Recovering Riemannian metrics in monotone families from boundary data

Gaburro, Romina and Lionheart, William R.B.

2008

MIMS EPrint: **2008.71**

Manchester Institute for Mathematical Sciences

School of Mathematics

The University of Manchester

Recovering Riemannian metrics in monotone families from boundary data

ROMINA GABURRO*AND WILLIAM R.B. LIONHEART †

Abstract. We discuss the inverse problem of determining the anisotropic conductivity of a body described by a compact, orientable, Riemannian manifold M with boundary ∂M, when measurements of electric voltages and currents are taken on all of ∂M. Specifically we consider a one parameter family of conductivity tensors, extending results obtained in [AG] where the simpler Euclidean case is considered. Our problem is equivalent to the geometric one of determining a Riemannian metric in monotone one parameter family of metrics from its Dirichlet to Neumann map on ∂M.

1 Introduction.

In electrical impedance tomography (EIT) one seeks to recover the interior electrical conductivity of an object from measurements of electrostatic potential and current density at the boundary of the object. In an anisotropic medium Ω, where $\Omega \subset \mathbb{R}^n$ is a domain, with conductivity tensor the symmetric, positive definite matrix $\sigma = \sigma(x)$, $x \in \Omega$, the electrostatic potential u in the medium satisfies

$$\text{div}(\sigma \nabla \text{u}) = 0, \quad \text{in} \quad \Omega. \tag{1.1}$$

Complete information about the relationship between applied surface current density and surface voltage is represented by the Dirichlet-to-Neumann map Λ_σ, associated with σ, defined by

$$\Lambda_\sigma : u|_{\partial\Omega} \mapsto \sigma \nabla u \cdot u|_{\partial\Omega},$$

---
*Department of Mathematics and Statistics, University of Limerick, Castletroy, Limerick, Ireland, Romina.Gaburro@ul.ie

†School of Mathematics, The University of Manchester, Alan Turing Building, Oxford Rd, Manchester M13 9PL, UK, bill.lionheart@manchester.ac.uk

for any solution u to (1.1). In other words the operator Λ_σ maps the Dirichlet data $u|_{\partial\Omega}$ (the boundary voltage) into the corresponding Neumann data (the boundary current density).

The inverse problem consists in determining σ from the knowledge of Λ_σ. It is well known that an isotropic (scalar) conductivity is uniquely determined by the boundary data (see [A], [KV1], [KV2], [SU]), while an anisotropic conductivity tensor is not uniquely determined by the boundary data (see [A], [AG], [L], [LU], [LaU], [N], [S]).

The physical problem of recovering the conductivity of a body by measurements of electric voltage and current density on its surface is closely related to the geometric problem of determining a Riemannian metric from its Dirichlet-to-Neumann map for harmonic functions (see [LU], [L]). For an orientable manifold M of dimension $n > 2$ the electric field is the 1-form $du \in \Omega^1(M)$ while the current density corresponds to an $(n-1)$-form and the conductivity tensor $\sigma \in \Omega^1(M) \otimes (\Omega^{n-1}(M))^*$, which can be viewed as a linear map taking electric field to current density (Ohm's law). The electrical power dissipation is then $du \wedge \sigma du \in \Omega^n(M)$ and must be a non-vanishing n-form that is symmetric: $\alpha \wedge \sigma\beta = \beta \wedge \sigma\alpha$ for all $\alpha, \beta \in \Omega^1(M)$. In dimension $n > 2$, the conductivity σ uniquely determines a Riemannian metric g such that

$$\sigma = *_g, \tag{1.2}$$

where $*_g$ is the *Hodge star* operator mapping 1-forms on M into $(n-1)$-forms (see [G1], [L], [LU]). The Dirichlet-to-Neumann map associated to σ is therefore defined as the operator Λ_σ mapping functions $u|_{\partial M} \in H^{1/2}(\partial M)$ into $(n-1)$-forms $\Lambda_\sigma(u|_{\partial M}) \in H^{-1/2}(\Omega^{n-1}(\partial M))$

$$\Lambda_\sigma(u|_{\partial M}) = i^*(\sigma du), \tag{1.3}$$

for any u, solution to

$$\Delta_g u = 0, \qquad \text{in} \quad M, \tag{1.4}$$

where $i : \partial M \to M$ is the inclusion mapping, i^* is the pull-back of i and $\Delta_g = -*_g d *_g d$ is the *Laplace Beltrami operator* on functions. (1.4) in coordinates becomes

$$\sum_{i\,j\,=\,1}^{n} (\det g)^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \left\{ (\det g)^{\frac{1}{2}}\, g^{i\,j}\, \frac{\partial u}{\partial x^j} \right\} = 0, \qquad \text{in} \quad M.$$

For the case $n = 2$ the situation is different as the two-dimensional conductivity determines a conformal structure of metrics under scalar

field, i.e. there exists a metric g such that $\sigma = \gamma *_g$, for a positive function γ.

