$, where subscript $_{\mathcal{D}}$ denotes localisation at the Serre subcategory (8.1) corresponding to $\mathcal{D}$.*

dualhomtp

Theorem 23.5. *Let $\mathcal{C}$ be finitely accessible additive category with products and let $M \in \mathcal{C}$. Suppose that p is a pp-type with free variable of sort $A \in \mathcal{C}^{\mathrm{fp}}$. Then there is a canonical isomorphism $p(M) \simeq \big(((-,A)\otimes -)/F_{Dp}, M\otimes -\big)$. More generally if the formula ϕ and pp-type p have the same sort then $(F_{D\phi/Dp}, M\otimes -) \simeq p(M)/p(M)\cap\phi(M)$. If $\mathcal{D}$ is a definable subcategory of $\mathcal{C}$ and $M \in \mathcal{D}$ then this, by 7.2, is also isomorphic to $((F_{D\phi/Dp})_{\mathcal{D}}, (M\otimes -)_{\mathcal{D}})$.*

24 Hulls of types and irreducible types

sechulls

Let M be an object of a definable category $\mathcal{D}$ and let a be an element of M of sort $F \in \mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ (if we identify this category with $\mathrm{fun}(\mathcal{D})$ then we mean literally $a \in \overrightarrow{F}M$). We regard $\mathcal{D}$ as a definable subcategory of a functor category Mod-$\mathcal{A}$ and then consider the embedding (see 9.4) of $\mathcal{D}$ into $\mathrm{Fun}^{\mathrm{d}}(\mathcal{D})$, which takes M to $(M\otimes_{\mathcal{A}} -)_{\tau^{\mathrm{d}}_{\mathcal{D}}}$ where $\tau^{\mathrm{d}}_{\mathcal{D}}$ is the torsion theory on $\mathrm{Fun}^{\mathrm{d}}$-$\mathcal{A}$ corresponding (8.1) to $\mathcal{D}^{(\mathrm{d})}$. We will write this functor simply as $M\otimes -$ for short. Since, 23.3, $\overrightarrow{F}M \simeq (dF, M\otimes -)$ we may regard a as a morphism, which we write $(a\otimes -) : dF \to (M\otimes -)$. Let N be a pure-injective hull of M (in $\mathcal{D}$, equally in Mod-$\mathcal{A}$) so, by 5.12, $N\otimes -$ is the injective hull of $M\otimes -$. Also by that result the injective hull of the image of $a\otimes -$ has the form $H(a)\otimes -$ for some pure-injective direct summand, $H(a)$, of N. One may check that, as is implied by the notation, this is unique to isomorphism over a. In fact one has the following result. A proof for the case of modules may be found at, for example, [77, 4.15] but it is easier to move, 5.12, to the functor category, and use 21.5 and properties of injectives in the functor category (this is done in [84, §§4.3.3, 4.3.5]).

pphull1

Proposition 24.1. *Let $\mathcal{D}$ be a definable category and let p be a pp-type (for the language based on $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$), of sort F say. Then there is $N \in \mathcal{D}$, which we may take to be pure-injective, and there is $a \in FN$ such that $\mathrm{pp}^N(a) = p$. Let*

$N, N' \in \mathrm{Pinj}(\mathcal{D})$ and suppose that a, respectively a' are elements of N, resp. N', of sort F with $\mathrm{pp}^N(a) = p = \mathrm{pp}^{N'}(a')$. Choose, as above, hulls $H(a)$, $H(a')$ of a and a' which are direct summands of N, resp. N'. Then there is a morphism from N to N' which restricts to an isomorphism from $H(a)$ to $H(a')$ taking a to a'.

Therefore these hulls depend only on pp-types and so, in the situation above, we say that $H(a)$ is a copy of the **hull** of p and write $H(p)$ for this.

pphull2

Corollary 24.2. *Let $N \in \mathrm{Pinj}(\mathcal{D})$ where $\mathcal{D}$ is a definable category and let b be an element of N of some sort. Set $p = \mathrm{pp}^N(b)$. Suppose that a is a realisation of p in $H(p)$. Then there is an embedding, f, of $H(p)$ as a direct summand of N with $fa = b$.*

Say that p is **irreducible** (some say "indecomposable") if $H(p)$ is indecomposable. The following criterion is an expression of the fact that the functor $N \otimes -$ is uniform (i.e. the intersection of any two non-zero subfunctors is non-zero).

zgcrit

Theorem 24.3. *(Ziegler's criterion [105, 4.4, 4.5]) Suppose that p is a pp-type (of sort σ, say) in the canonical (or any other) language for the definable category $\mathcal{D}$. Then p is irreducible iff for all pairs, ψ_1, ψ_2, of pp formulas, with free variables of sort σ and with $\psi_1 \notin p$ and $\psi_2 \notin p$, there is $\phi \in p$ such that $\phi \wedge \psi_1 + \phi \wedge \psi_2 \notin p$.*

Proof. Any of the usual proofs will work. For a model-theoretic proof see the original reference [105] or [77, 4.29]: for a functorial one see [84, 4.3.49]. $\square$

Note that if N is an indecomposable pure-injective then it is the hull of each of its non-zero elements.

divide

Proposition 24.4. *Let N be an indecomposable pure-injective object of a definable category $\mathcal{D}$. Suppose that N is a direct summand of an object of the form $M_1 \oplus M_2$. Then at least one of the projections $M_1 \oplus M_2 \longrightarrow M_i$ induces an embedding of N as a direct summand of M_1 or M_2.*

Proof. Let a be a non-zero element of N, of any sort F say, and let $p = \mathrm{pp}^N(a)$. Since the embedding of N into $M_1 \oplus M_2$ is pure, $FN \leq F(M_1 \oplus M_2) = FM_1 \oplus FM_2$ (we will write F rather than $\overrightarrow{F}$), say $a = a_1 + a_2$ with $a_1 \in FM_1$ and $a_2 \in FM_2$. Set $\mathrm{pp}^{M_1}(a_1) = p_1$ and $\mathrm{pp}^{M_2}(a_2) = p_2$. Clearly, as sets of formulas, $p = p_1 \cap p_2$. If each inclusion $p \subseteq p_i$ were proper then there would be $\psi_i \in p_i \setminus p$. By Ziegler's criterion there is $\phi \in p$ such that $\phi \wedge \psi_1 + \phi \wedge \psi_2 \notin p$. But $\phi \wedge \psi_i \in p_i$ and $a = a_1 + a_2$ so this is a contradiction. Therefore $p_1 = p$ say. By 24.2 there is an embedding of N as a direct summand of M_1 (taking a to b) and, if one needs the stronger statement (that the embedding be induced by the projection), then [77, 4.14] gives it. $\square$

