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Nonuniqueness in diffusion-based optical tomography

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A condition on nonuniqueness in optical tomography is stated. The main result applies to steady-state (dc) diffusion-based optical tomography, wherein we demonstrate that simultaneous unique recovery of diffusion and absorption coefficients cannot be achieved. A specific example of two images that give identical dc data is presented. If the refractive index is considered an unknown, then nonuniqueness also occurs in frequency-domain and time-domain optical tomography, if the underlying model of the diffusion approximation is employed.   © 1998 Optical Society of America

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Optical tomography is the method of using light in a narrow-wavelength band in the near infrared ($\sim 700 - \sim 1000$ nm) to transilluminate tissue and using the resulting measurements of intensity on the tissue boundary to reconstruct a map of the optical properties within the tissue. Recent surveys of optical tomography can be found in Refs. 1 and 2. Since there are several optical properties on which the data depend, the image reconstruction process can be regarded as an inverse problem in the recovery of multiple coefficients from boundary data. The data-acquisition method can use steady-state, time-resolved, or intensity-modulated measurements, each of which has been used to generate reconstructed images.[3–5]

The predominant reconstruction method is based on recovering the coefficients of a partial differential equation that is either elliptic (in the steady-state and the intensity-modulated cases) or parabolic (in the time-resolved case). A central issue in problems of this type is uniqueness, i.e., the ability to demonstrate that only one solution exists that is consistent with the data.[6,7] Whereas some standard results can be applied in optical tomography, little research has specifically addressed the particular problems in this application. A result showing the nonuniqueness of the inverse source problem was recently presented,[8] but the extension to the inverse parameter problem depended on a Born approximation and a finite number of sources. The similarity of absorption and scattering perturbation was shown for a single source, also by use of a Born approximation.[9] In this Letter we present a simple proof that for steady-state measurements there exists an infinite set of optical parameters that give rise to identical data. This result is independent of the specific location of source and measurement points and applies in the theoretical limit of complete data. The proof is based on fundamental properties of the underlying partial differential equation and does not depend on a linearization of the inverse problem.

We assume here that the light-propagation model is the diffusion approximation to the radiative transfer equation, which in the frequency domain is

$$-\nabla \cdot \kappa \nabla \hat{\Phi}(\omega) + \left(\mu_a + \frac{i\omega n}{c} \right) \hat{\Phi}(\omega) = \hat{q}_0(\omega), \quad (1)$$

where $\hat{\Phi}$ is the isotropic photon density, $\hat{q}_0$ is an isotropic source distribution, c is the speed of light in vacuum, n is the refractive index, and the diffusion coefficient κ is given by $\kappa = [3(\mu_a + \mu_s')]^{-1}$, where μ_a and μ_s' are the absorption and the reduced scattering coefficient, respectively. We consider the problem in a domain Ω, with boundary $\partial\Omega$. We assume that $\Omega = \Omega_1 \cup \Omega_0$, with $\partial\Omega_0$ strictly contained in the interior of Ω as shown in Fig. 1. The important aspect of this decomposition is that we argue that q_0 is always zero within Ω_0. This argument is equivalent to making the standard assumption that the source is localized to within a distance z_0 of the boundary (usually taken as $z_0 = 1/\mu_s'$) so that irrespective of the source's location on the boundary its isotropic representation is wholly contained within Ω_1. The boundary measurement $\Gamma(\mathbf{b})$ at $\mathbf{b} \in \partial\Omega$ is related to $\Phi(\mathbf{r})$ by

$$\Gamma(\mathbf{b}) = -c\kappa(\mathbf{b})\mathbf{b}_\perp \cdot \nabla\Phi(\mathbf{b}), \quad (2)$$

where $\mathbf{b}_\perp$ is the outer normal of $\partial\Omega$ at $\mathbf{b}$.