In the case of a non-orientable manifold the current density $-\sigma du$ must be considered as a twisted $(n-1)$-form, that is it takes its values in the (non-trivial) orientation line bundle. We omit the non-orientable case from this paper for the sake of clarity.

The problem of recovering the Riemannian metric by boundary data in the inverse conductivity problem has been studied in the past and in recent years. Kurylev gave a fruitful insight on the study of inverse problems on Riemannian manifolds in [K], where the problem of reconstructing the coefficients of an elliptic operator from its boundary spectral data is presented. We also refer to [KKL], where the authors investigated whether the so-called boundary distance representation of a Riemannian manifold determines the Riemannian manifold. See also [LSU]. Lassas and Uhlmann [LaU] recovered a connected compact real-analytic Riemannian manifold (M, g) with boundary by making use of the Green's function of Δ_g. See also [LaTU].

In [AG] the case where the anisotropic conductivity tensor σ is *a priori* known to be of type $\sigma(x) = \sigma(x, a(x))$, is considered, where the one parameter matrix valued functions $t \longrightarrow \sigma(x, t)$ is *a priori* known to satisfies the so-called monotonicity assumption

$$D_t\, \sigma(\cdot,\, t) \geq \text{Const.} I > 0. \tag{1.5}$$

The aim of this paper is to consider the more general case of a Riemannian manifold (M, g_0) of dimension $n \geq 3$, where a one parameter family of metrics of type

$$t \longrightarrow g_t(x) := g(x, t),$$

is prescribed on M, for any $t \in [\lambda^{-1},\, \lambda]$, with $\lambda > 0$ constant and such that $g(x,\, 0) = g_0(x)$. Denoting by $*_t$ the *Hodge star* operator associated to the metric g_t, we assume that the following *monotonicity condition* is satisfied

$$*_0 \left((D_t *_t)\ \theta \wedge \theta \right) \geq Const. *_0 \left(*_0\, \theta \wedge \theta \right), \qquad \text{for any} \quad \theta \in \Omega^1(M). \tag{1.6}$$

The results obtained in [AG] are given in terms of the Euclidean metric $(g_0)_{ij} = \delta_{ij}$, here we allow g_0 to be a general Riemannian metric and condition (1.6) is given in terms of it. The case of a manifold with a flat metric g_0 will be still more general than the one treated in [AG].

Results of stability and uniqueness at the boundary and then global uniqueness in the interior are proven in the present paper.

The paper is organized as follows. Section 2 contains the statements of the main results (Theorems 2.3, 2.4, 2.5 and Corollary 2.6). In Section 3 we prove results of the existence of singular solutions on a Riemannian manifold. In Section 4 we give the proofs of the main results. For sake of brevity we only give the proof of Theorem 2.3, 2.4 as proofs of Theorem 2.5, and Corollary 2.6 follow the same line of proof of Theorems 2.3, 2.4 and the arguments used in [A], [AG].

2   Main results.

Let (N, g_0) be a C^∞ open, bounded Riemannian manifold of dimension $d \geq 3$.

DEFINITION 2.1. *For any $x^0 \in N$, $v \in T_{x^0}N$, we denote by $\rho_{v, x^0}(s)$ the geodesic of length s, starting at x^0 with direction v.*

DEFINITION 2.2. *For any $x^0 \in N$, we denote by $B_{N, r}(x^0)$ the geodesic ball*

$$B_{N, r}(x^0) = \left\{ x \in N \,|\, d(x, x^0) < r \right\},$$

where $d(\cdot, \cdot)$ is the geodesic distance on N induced by g_0.

Let $M \subset N$ be a compact submanifold of N, of dimension $3 \leq n \leq d$, with Lipschitz boundary ∂M; the definition of Lipschitz boundary we will be using is the one formulated below.