25 Interpretation functors

secinterp

We recall the general model-theoretic notion of interpretation of structures. Start with two structures M and N, usually structures for different languages, say M is an $\mathcal{L}$-structure and N is an $\mathcal{L}_1$-structure. We say that M can be **interpreted in** N if it can be found "definably contained within" N. More precisely, there should be a sort σ/ϵ of $\mathcal{L}_1^{\mathrm{eq}}$ (the general imaginaries language, not restricted to pp definitions and allowing definable relations and constants as well as functions) modulo the theory of N, that is, σ is an *arbitrary* formula and ϵ is a formula defining, modulo the theory of N, an equivalence relation on the solution set of σ, such that the following hold.
There exists a bijection $\alpha : M \to \sigma(N)/\epsilon(N)$ and, moreover, for every basic (function, relation, constant) symbol f, R, or c of $\mathcal{L}$ there should be a corresponding formula η_f, ζ_R or κ_c of $\mathcal{L}_1^{\mathrm{eq}}$ which defines the appropriately sorted of function, relation or constant on $\sigma(N)/\epsilon(N)$ and all this should be such that, if $\sigma(N)/\epsilon(N)$ is regarded as an $\mathcal{L}$-structure in this way, then α is an isomorphism. This is the definition assuming that $\mathcal{L}$ is a 1-sorted language: the many-sorted case is an obvious modification (and is done below).

That phrasing of the definition emphasises just the two structures involved but we can move part of the way to something more functorial as follows.

The data for an interpretation of certain $\mathcal{L}$-structures in certain $\mathcal{L}_1$-structures is as follows.
(i) For each sort s of $\mathcal{L}$ there is a pair σ_s/ϵ_s of formulas of $\mathcal{L}_1$ such that, if x is the free variable of σ_s (as usual we can simplify by assuming that there's just one free variable of a suitable product sort of $\mathcal{L}_1^{(\mathrm{eq})}$) then the free variables of ϵ_s are x, y where y has the same sort as x.
(ii) For each constant symbol c, of sort s, in $\mathcal{L}$ there is a formula κ_c of $\mathcal{L}_1$ with free variable having the same sort as that of σ_s.
(iii) For each function symbol f, from sort s to sort t, in $\mathcal{L}$ (let's simplify by assuming that $\mathcal{L}$ already has product sorts) there is a formula ϕ_f of $\mathcal{L}_1$ with free variables (x, y) where x is of sort that of the free variable of σ_s and y has sort that of the free variable of σ_t.
(iv) For each relation symbol R, of sort s, in $\mathcal{L}$ there is a formula ρ_R of $\mathcal{L}_1$ with free variable having sort that of the free variable of σ_s.

So far all we have done is to assign formulas of $\mathcal{L}_1$ to the basic symbols of $\mathcal{L}$. Next we have to give the conditions on an $\mathcal{L}_1$-structure N for these to pick out an $\mathcal{L}$-structure sitting inside N^{eq}. Namely, let $\mathcal{N}$ denote the class of those $\mathcal{L}_1$-structures N such that on N:
(i) For each sort s of $\mathcal{L}$, ϵ_s defines an equivalence relation on $\sigma_s(N)$. It is no loss in generality, and it keeps things simple, if we replace $\epsilon_s(x, y)$ by $\epsilon_s(x, y) \wedge \sigma_s(x) \wedge \sigma_s(y)$.
(ii) For each c as above the solution set $\kappa_c(N)$ is a single $\epsilon_s(N)$-equivalence class.
(iii) For each f as above the formula ϕ_f well-defines a function from $\sigma_s(N)/\epsilon_s(N)$ to $\sigma_t(N)/\epsilon_t(N)$ (that is, it takes elements in $\sigma_s(N)$ to elements in $\sigma_t(N)$, takes

any $\epsilon_s(N)$-equivalence class into a single $\epsilon_t(N)$ class and, without loss of generality (modifying ϕ_f if necessary) is a total relation on $\sigma_s(N)$).
(iv) No condition is needed on relations (though we could be tidy and suppose, as for functions, that if the relation is defined on one member of a $\sigma_s(N)$-equivalence class then it is defined on all members of that class).

All these conditions are expressible by sentences of $\mathcal{L}_1$ so the class $\mathcal{N}$ is elementary. Clearly, if $N \in \mathcal{N}$ then the above data defines, sitting within N^{eq}, an $\mathcal{L}$-structure: the $\mathcal{L}$-structure so **interpreted in** $\mathcal{N}$. Let $\mathcal{M}$ be the class of $\mathcal{L}$-structures which arise in this way: simple examples (13.5) show that this is not, in general, an elementary class, but no matter: we say that there is an **interpretation of** $\mathcal{M}$ **in** $\mathcal{N}$. Note, however, that the function involved actually goes the other way: given $N \in \mathcal{N}$ we obtain an object of $\mathcal{M}$. We hope that this explains the form of the definition, coming up soon, of interpretation functor.

Everything above applies to any structures: now we return to additive ones so, of course, pp formulas replace general formulas.