We can consider the model to be characterized by the three spatially varying functions n, μ_a, and κ. These functions are strictly positive everywhere, and in addition, $n \geq 1$; in practice upper and lower bounds will also exist for each of these functions.
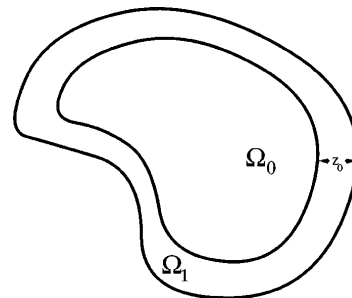

Fig. 1. Definition of Ω_0 and Ω_1. Ω_0 is a region with every point at least a distance z_0 from the domain boundary $\partial\Omega$. All isotropic sources are wholly contained within Ω_1.

A simplification in these types of problem[6] is to make the change of variables $\gamma^2 = \kappa$ and $\Psi = \gamma\Phi$. Writing Eq. (1) as

$$-\gamma^2\nabla^2\hat{\Phi} - 2\gamma\nabla \cdot \gamma\nabla\hat{\Phi} + \mu_a\hat{\Phi} + \frac{i\omega n}{c}\hat{\Phi} = \hat{q}_0 \qquad (3)$$

and using $\nabla^2\Psi = \gamma\nabla^2\Phi + 2\nabla\Phi \cdot \nabla\gamma + \Phi\nabla^2\gamma$ leads to the Helmholtz-type equation

$$-\nabla^2\hat{\Psi}(\omega) + \hat{\eta}(\omega)\hat{\Psi}(\omega) = \frac{\hat{q}_0(\omega)}{\gamma}, \qquad (4)$$

where $\hat{\eta}(\omega) = \eta_0 + i\omega\xi$, with

$$\eta_0 = \left(\frac{\nabla^2\gamma}{\gamma}\right) + \frac{\mu_a}{\gamma^2}, \qquad \xi = \frac{n}{c\gamma^2}. \qquad (5)$$

If it were the case that the source $\hat{q}_0$ in Eq. (4) were of Neumann type[10] and we had complete knowledge of all pairs of Dirichlet and Neumann data on the boundary, then one could extend the result found by Sylvester and Uhlmann[6] to the complex case (Ref. 11, theorem 5.2.2) to prove that $\hat{\eta}$ is uniquely determined. For the case of an interior source, as is generally assumed in diffusion-based optical tomography, to the best of our knowledge the equivalent uniqueness result is still an open question. In either case, in the practical situation when we have a finite number of sources and detectors we can seek to recover only a finite number of parameters in an infinite dimensional solution space; one hopes that this number would increase with the number of independent measurements. However, as in any ill-posed problem, the number of parameters that one can recover at a given accuracy is limited by the accuracy as well as the number of independent measurements.

Suppose that we have two ordered sets of functions (κ, μ_a, n) and $(\tilde{\kappa}, \tilde{\mu}_a, \tilde{n})$ with the equivalent canonical parameters $\hat{\eta}$ and $\hat{\tilde{\eta}}$. We can state two conditions:

Condition 1: $\hat{\tilde{\eta}} = \hat{\eta}$ everywhere in Ω,
Condition 2: $\tilde{\kappa} = \kappa$ everywhere in Ω_1.

If these conditions are met, then it follows immediately that both sets will give rise to the same solution Ψ and thus to the same measured data Γ.

Consider the dc case ($\omega = 0$). Suppose that we add a function α to κ and β to μ_a:

$$\tilde{\kappa} = \kappa + \alpha, \qquad \tilde{\mu}_a = \mu_a + \beta. \qquad (6)$$

Condition 1 is now just $\tilde{\eta}_0 = \eta_0$. If condition 2 also holds, then the identical data will be measured if

$$\left[\frac{\nabla^2(\kappa + \alpha)^{1/2}}{(\kappa + \alpha)^{1/2}}\right] + \frac{\mu_a + \beta}{(\kappa + \alpha)} = \left(\frac{\nabla^2\kappa^{1/2}}{\kappa^{1/2}}\right) + \frac{\mu_a}{\kappa}. \qquad (7)$$

Thus for any α satisfying $\alpha = 0$ in Ω_1, we can find β from

$$\beta = (\kappa + \alpha)\left\{\left(\frac{\nabla^2\kappa^{1/2}}{\kappa^{1/2}}\right) + \frac{\mu_a}{\kappa} - \left[\frac{\nabla^2(\kappa + \alpha)^{1/2}}{(\kappa + \alpha)^{1/2}}\right]\right\} - \mu_a. \qquad (8)$$