DEFINITION 2.3. *Given positive numbers L, r, h satisfying $h \geq Lr$, we say that a compact manifold $M \subset N$ has Lipschitz boundary if, for every $x^0 \in \partial M$, there exists a chart $(U, \{x_i\}_{i=1}^n)$ around x^0 in N and an (n-1)-dimensional submanifold $\mathcal{M} \subset U$, with $x_n = 0$, such that $x^0 \in \mathcal{M}$ and such that $\partial M \cap C_{r, h}$ is the graph of a Lipschitz function $f : \mathcal{M} \longrightarrow \mathbb{R}$ which satisfies*

$$|f(x') - f(y')| \leq L \, d(x', y'),$$

for any $x', y' \in \mathcal{M} \cap C_{r, h}$, where $\tilde{\nu} = -\frac{\partial}{\partial x_n}$ on $\partial M \cap U$ and

$$C_{r, h} = \left\{ x = \rho_{\tilde{\nu}, y}(s) \,|\, y \in B_{\mathcal{M}, r}(x^0), \, -h < s < h \right\}$$

is the geodesic cylinder in N of base $B_{\mathcal{M}, r}(x^0)$ and height h. Moreover

$$M \cap C_{r, h} = \left\{ x \in C_{r, h} \,|\, y \in B_{\mathcal{M}, r}(x^0), \, -h < s < 0 \right\}.$$

Let us denote by μ_{g_0} the volume form associated to the metric g_0 and by ∇ the Levi-Civita connection on (N, g_0); the class $\mathcal{H}$ of metrics $g_t(x) := g(x, t)$ admissible for our problem is given by the following definition. In the sequel we will make use of both notations $g_t(x)$ and $g(x, t)$, depending on the contest.

DEFINITION 2.4. *Given $p > n$, λ, E, $\mathcal{E}$, $\mathcal{F} > 0$, and denoting by $T_0^2(M)$ the bundle of covariant tensors of type (2, 0) on M, we say that the metric $g(\cdot, \cdot) \in \mathcal{H}$ if it satisfies the following conditions*

$$g_t \in W^{1,p}\big(M \times [\lambda^{-1}, \lambda], T_0^2(M)\big); \tag{2.1}$$

$$D_t g_t \in W^{1,p}\big(M \times [\lambda^{-1}, \lambda], T_0^2(M)\big); \tag{2.2}$$

$$\text{Ess sup}_{t \in [\lambda^{-1}, \lambda]} \Big(\| g_t(\cdot) \|_{L^p(M, \mu_{g_0})} + \| \nabla_X g_t(\cdot) \|_{L^p(M, \mu_{g_0})}$$
$$+ \|D_t g_t(\cdot)\|_{L^p(M, \mu_{g_0})} + \|D_t \nabla_X g_t(\cdot)\|_{L^p(M, \mu_{g_0})} \Big) \leq \mathcal{E},$$
for any smooth vector field $X \in C^\infty(TM)$, with $\|X\|_{L^\infty(M, \mu_{g_0})} = 1$ $\quad(2.3)$

$$\lambda^{-1} |\xi|^2 \leq g^{ij}(x)\xi_i\xi_j \leq \lambda |\xi|^2, \qquad \text{for almost every } x \in \Omega,$$
$$\text{for every } t \in [\lambda^{-1}, \lambda], \ \xi \in \mathbb{R}^n. \quad (2.4)$$

$$*_0 \big(\big(D_t *_{g(x, t)} \big) \ \theta \wedge \theta \big) \geq E^{-1} \ *_0 \left(*_0 \theta \wedge \theta \right), \qquad \text{for almost every } x \in \Omega,$$
$$\text{for every } t \in [\lambda^{-1}, \lambda],$$
$$\text{for every } \xi \in \mathbb{R}^n. \quad (2.5)$$

(2.4) and (2.5) are a condition of uniform ellipticity and a condition of monotonicity with respect to the variable t (see [AG]).

Remark 2.1. *The volume form associated to the metric g_0 is specified in (2.3), but, since M is compact, all the L^p-norms related to different volume forms are equivalent, therefore a different choice of the volume form will maintain* Ess sup *appearing in (2.3) bounded, although constant $\mathcal{E}$ will depend on the volume form. For sake of brevity we will denote any L^p norm by omitting to specify the volume form μ_{g_0} for now on, by meaning that these norms are calculated in terms of μ_{g_0}.*

Remark 2.2. *Conditions (2.1)-(2.3), combined together with the Sobolev imbedding theorems for $p > n$ on manifolds with Lipschitz boundary (see [GT, chapter 7, p. 158]), lead to*

$$g_t^{-1} \in W^{1,\,p}\big(M \times [\lambda^{-1},\,\lambda],\,T_0^2(M)\big); \qquad (2.6)$$

$$D_t g_t^{-1} \in W^{1,\,p}\big(M \times [\lambda^{-1},\,\lambda],\,T_0^2(M)\big). \qquad (2.7)$$