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are definable additive categories. An **interpretation functor** $I : \mathcal{C} \longrightarrow \mathcal{D}$ is defined in terms of certain data which depends on the languages that we are using for $\mathcal{C}$ and $\mathcal{D}$, though the functor itself is independent of choice of language. So assume first that $\mathcal{L}_{\mathcal{C}}$ and $\mathcal{L}_{\mathcal{D}}$ are any of the languages of the kinds discussed in Sections 18 and 22. Then an interpretation functor from $\mathcal{C}$ to $\mathcal{D}$ is given by the following data.
For each sort σ of $\mathcal{L}_{\mathcal{D}}$, a pp-pair ϕ_σ/ψ_σ in $\mathcal{L}_{\mathcal{C}}$ and, for each function symbol f of $\mathcal{L}_{\mathcal{D}}$ from sort σ to sort σ', a pp formula $\rho_f(\overline{x}, \overline{y})$ where the sort of $\overline{x}$ is that of the free variable sequence of ϕ_σ (hence also of ψ_σ) and the sort of $\overline{y}$ is that of the free variable sequence of $\phi_{\sigma'}$. It is a requirement that in each object C of $\mathcal{C}$, ρ_f define a map from $\phi_\sigma(C)/\psi_\sigma(C)$ to $\phi_{\sigma'}(C)/\psi_{\sigma'}(C)$. It is also required that every sentence in the theory, $\mathrm{Th}(\mathcal{D})$, of $\mathcal{D}$ in $\mathcal{L}_{\mathcal{D}}$ translate, *via* $f \mapsto \rho_f$, to a sentence true in $\mathcal{C}$; of course, it is sufficient that a set of axioms for the theory of $\mathcal{D}$ so translate.
The functor itself is that which takes an object C of $\mathcal{C}$ to the $\mathcal{L}_{\mathcal{D}}$-structure with sorts the $\phi_\sigma(C)/\psi_\sigma(C)$ and with the interpretation of each function symbol f being the function defined by $\rho_f(C)$: our requirements ensure that this is an object of $\mathcal{D}$ (that is, an $\mathcal{L}_{\mathcal{D}}$-structure which satisfies the axioms for being in $\mathcal{D}$). The action of this interpretation functor on morphisms is just restriction; this is well-defined because our interpretation data is given by pp formulas.

For instance if $\mathcal{C} = \text{Mod-}R$ and $\mathcal{D} = \text{Mod-}S$ are module categories and the languages chosen are the usual (1-sorted) ones for modules, based on R, respectively S, then an interpretation functor from Mod-R to Mod-S is given by a pp pair ϕ/ψ in the language of R-modules and, for each element $s \in S$, a pp formula ρ_s with $2n$ free variables, where n is the number of free variables of ϕ, such that, for each R-module M, ρ defines a function from $\phi(M)/\psi(M)$. Then the requirement is that "the addition and multiplication tables of S be preserved" by $s \mapsto \rho_s$ (no further axioms need be satisfied in this case since $\mathcal{D}$ is the category of all S-modules). This can be said neatly in terms of rings of definable scalars: the map $s \mapsto \rho_s$ is a ring homomorphism from S to the ring of definable scalars for R-modules in sort ϕ/ψ. For more on this see [81].

If the languages chosen are the imaginaries languages, $\mathcal{L}(\mathcal{C})^{\mathrm{eq}+}$, $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$ which, recall, are based on the categories $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+} \simeq \mathrm{fun}(\mathcal{C})$ and $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+} \simeq \mathrm{fun}(\mathcal{D})$ (22.2), then the definition of interpretation functor takes the following form.

To each sort σ of $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$, we assign a sort τ_σ in $\mathcal{L}(\mathcal{C})^{\mathrm{eq}+}$ and to each function symbol f of $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$ from sort σ to sort σ', we assign a pp formula $\rho_f(x,y)$ of $\mathcal{L}(\mathcal{C})^{\mathrm{eq}+}$ with x a variable of sort τ_σ and y a variable of sort $\tau_{\sigma'}$; since $\mathcal{L}(\mathcal{C})^{\mathrm{eq}+}$ is closed under taking finite products of sorts, single variables suffice. Also, since $(\mathbb{L}^{\mathrm{eq}+})^{\mathrm{eq}+}$ is naturally equivalent to $\mathbb{L}^{\mathrm{eq}+}$, there will be a morphism of this category for which ρ_f is (definably equivalent to) the corresponding function symbol of the language. So, from this data, we have, for each sort σ, now thought of as an object of the category $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$, an object τ_σ of $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$, and for each morphism f of $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$, a morphism of $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$. That this assignment is a functor is immediate from our requirement that the theory of $\mathcal{D}$ be preserved. But more is true: this functor from $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ to $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ is exact since for each short exact sequence in $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ there is a sentence in the $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$-theory of $\mathcal{D}$ which expresses this and whose translation, being true in $\mathcal{C}$, expresses exactness of the image sequence in $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$.

Thus, from the data of an interpretation functor from $\mathcal{C}$ to $\mathcal{D}$ we obtain an exact functor from $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+} \simeq \mathrm{fun}(\mathcal{D})$ to $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+} \simeq \mathrm{fun}(\mathcal{C})$. It follows by 13.1 that there is induced a functor $I : \mathcal{C} \to \mathcal{D}$ which commutes with direct limits and products; it follows directly from the definitions that this is just the interpretation functor itself. Conversely, given a functor $I : \mathcal{C} \to \mathcal{D}$ which commutes with direct limits and products, one has the corresponding exact functor, I_0 in the notation of Section 13, giving the data of an interpretation functor from $\mathcal{C}$ to $\mathcal{D}$; since the theory of $\mathcal{D}$ is axiomatised by the closure of certain pp pairs (19.4) this theory is indeed preserved because of exactness of I_0. Therefore we have the following result.

interpeqexstar

Theorem 25.1. *Let $\mathcal{C}$, $\mathcal{D}$ be definable categories. Then there is a natural bijection between interpretation functors I from $\mathcal{C}$ to $\mathcal{D}$ and exact functors I_0 from $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ to $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$, that is, from $\mathrm{fun}(\mathcal{D})$ to $\mathrm{fun}(\mathcal{C})$.*

In the terminology used before, this gives an **interpretation of** the (not necessarily definable and not necessarily full) subcategory $I\mathcal{C}$ of $\mathcal{D}$ **in** $\mathcal{C}$.

The example (of categories of modules) above illustrates that, in practice, an interpretation functor often is given in terms of a language much less rich than (though definably equivalent to) the full imaginaries language. In the case of $\mathcal{C} = \text{Mod-}R$ there are three obvious natural languages: the usual 1-sorted language based on $\{R\}$; the natural language of the category (§18) with sorts corresponding to the finitely presented modules; the full imaginaries language. Each of these corresponds to a certain full subcategory of $\mathbb{L}(\text{Mod-}R)^{\mathrm{eq}+}$ which in some sense generates the whole category and is such that the data of interpretation together with suitable exactness requirements, corresponding to an appropriate axiomatisation, is enough to determine an exact functor from $\mathbb{L}(\text{Mod-}R)^{\mathrm{eq}+}$.

The next result follows directly from 4.1 and is a category-theoretic formulation of data sufficient to define an interpretation functor.