In the case when $\omega \neq 0$ we consider also that a function ν is added to n:

$$\tilde{n} = n + \nu. \qquad (9)$$

Then, in addition to Eq. (7) we require that $\tilde{n} = (\tilde{\kappa}/\kappa)n$, leading to the condition

$$\nu = (\alpha/\kappa)n. \qquad (10)$$

Thus in general we may state that nonuniqueness can occur in the frequency-domain case if the refractive index is allowed to vary. Although in reality the bounds on κ, μ_a, and n rule out some possibilities, the above result proves that an infinite set of functions exists that give rise to the same data. Only if n is known in Ω and $\omega \neq 0$ and $\hat{\eta}$ can be found uniquely can we state that κ and μ_a can be determined uniquely from

$$\kappa = \frac{\omega n}{c\, I[\hat{\eta}]}, \qquad \mu_a = \kappa\left[\mathcal{R}[\hat{\eta}] - \left(\frac{\nabla^2\kappa^{1/2}}{\kappa^{1/2}}\right)\right]. \qquad (11)$$

We give a simple example of equivalent solutions for the dc case. If μ_a and κ are constant and α is a Gaussian, $\alpha = A\kappa\exp[-(x^2 + y^2)/2a^2]$, then, putting $r^2 = x^2 + y^2$, we have

$$\beta = \frac{1}{4a^4\left[A + \exp\left(\frac{r^2}{2a^2}\right)\right]}\left\{4a^2A(\kappa + a^2\mu)\left(1 + \frac{\alpha}{\kappa}\right)\right.$$

$$\left. - \left[A + 2\exp\left(\frac{r^2}{2a^2}\right)\right]\alpha r^2\right\}. \qquad (12)$$

We used these functions as the coefficients in a finite-element implementation of the diffusion equation[11] for the particular case $\kappa = 0.164609$ mm, $\mu_a = 0.025$ mm^{-1}, and $\mu_s' = 2$ mm^{-1}, with the Gaussian standard deviation $a = 3.57$ mm. Constant $n = 1.4$ was assumed, and the domain Ω was a mesh approximation to a circle of diameter 50 mm with 13,207 nodes and 26,040 elements. Figure 2 shows the mesh together with cross sections showing the functions $\tilde{\kappa}$, $\tilde{\mu}_a$, and $\tilde{\mu}_s'$, with the peak of $\tilde{\kappa}$ located at (6.25, 6.25). Note that $\tilde{\mu}_a$ is not a Gaussian but has negative sidelobes. Strictly, the support of α is infinite and does not satisfy condition 2; however, in this example $z_0 = 0.5$ mm, and the boundary of Ω_0 is sufficiently far away that $\tilde{\kappa}$ and κ are practically identical in Ω_1.

Steady-state dc intensity data were generated for the homogeneous case (μ_a, κ), the μ_a-only case ($\tilde{\mu}_a$, κ), the κ-only case (μ_a, $\tilde{\kappa}$), and the null-space case ($\tilde{\mu}_a$, $\tilde{\kappa}$). The source was placed on the boundary at position (25.0, 0) ($\theta = 0$ in Fig. 2), and the data were sampled at 380 angular locations around the boundary. Figure 3 shows the relative difference $(\Gamma_{inhom} - \Gamma_{hom})/\Gamma_{hom}$ for the three inhomogeneous cases. In the μ_a-only case the increased absorption leads to a reduction in intensity, whereas for the κ-only case the intensity is increased. The relative changes are of the same order as the data itself, but
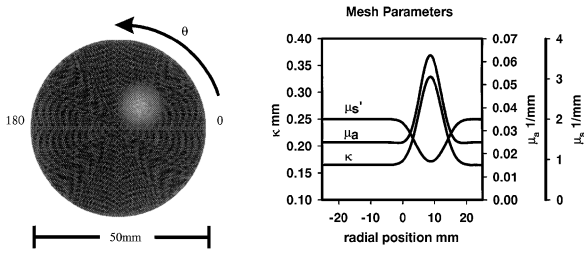
Fig. 2. Mesh used for the example data, together with cross sections showing the functions in κ, μ_a, and μ_s' that give rise to the same boundary data as constant values.
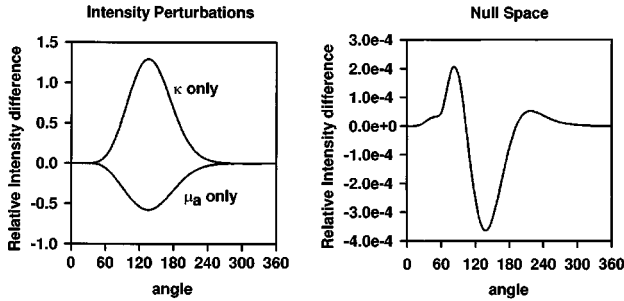