Furthermore

$$\mathrm{Ess\ sup}_{t\in[\lambda^{-1},\,\lambda]}\Big(\ \|\,g_t^{-1}(\cdot)\,\|_{L^p(M)} + \|\,\nabla_X g_t^{-1}(\cdot)\,\|_{L^p(M)}$$

$$+\,\|D_t g_t^{-1}(\cdot)\|_{L^p(M)} + \|D_t \nabla_X g_t^{-1}(\cdot)\|_{L^p(M)}\Big) \le \mathcal{F}(\mathcal{E},\,n),$$

for any smooth vector field $X \in C^\infty(TM)$, with $\|X\|_{L^\infty(M)} = 1$, (2.8)

where $\mathcal{F}(\mathcal{E},\,n) > 0$ is a constant depending on $\mathcal{E}$, n only. Moreover, if we define $G_t(x) := |g_t(x)|^{\frac{1}{2}} g_t^{-1}(x)$, then

$$G_t \in W^{1,\,p}\big(M \times [\lambda^{-1},\,\lambda],\,T_0^2(M)\big); \qquad (2.9)$$

$$D_t G_t \in W^{1,\,p}\big(M \times [\lambda^{-1},\,\lambda],\,T_0^2(M)\big); \qquad (2.10)$$

$$\mathrm{Ess\ sup}_{t\in[\lambda^{-1},\,\lambda]}\Big(\ \|\,G_t(\cdot)\,\|_{L^p(M)} + \|\,\nabla_X G_t(\cdot)\,\|_{L^p(M,\,\mu_{g_0})}$$

$$+\,\|D_t G_t(\cdot)\|_{L^p(M)} + \|D_t \nabla_X G_t(\cdot)\|_{L^p(M)}\Big) \le C(\mathcal{E},\,\mathcal{F}),$$

for any smooth vector field $X \in C^\infty(TM)$, with $\|X\|_{L^\infty(M)} = 1$, (2.11)

where $C(\mathcal{E},\,\mathcal{F}) > 0$ is a constant depending on $\mathcal{E}$, $\mathcal{F}$ only.

We shall denote by $\|\cdot\|_\star$ the norm of bounded linear operators between $H^{\frac{1}{2}}(\partial M)$ and $H^{-1/2}(\Omega^{n-1}(\partial M))$.

The first result is a stability result of the metrics at the boundary.

THEOREM 2.3. (Lipschitz stability at the boundary). *Let $(N,\,g_0)$ be a C^∞ open, bounded n-dimensional Riemannian manifold. Given $p > n$, let $M \subset N$ be a compact submanifold of N of dimension $n \ge 3$, with Lipschitz boundary ∂M. Suppose a and b are two functions on M satisfying*

$$\lambda^{-1} \le a(x),\ b(x) \le \lambda, \qquad \text{for each} \quad x \in M, \tag{2.12}$$

$$\parallel a \parallel_{W^{1,\,p}(M)},\ \parallel b \parallel_{W^{1,\,p}(M)} \le E \tag{2.13}$$

and $g(x,\ t) \in \mathcal{H}$. *Then we obtain*

$$\parallel g(x,\ a(x)) - g(x,\ b(x)) \parallel_{L^\infty(\partial M)} \le C \parallel \Lambda g(x,\ a) - \Lambda_{g(x,\,b)} \parallel_\star, \tag{2.14}$$

where C is a positive constant depending only on n, p, L, r, h, $diam(M)$, λ, E and $\mathcal{E}$.

THEOREM 2.4. (Hölder stability of derivatives at the boundary). *Given p, n, M, $(N,\ g_0)$ as in Theorem 2.3, let a, b satisfy (2.12), (2.13) and $g \in \mathcal{H}$. Suppose there exist a point $y \in \partial M$ and a neighborhood U of y in M, a positive integer k and some $\alpha, 0 < \alpha < 1$ such that*

$$g(x,\ t) \in C^{k,\alpha}(\bar{U} \times [\lambda^{-1},\ \lambda],\ T^{2,\,0}(M)), \tag{2.15}$$

$$(a - b) \in C^{k,\,\alpha}(\bar{U}). \tag{2.16}$$

Then, for any neighborhood W of y in M such that $W \subset U$ and any smooth vector field $Z \in C^\infty(TM)$, with $\parallel Z \parallel_{L^\infty(M)} = 1$, we have

$$\parallel \nabla_Z^k \big(g(x,\ a) - g(x,\ b) \big) \parallel_{L^\infty(\partial M \cap \bar{W})} \le C \parallel \Lambda_{g(x,\,a)} - \Lambda_{g(x,\,b)} \parallel_\star^{\delta_k \alpha}, \tag{2.17}$$

where ∇_Z^k is the kth covariant derivative with respect to the vector field Z and $\delta_k = \prod_{j=0}^k \frac{\alpha}{\alpha+j}$. Here $C > 0$ is a constant which depends only on n, p, L, r, h, $diam(M)$, $dist(W \cap \partial M, M \setminus U)$, λ, E, $\mathcal{E}$, α and k.