Proposition 25.2. *Suppose that* $\mathcal{C}$ *and* $\mathcal{D}$ *are definable categories. Let* $\mathcal{A}'$ *be a subcategory of* $\mathrm{fun}(\mathcal{D})$ *such that, if* $\mathcal{E}$ *is the exact structure on* $\mathcal{A}'$ *induced by its inclusion in* $\mathrm{fun}(\mathcal{D})$*, then* $\mathrm{Ab}(\mathcal{A}', \mathcal{E}) = \mathrm{fun}(\mathcal{D})$.

Let $I_0' : (\mathcal{A}', \mathcal{E}) \longrightarrow \mathrm{fun}(\mathcal{C})$ *be an exact functor. Then there is a unique extension to an exact functor* $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \mathrm{fun}(\mathcal{C})$ *and hence a corresponding interpretation functor* $I : \mathcal{C} \longrightarrow \mathcal{D}$.

With 13.1, 25.1 gives the following.

interpeqdef

Corollary 25.3. *Let* $\mathcal{C}$, $\mathcal{D}$ *be definable categories. Then a functor from* $\mathcal{C}$ *to* $\mathcal{D}$ *is an interpretation functor iff it is a definable functor iff it commutes with products and direct limits.*

interpeqdef1

Corollary 25.4. *Let* $\mathcal{C}$, $\mathcal{D}$ *be definable categories. There is a natural bijection between interpretation functors from* $\mathcal{C}$ *to* $\mathcal{D}$ *and exact functors from* $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ *to* $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$.

Any such functor $I : \mathcal{C} \longrightarrow \mathcal{D}$, as well as inducing a map from $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ to $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$, namely the corresponding (13.1) exact functor $I_0 : \mathrm{fun}(\mathcal{D}) = \mathbb{L}(\mathcal{D})^{\mathrm{eq}+} \longrightarrow \mathbb{L}(\mathcal{C})^{\mathrm{eq}+} = \mathrm{fun}(\mathcal{C})$, defines a map of languages from $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$ to $\mathcal{L}(\mathcal{C})^{\mathrm{eq}+}$. Namely, to each sort σ of $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$ corresponds the sort $I_0\sigma$ of $\mathcal{L}(\mathcal{C})^{\mathrm{eq}+}$ and to each function symbol f corresponds $I_0 f$ (thought of now as a function symbol of a language rather than a morphism). The extension to formulas is defined to be simply replacement of function symbols and re-typing of variables. Thus the action on formulas is defined to commute with the propositional connectives $\wedge, \vee, \neg$ and so clearly this commutes with $+$ on pp formulas. It seems reasonable to use the notation $I_0\chi$ for the formula obtained from χ under this translation.

(Some of the awkwardness in saying this precisely is due to the fact that languages are not usually defined as mathematical "objects". Our introduction of the category $\mathbb{L}^{\mathrm{eq}+}$ is partly to remedy that.)

It is easy to check, using the formula $(*)$ before 13.1, that for any pp formula ϕ we have $F_{I_0\phi} = I_0 F_\phi$. Therefore if $\psi(D) \le \phi(D)$ for every $D \in \mathcal{D}$ then $I_0\psi(C) \le I_0\phi(C)$ for all $C \in \mathcal{C}$.

Example 25.5. We consider a functor from Mod-$k[T]$ to Mod-$k\widetilde{A_1}$, where k is a field, $k[T]$ is the polynomial ring and $k\widetilde{A_1}$ is the path algebra of the Kronecker quiver $\widetilde{A_1}$. This is the quiver with two vertices, $1, 2$ say, and two arrows, α, β say, from vertex 1 to vertex 2.

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$$

(One may note that the inverse equivalences between the category of modules over the path algebra and the category of representations of the quiver are examples of interpretation functors.) The functor, I, is that which takes a $k[T]$-module M to the $k\widetilde{A_1}$-module which has underlying k-space $M \oplus M = Me_1 \oplus Me_2$, where e_i is the idempotent of $k\widetilde{A_1}$ corresponding to vertex i and where the action of α is defined as $(m,m')\alpha = (0,m)$ and β is defined by $(m,m')\beta = (0,mT)$. The image of I is the subcategory $\mathcal{D}$ of Mod-$k\widetilde{A_1}$ defined by the condition that α be invertible.

Clearly I commutes with direct limits and products, hence corresponds to an exact functor $I_0 : \mathrm{fun}(\mathcal{D}) \longrightarrow \text{fun-}k[T] = (\mathrm{Ab}(k[T]))^{\mathrm{op}}$. In the latter category denote by σ_i the sort such that $\sigma_i(M) = Me_i$. We also have the definable functions α, α^{-1} and β in $\mathrm{fun}(\mathcal{D}) \simeq \mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$. Then I_0 takes both σ_1 and σ_2 to the "home sort" $(x = x)/(x = 0)$ in $\mathbb{L}(\text{Mod-}k[T])^{\mathrm{eq}+}$, it takes both α and α^{-1} to the identity map of that sort, and it takes β to the (definable) map multiplication by T. The map on formulas determined by this functor I_0 then takes, for instance, the formula $\exists x_1(x_1\beta = x_2)$ in $\mathcal{L}(\mathcal{D})$ to $\exists x'(x'T = x)$ in the language for $k[T]$-modules.

compimsort

Remark 25.6. Note how an element of IC ($C \in \mathcal{C}$) can be viewed as an element of $C^{\mathrm{eq}+}$. Namely, if $a \in \tau(IC)$, where τ is a sort of $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$, so $a \in F_\tau(IC)$ where F_τ is the functor in $\mathrm{fun}(\mathcal{D})$ corresponding to τ in $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$, then (see just before 13.1) $F_\tau(IC) = I_0F \cdot C$, so a may be regarded as an element, a' say, of C of the sort $\sigma \in \mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ corresponding to $I_0F_\tau \in \mathrm{fun}(\mathcal{C})$. Also note how this is a relation between $\mathrm{pp}^{IC}(a)$ and $\mathrm{pp}^{C}(a')$, namely if $a \in \phi(IC)$, i.e. $a \in F_\phi(IC)$, then $a' \in I_0F_\phi(C)$, that is $a' \in (I_0\phi)(C)$.