Fig. 3. Relative difference of the intensity data $\Gamma_{\tilde{\mu}_a, \kappa} - \Gamma_{\mu_a, \kappa}/\Gamma_{\mu_a, \kappa}$ (μ_a only), $\Gamma_{\mu_a, \tilde{\kappa}} - \Gamma_{\mu_a, \kappa}/\Gamma_{\mu_a, \kappa}$ (κ only), and $\Gamma_{\tilde{\mu}_a, \tilde{\kappa}} - \Gamma_{\mu_a, \kappa}/\Gamma_{\mu_a, \kappa}$ (null space), for a source at $\theta = 0$.

note that they are not equal and opposite, because the additive effect of the μ_a and the κ inhomogeneities is not linear. The null-space case cancels out the change in intensity almost exactly, to within an order of less than 0.05%. There is an inherent inaccuracy in these calculations in that the smooth functions α and β are represented by piecewise linear approximations. Nevertheless the order of 0.05% error that results from round-off error in the subtraction of two similar numbers at machine precision is within the level of the previously reported numerical accuracy of the finite-element method impementation.[12] We repeated the calculations with the source at different angular positions on the boundary, with similar results in all cases.

We can also consider the time-domain problem for which the governing equation is

$$-\nabla \cdot \kappa \nabla \Phi(t) + \mu_a \Phi(t) + \frac{n}{c}\frac{\partial \Phi(t)}{\partial t} = q_0(t), \qquad (13)$$

which has the canonical form

$$-\nabla^2 \Psi(t) + \eta_0 \Psi(t) + \xi \frac{\partial \Psi(t)}{\partial t} = \frac{q_0(t)}{\gamma}. \qquad (14)$$

Using the same notation for $\tilde{\Psi}$, $\tilde{\eta}_0$, and $\tilde{\xi}$, imposing the same relationships on α, β, and ν given in Eqs. (7) and (10), and subtracting, we get

$$-\nabla^2[\tilde{\Psi}(t) - \Psi(t)] + \eta_0[\tilde{\Psi}(t) - \Psi(t)] +$$

$$\xi\frac{\partial[\tilde{\Psi}(t) - \Psi(t)]}{\partial t} = 0. \qquad (15)$$

Equation (15) can be satisfied only by $\tilde{\Psi}(t) = \Psi(t)$, leading once again to nonuniqueness if n is assumed to be unknown.

We have shown that if the diffusion equation is assumed as the model of light propagation in optical tomography, then in the general case when three parameters n, μ_a, and κ are unknown there exists no unique solution to the inverse problem. For the steady-data (dc) case an infinite set of functions can be found that are consistent with the data. One cannot improve this result by taking more sources and detectors. If n is assumed to be known, then uniqueness can be demonstrated in the limit of continuous measurement and source distributions, provided that either time-resolved or intensity-modulated measurements are employed. It is quite likely, however, that there exist functions that, although distinguishable, have only small differences in the data that they produce.

The implication for the inverse problem is that the functions derived lie in the null space of the forward problem and therefore make the inverse problem excessively ill posed. Note that the statement presented here is about the nonlinear problem (forward and inverse) and does not depend on a particular discretization or linearization. In practice, in inverse problems one solves a regularized problem wherein a combination of a likelihood term and a term representing some constraints on the solution is optimized. Under these circumstances functions such as those in Fig. 2 will have different energies and may well be distinguishable. Nevertheless, the presence of the fundamental nonuniqueness may pose a difficulty.

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