Consider a definable category $\mathcal{D}$ and let Σ be a collection of objects of $\mathrm{fun}(\mathcal{D})$, that is (22.2), a collection of sorts of $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$. If $D \in \mathcal{D}$ then there are various ways of thinking about the restriction, write it as $D \restriction \Sigma$, of D to the full subcategory, $\mathcal{A}_\Sigma$, of $\mathrm{fun}(\mathcal{D})$ on Σ: we may view this as the restriction of the exact functor evaluation at D, ev_D, to $\mathcal{A}_\Sigma$; a more model-theoretic view is to see this as the collection of sets $\sigma(D)$, ($\sigma \in \Sigma$) with the morphisms of $\mathcal{A}_\Sigma$ giving definable maps between these sets. However we view this, there may have been some loss of implicit structure.

exA2sort

Example 25.7. Consider the path algebra, $R = kA_2$, over a field k, of the quiver A_2 (see 3.1, 13.5). If e_i is the idempotent of R corresponding to vertex i of the quiver ($i = 1,2$) then there are sorts σ_i such that, for $M \in$ Mod-R $\sigma_iM = Me_i$. Take $\Sigma = \{\sigma_2\}$. The only endomorphisms of this sort are the multiplications by the scalars of k, hence the only structure on the restriction $M \restriction \Sigma$ is that of a k-vectorspace. In particular the image of the arrow from vertex 1 to vertex 2 has been lost as a definable subset of Me_2.

The restricted language, $\mathcal{L}(\mathcal{D}) \restriction \mathcal{A}_\Sigma$, contains only the sorts in Σ and the function symbols for morphisms in $\mathcal{A}_\Sigma$. We may restore the lost implicit structure as follows. For each pp formula $\phi \in \mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$ with free (but not necessarily bound) variables having sorts in Σ take a relation symbol, R_ϕ, with the same free variables as ϕ. Denote by $(\mathcal{L}(\mathcal{D}) \restriction \Sigma)^{\mathrm{full}}$ the language obtained from $\mathcal{L}(\mathcal{D}) \restriction \mathcal{A}_\Sigma$ by adding all these relation symbols. The restriction of any $D \in \mathcal{D}$ to $\mathcal{A}_\Sigma$ naturally becomes a structure for this language by interpreting R_ϕ as $\phi(D)$. We call this the **full induced structure** on D.

In the example above it would be enough to add a relation symbol for the image of the arrow from vertex 1 to vertex 2.

Let us say that a set Γ of objects/sorts of $\mathrm{fun}(\mathcal{D})$ **subgenerates** $\mathrm{fun}(\mathcal{D})$ if every $F \in \mathrm{fun}(\mathcal{D})$ is a subquotient of a finite product of objects in Γ. If Γ is so then, if $D \in \mathcal{D}$, $F \in \mathrm{fun}(\mathcal{D})$ and $a \in FD$ (strictly, $\in \overrightarrow{F}D$), there are $G_1, \dots, G_n \in \Gamma$ and there is $a' \in G_1(D) \oplus \cdots \oplus G_n(D)$ and there is $H \leq G_1 \oplus \cdots \oplus G_n$ (write $\times$ instead of $\oplus$ if one thinks of these as sorts rather than functors), such that there is an inclusion $i : F \longrightarrow (G_1 \oplus \cdots \oplus G_n)/H$ and such that, if $\pi : G_1 \oplus \cdots \oplus G_n \longrightarrow (G_1 \oplus \cdots \oplus G_n)/H$ is the projection, then $i_D(a) = \pi_D(a')$. (In the classical imaginaries/$^{\mathrm{eq}}$ construction a would be a tuple from a power of the home sort which maps to a given element, a', of the quotient of this power by a definable equivalence relation.) One may check that, for such Γ, the restriction of any $D \in \mathcal{D}$ to the full subcategory $\mathcal{A}_\Gamma$ with objects from Γ determines the whole of $\mathcal{D}$, in particular, the language $\mathcal{L}(\mathcal{D}) \restriction \Gamma$ already gives the full structure on $D \restriction \mathcal{A}_\Gamma$ (model-theoretically, $(\mathcal{L}(\mathcal{D}) \restriction \Gamma)^{\mathrm{full}}$ is a conservative extension of $\mathcal{L}(\mathcal{D}) \restriction \Gamma$).

liftend

Lemma 25.8. *([79, A2.1] for modules) Let N be a pure-injective object of a definable category $\mathcal{D}$. Let Σ be a set of sorts of $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$. Let $N \restriction \Sigma$ denote the restriction of N to this set of sorts and give $N \restriction \Sigma$ the full structure induced, as above, by $\mathcal{L}(\mathcal{D})$. Then the natural (restriction) map $\mathrm{End}(N) \to \mathrm{End}(N \restriction \Sigma)$ is surjective.*

Proof. Let f be an endomorphism of $N \restriction \Sigma$ where this is given the full induced structure.

Choose a subgenerating set Γ for $\mathrm{fun}(\mathcal{D})$ (we could take this to be the whole of $\mathrm{fun}(\mathcal{D})$ but often, such as with modules, there is a natural choice of "small" subgenerating set). Enumerate the elements of $\bigcup\{\sigma(N) : \sigma \in \Sigma\}$ as the (probably infinite) tuple $\overline{a}$. Also enumerate, as $\overline{c}$, the set $\bigcup\{\gamma(N) : \gamma \in \Gamma\}$. Let $p = p(\overline{x}, \overline{a})$ be the pp-type of $(\overline{c}, \overline{a})$ in N (regarded as an $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$-structure). Let $q = p(\overline{x}, f\overline{a})$. We claim this is a consistent set of pp formulas (with parameters).

To see this, take $\phi(\overline{x}, \overline{a}) \in p$ (of course only finitely many variables and constants actually appear in ϕ). Then $N \models \exists \overline{x}\, \phi(\overline{x}, \overline{a})$. Note that $\exists \overline{z}\, \phi(\overline{x}, \overline{y})$ has all its free variables of sorts in Σ so the language for the full induced structure includes a relation symbol R such that $R(N \restriction \Sigma) = \{\overline{b} : N \models \exists \overline{x}\, \phi(\overline{x}, \overline{b})\}$. This set must be preserved by any endomorphism, such as f, of the full induced structure, so $N \models \exists \overline{x}\, \phi(\overline{x}, f\overline{a})$. This shows consistency of q.

Since N is pure-injective = algebraically compact (21.1) there is a solution of q in N (although the definition of algebraically compact deals with consistent sets of formulas where there are finitely many free variables but the extension to arbitrary sets of pp formulas is an easy transfinite induction). That is, there is $\overline{d}$ in N such that $N \models p(\overline{d}, f\overline{a})$. Define a map on $N \restriction \Gamma$ by $\overline{c} \mapsto \overline{d}$. By construction this is an endomorphism of $N \restriction \Gamma$ and hence, as remarked above, defines an endomorphism of N, regarded as an $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$-structure, which restricts to f on $N \restriction \Sigma$, as required. $\square$

Given an interpretation functor $I : \mathcal{C} \longrightarrow \mathcal{D}$ there is the question, see [81, 3.7], as to whether it preserves all induced structure. That is, for each object $C \in \mathcal{C}$ the image, IC may be regarded as part of $C^{\mathrm{eq}+}$, namely the restriction of C, regarded as an exact functor from $\mathrm{fun}(\mathcal{C})$ to $\mathbf{Ab}$, to the image of I_0. This may be given the full induced structure. On the other hand, regarded (see 25.6) as an object of $\mathcal{D}$, IC has only the $\mathcal{D}$-definable structure. We say that I **preserves all induced structure** if the full induced structure on IC is definable from the $\mathcal{D}$-structure, that is, is definable using only the morphisms in $\mathrm{im}(I_0)$.

Recall (§13) that I is said to be full on pure-injectives if it is full when restricted to $\mathrm{Pinj}(\mathcal{C})$.

fullindstr

Theorem 25.9. *([81, p. 203, 3.17]) Let $I : \mathcal{C} \longrightarrow \mathcal{D}$ be an interpretation functor between definable categories. Then I preserves all induced structure iff I is full on pure-injectives. If this is so then I_0 is full.*

Proof. The proof uses Svenonius' Theorem (e.g. [15, 5.3.3] or [44, 10.5.2]) from model theory so we recall what that says. Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be languages. The inclusion means that every sort of $\mathcal{L}_1$ is also one of $\mathcal{L}_2$ and every symbol of $\mathcal{L}_1$ is also one (with the same sorting) of $\mathcal{L}_2$. Suppose that M is an $\mathcal{L}_2$-structure which is "sufficiently saturated" (in our situation every structure is stable so has a "sufficiently saturated" elementary extension, see, e.g., [44, 10.2.7]). Denote by $M \upharpoonright \mathcal{L}_1$ the restriction of M to an $\mathcal{L}_1$-structure Let $\eta(x)$ be a formula of $\mathcal{L}_2$ and suppose that the solution set $\eta(M)$ is preserved, as a set, by every automorphism of $M \upharpoonright \mathcal{L}_1$. Then the theorem is that there is a formula ξ of $\mathcal{L}_1$ such that $\xi(M) = \eta(M)$ (hence also the same will be true with M replaced by any $\mathcal{L}_2$-structure elementarily equivalent to M).

For example, suppose that $\alpha : R \to S$ is a ring morphism which, for purposes of this discussion, we may suppose to be monic (replace R by αR). This induces an inclusion (strictly, an embedding) of the language, $\mathcal{L}_R$, for R-modules into that for S-modules, whereby every function symbol r of the first is replaced by αr. Then if M is an S-module, $M \upharpoonright \mathcal{L}_R$ is M_R: M regarded as an R-module by restriction of scalars along α. Svenonius' Theorem implies that if ϕ is a pp formula in $\mathcal{L}_S$ and if M_S is sufficiently saturated then, if $f\phi(M) \subseteq \phi(M)$ for every $f \in \mathrm{Aut}(M_R)$ then there is a formula ψ of $\mathcal{L}_R$ such that $\phi(M) = \psi(M)$ (one may check that ψ may be taken to be pp if $M \equiv M^2$) so then ϕ and ψ will define the same set in any module in the definable category, $\langle M \rangle$, generated by M.

For the proof of our theorem, $\mathcal{L}_1$ is the language based on the category $\mathrm{im}(I_0) = I_0(\mathbb{L}(\mathcal{D})^{\mathrm{eq}+})$ and $\mathcal{L}_2$ is the language for the full induced (by $\mathcal{L}(\mathcal{C})$) structure on the image of I_0, that is, the language based on the full subcategory of $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ which has, for its objects, those in the image of I_0 and where all necessary relation symbols are added as described above.

Suppose first that I is full on pure-injectives. Let ϕ be any pp formula of $\mathcal{L}_2$. Let M be an object of $\mathcal{C}$ such that $\langle M \rangle = \mathcal{C}$, i.e. $\mathrm{supp}(M) = \mathrm{Zg}(\mathcal{C})$ (this is defined just before 21.14) and such that M is sufficiently saturated, in particular (21.1) is pure-injective. Let M_2 denote the restriction of M to $\mathcal{L}_2$ (that is, IM equipped with its full induced structure): clearly this also is sufficiently saturated. Since I is full on pure-injectives $I : \mathrm{End}(M) \to \mathrm{End}(IM)$ is surjective so every automorphism of IM lifts to an endomorphism of M and that restricts to an endomorphism of M_2. Since the formula ϕ is pp, $\phi(M)$ is preserved by every endomorphism of M_2. Therefore $\phi(M)$ it is preserved by every automorphism of IM. By Svenonius' Theorem it follows that there is a (necessarily pp) formula of $\mathcal{L}_1$ which is equivalent to ϕ in every object of $I\mathcal{C}$. That formula, being in $\mathcal{L}_1$, can be pulled back to a formula of $\mathcal{L}(\mathcal{D})$ (choose a preimage in $\mathcal{L}(\mathcal{D})$ for every symbol) which, therefore, is equivalent to ϕ in every object of the image of I (and hence in the definable subcategory of $\mathcal{D}$ generated by this image). Thus I preserves all induced structure.

For the converse suppose that I preserves all induced structure. In order to check that I is full on pure-injectives it is enough to show that if $N \in \mathrm{Pinj}(\mathcal{D})$ then $I : \mathrm{End}(N) \to \mathrm{End}(IN)$ is full. By assumption every endomorphism of IN also is an endomorphism of the full induced structure on IN, that is, of IN regarded as a structure for the language $\mathcal{L}_2$ above. That every such endomorphism lifts to an endomorphism of N is the content of 25.8 which, therefore, finishes the proof.

The last statement is clear since morphisms of $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ between objects of $\mathrm{im}(I_0)$ must, in this case, be $\mathcal{D}$-definable, hence be in the image of I_0. $\square$

Corollary 25.10. *Let $I : \mathcal{C} \longrightarrow \mathcal{D}$ be an interpretation functor between definable categories. If I is full then I preserves all induced structure.*

exA2sort2

Example 25.11. The condition of I_0 being full is not enough to imply that I preserves all induced structure. This can be seen by continuing 25.7. Consider the functor $I : \text{Mod-}kA_2 \longrightarrow \text{Mod-}k$ which takes M to Me_2. Clearly this is an interpretation functor and it is easy to see, continuing from 25.7, that I is not full (on pure-injectives but, over this ring of finite representation type, every module is pure-injective). The functor I_0, however, is full. By 12.5, fun-$k \simeq$ mod-k. Let Σ be the set of sorts in the image of I_0: this is just σ_2 (in the notation of 25.7) and its powers. Since σ_2 is, under the equivalence 22.2, easily seen to be identified with the functor $(S_2, -)$ where S_2 is the simple module at vertex 2 of the quiver A_2, it follows easily that there are no morphisms in fun-$kA_2 \restriction \Sigma$ other than those in the image of I_0 which is, therefore, full.

26 Stability

secstab

Just as for ordinary modules one has the usual criteria for ω-stability (more generally, total transcendality) and superstability. The proofs are essentially the same, the only (very slight) complication being that one must take account of the multi-sorted nature of the language. In fact, it is more in the definitions than in the proofs that this has its effect. For example, one should define an object M of a definable category $\mathcal{D}$ to be **finite** if it is finite in each sort, that is, if we regard $\mathcal{D}$ as a definable subcategory of a functor category Mod-$\mathcal{A}$ then each morphism group $((-, A), M)$, that is MA, should be finite. (Clearly this is the right definition since the finite structures should be those with no proper elementary extensions.) We should really prove that this definition is independent of representation/language. An argument is as follows. If each sort (A, M) is finite then for every $C \in \text{mod-}\mathcal{A}$ the group (C, A) also will be finite (since there is an epimorphism from a finite direct sum of representables to C) and hence, for every functor $F \in \text{fun}(\mathcal{D})$, the group FC must be finite (since there is a presentation of F by representable functors $(D, -) \to (C, -) \to F \to 0$ with $C, D \in \text{mod-}\mathcal{A}$). Under any other representation, each sort of M is the value of $\overrightarrow{F}M$ for some such F. So it is proved. The same sort of argument shows independence of representation of other notions such as κ-stability and also the various chain conditions on pp-definable subgroups that we will see below. (A primitive version of this invariance is the observation that the various finiteness conditions in the model theory of modules and others could be imposed only on 1-types and then they followed invariably for n-types: the explanation for this loses all trace of being *ad hoc* when we work in this general context).

addstable

Theorem 26.1. *Every object of a definable additive category is stable.*

The proof is an easy consequence of pp-elimination of quantifiers.

sschar

Theorem 26.2. *Let M be an object of a definable category. Then M is superstable iff for every sort of M and for every descending chain, $\phi_0(M) > \phi_1(M) > \cdots > \phi_i(M) > \dots$, of pp-definable subgroups of the value of that sort on M, there is n such that each quotient $\phi_i(M)/\phi_{i+1}(M)$, for $i \geq n$ is finite (in the usual sense!).*

Superstability *per se* is, algebraically, not that interesting a condition in this context: in so far as it differs from total transcendentality it depends on certain pp-quotients being finite, and this is not a condition which is preserved under, for example, forming infinite direct sums. For instance, if M is totally transcendental then so is every object in the definable subcategory, $\langle M \rangle$, generated by M, whereas every object of $\langle M \rangle$ is superstable iff M already is totally transcendental. More interesting is the kind of condition that we get by closing superstability under direct sums, see Section 27 below.

Say that M **has the descending chain condition** on pp-definable subgroups if each sort (A, M) satisfies this condition. Also, as usual, say that M is **Σ-pure-injective** if every direct sum, $M^{(I)}$, of copies of M is pure-injective. The proof of the next result is, with trivial modifications, as for modules.

Theorem 26.3. *Let M be an object of a definable category. Then M is totally transcendental iff M has the descending chain condition on pp-definable subgroups iff M is Σ-pure-injective iff $M^{(\aleph_0)}$ is pure-injective.*

We also say, following [19], that an object M of a definable category $\mathcal{D}$ has **finite endolength** if each sort, $\overrightarrow{F}M$, of M has finite length when considered as a module over the endomorphism ring (in $\text{fun}(\mathcal{D})$), $\text{End}(F)$, of that sort. As for finiteness, this implies the condition for every sort $F \in \text{fun}(\mathcal{D})$, of $M^{\text{eq}+}$, even if the original language is based on a subcategory. Since pp-definable subgroups are invariant under endomorphisms an object of finite endolength is Σ-pure-injective.

27 Ranks

secrk

In the additive context the usual hierarchies of ranks and complexity in model-theoretic stability theory turn out, once they go beyond totally transcendental theories, to be over-subtle. There are, however, other ranks and dimensions which do give useful and significant gradations of complexity in additive categories. We say just a little about them here since there are already accounts in [77] and, especially, [84]. The first, elementary Krull dimension, was introduced by Garavaglia [31]. This is simply the usual (Gabriel-Rentschler) Krull dimension for the lattice of pp formulas. For instance, a superstable object has elementary Krull dimension (≤ 1). That dimension has a "direction" (artinian lattices have Krull dimension 0 whereas noetherian, non-artinian lattices have dimension (defined and) at least 1) and it turns out that a refinement of this dimension (slower-growing but "equi-existent") is more useful. This dimension, now called m-dimension, was introduced in Ziegler's paper [105].

Another dimension, appearing in [105] and termed width (a variant, but essentially equivalent, notion of breadth is defined at [77, §10.2]), also has proved to be useful, though it is a much coarser measure. At least for modules, the actual value of m-dimension matters (this fineness makes it better than elementary Krull dimension) but usually what one wants to know about width is whether it is defined (ordinal-valued) or not ("∞"). There are a number of results relating these dimensions to the structure of pure-injective objects ([105], [77, Chpt. 10], [84, Chpts. 7, 13]).

One may apply the Cantor-Bendixson analysis to the Ziegler spectrum, $\mathrm{Zg}(\mathcal{D})$, of $\mathcal{D}$ (§14) (that analysis strips away all the isolated points, then repeats the process with what remains, transfinitely). In the context of the Ziegler spectrum, isolation of points in closed subsets turns out to be a significant property (see [84, Chpt. 5 esp.]) and so, therefore, does the CB-rank of the spectrum (and the CB ranks of its various closed subsets).

The final dimension, Krull-Gabriel dimension, came from Lenzing and his students, see [33], also [46]. That dimension, we denote it KGdim, is a modified version of Gabriel dimension: it is applied to the functor category and, at each stage, one localises away only the *finitely presented* simple objects (Burke showed, see [13, 5.1] that Gabriel and Krull-Gabriel dimension co-exist though the former generally grows more quickly since, at each stage, it localises away *all* simple objects).

The definitions being available elsewhere, and in forms easily generalised from modules to definable categories, we do not give them here but we do state the main connections between them. As usual, one may give proofs which are not substantially different from those for the original modules case.

We have already mentioned that m-dimension and elementary Krull dimension co-exist.

If $\mathcal{D}$ is a definable subcategory of an ordinary module category, Mod-R, then the **m-dimension** of $\mathcal{D}$ is defined to be the m-dimension of the lattice of pp formulas for $\mathcal{D}$ in one variable (of the unique sort of the usual 1-sorted language, based on R). Equivalently it is the m-dimension of the lattice of finitely generated subfunctors of the localisation, $(R_R, -)/\mathcal{S}_\mathcal{D}$, of the forgetful functor by the Serre subcategory corresponding to $\mathcal{D}$. You can see the problem in generalising this: there is, in general, no distinguished sort, so one has, for each sort, the m-dimension of pp formulas in that sort, equivalently, for each object of $\mathrm{fun}(\mathcal{D})$, the m-dimension of the lattice of finitely generated subobjects of that object. We note, however, that, in the usual 1-sorted language of R-modules, the sort corresponding to R_R generates all the others in $\mathbb{L}_R^{\mathrm{eq}+} \simeq \text{fun-}R$. In particular, it is the least upper bound on the m-dimensions of all sorts. So we may reasonably define the m-dimension of $\mathcal{D}$ in the general case simply to be the supremum of m-dimensions of all sorts: clearly this can be measured on any generating subcategory. Certainly it can happen, if there is no finite generating set of sorts, that this upper bound is a limit ordinal which is not actually attained (a similar comment will apply to Cantor-Bendixson rank of the spectrum since that need not be compact outside the context of modules over rings).

Like m-dimension, width is a dimension of modular lattices, applied to the lattice of pp formulas of a sort, equivalently the lattice of finitely generated subfunctors of a finitely presented functor. Whereas m-dimension is defined, inductively, by collapsing two-point intervals, width is defined by collapsing intervals which are chains, so certainly one has width $\leq$ m-dimension.

mdimeqCB

Theorem 27.1. *Let $\mathcal{D}$ be a definable category. If* $\mathrm{mdim}(\mathcal{D}) < \infty$ *then* $\mathrm{CB}(\mathcal{D}) = \mathrm{mdim}(\mathcal{D})$. *If $\mathcal{L}(\mathcal{D})$ is countable then the converse is true.*

These are results, [105, 8.6, 8.3], of Ziegler (for modules). In fact, in the situation of the theorem, the successive quotient lattices and the successive CB derivatives correspond (see [77, §10.4] also [84, §5.3.6]).

It is not known whether or not the converse in 27.1 is true in general but, in many special cases, it is (see, e.g. [84, §5.3.2]).

Suppose that $\mathcal{C}$ is a locally finitely presented category. Let $\mathcal{S}_0$ be the Serre subcategory generated by the finitely presented simple objects of $\mathcal{C}$. Let $\mathcal{T}_0$ be the closure of $\mathcal{S}_0$ under direct limits: then $\mathcal{T}_0$ is the hereditary, finite type, torsion class generated by $\mathcal{S}_0$. Let $Q_0 : \mathcal{C}_0 = \mathcal{C} \longrightarrow \mathcal{C}_1 = \mathcal{C}/\mathcal{T}_0$ be the corresponding localisation. Inductively define $\mathcal{T}_{\alpha+1}$ to be the hereditary, finite type, torsion subclass of $\mathcal{C}$ such that localisation at $\mathcal{T}_{\alpha+1}$, denote it $Q_{\alpha+1} : \mathcal{C} \longrightarrow \mathcal{C}_{\alpha+2} = \mathcal{C}/\mathcal{T}_{\alpha+1}$, is the localisation $Q_\alpha : \mathcal{C} \longrightarrow \mathcal{C}_{\alpha+1}$ followed by the localisation $\mathcal{C}_{\alpha+1} \longrightarrow (\mathcal{C}_{\alpha+1})_1$. At limit ordinals λ, set $\mathcal{T}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{T}_\alpha$ and let Q_λ be the corresponding localisation. The **Krull-Gabriel dimension**, $\mathrm{KGdim}(\mathcal{C})$ is the least α such that $\mathcal{C}_{\alpha+1}$ is the trivial (all objects zero) category, if such an ordinal exists, otherwise set $\mathrm{KGdim}(\mathcal{C}) = \infty$. If $\mathcal{D}$ is a definable category then this analysis applies to the functor category, $\mathrm{Fun}(\mathcal{D})$, which is locally coherent (by 6.1 and 7.3). For modules the equality of m-dimension and Krull-Gabriel dimension may be found at [13, §4]. Since these functor categories, and all their localisations, are locally coherent, the Krull-Gabriel filtration of $\mathrm{Fun}(\mathcal{D})$ can be seen by its effect on $\mathrm{fun}(\mathcal{D})$. So the duality of $\mathrm{fun}(\mathcal{D})$ and $\mathrm{fun}^{\mathrm{d}}(\mathcal{D})$, which preserves simple objects in particular, shows that $\mathcal{D}$ and $\mathcal{D}^{\mathrm{d}}$ have the same Krull-Gabriel dimension.

Theorem 27.2. *Let $\mathcal{D}$ be a definable category. Then* $\mathrm{KGdim}(\mathrm{Fun}(\mathcal{D})) = \mathrm{mdim}(\mathcal{D}) = \mathrm{mdim}(\mathcal{D}^{\mathrm{d}}) = \mathrm{KGdim}(\mathrm{Fun}^{\mathrm{d}}(\mathcal{D}))$.

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