The following uniqueness result can be obtained under a slightly weaker assumption.

THEOREM 2.5. (Uniqueness at the boundary). *Let p, n, M, $(N,\ g_0)$, a, b, g as in Theorem 2.3. Suppose there exist a point $y \in \partial M$, a neighborhood U of y in $\bar{M}$ and a positive integer k such that*

$$a - b \in C^k(\bar{U}). \tag{2.18}$$

If

$$\Lambda_{g(x,\,a)} = \Lambda_{g(x,\,b)},$$

then

$$D^j(a - b) = 0 \quad on \quad \partial M \cap \bar{U}, \quad for \ any \ j \leq k. \qquad (2.19)$$

If in addition

$$g(x, \, t) \in C^k\big(\bar{U} \times [\lambda^{-1}, \, \lambda], \, T^{2, \, 0}(M)\big), \qquad\qquad (2.20)$$

then, for any neighborhood W such that $\bar{W} \subset U$ and any smooth vector field $Z \in C^\infty(TM)$, we have

$$\nabla_Z^j\big(g(x, \, a)\big) = \nabla_Z^j\big(g(x, \, b)\big), \qquad for \quad any \quad j \leq k. \qquad (2.21)$$

The following corollary is a well-known consequence of the previous theorem in the Euclidean case (see [A], [AG]) and can be easily adapted to the case of a Riemannian manifold.

COROLLARY 2.6. (Uniqueness in the interior). *Let n, M, $(N, \, g_0)$ be as in Theorem 2.3. Let a, b be two functions satisfying (2.12) and (2.13) with $p = \infty$. Let $g(x, \, t) \in \mathcal{H}$ and in addition $g \in W^{1, \, \infty}(M \times [\lambda^{-1}, \, \lambda, \, T_0^2(M)])$. Suppose that M can be partitioned into a finite number of Lipschitz submanifolds, $\{A_j\}_{j=1}^N$, such that*

$$a - b \qquad is \, analytic \, on \quad \bar{A}_j, \qquad for \, any \quad j = 1 \ldots n.$$

If $\Lambda_{g(x, \, a)} = \Lambda_{g(x, \, b)}$, then

$$g(x, \, a) = g(x, \, b) \qquad on \quad M. \qquad\qquad (2.22)$$

3   Singular solutions.

Let $(N, \, g_0)$ be the C^∞ orientable Riemannian manifold of dimension $n \geq 3$, introduced in Section 2 and let g be a metric on N satisfying

$$\| \, g^{ij} \, \|_{W^{1, \, p}(N)} \leq E, \qquad i, \, j = 1, \ldots, n, \qquad\qquad (3.1)$$

where $p > n$ and E is a positive constants. Let us consider the Laplace Beltrami operator on functions, associated to g, $\Delta_g = - *_g d *_g d$, which in coordinates is

$$\Delta_g = - \sum_{i \, j \, = \, 1}^n |g|^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \Big\{ |g|^{\frac{1}{2}} \, g^{i \, j} \, \frac{\partial}{\partial x^j} \Big\}, \qquad on \quad N, \qquad (3.2)$$

where $|g|$ denotes the determinant of g_{ij}. Clearly, for any chart on N, there exists a positive constant λ such that Δ_g satisfies the ellipticity condition

$$\lambda^{-1}\,|\xi|^2 \leq g^{ij}(x)\xi_i\xi_j \leq \lambda\,|\xi|^2, \tag{3.3}$$

for all x in the domain of the chart and all $\xi \in \mathbb{R}^n$. Here we denote the Euclidean norm on $\mathbb{R}^n$ simply by $|\cdot|$. Let us also consider the geodesic ball

$$B_{N,\,r}(\bar{x}) = \big\{ x \in N \mid d(x,\,\bar{x}) \, < \, r \big\},$$

where d is the geodesic distance induced by g_0 and $\bar{x} \in N$. We will simply denote $B_{N,\,r}$ by B_r when it will be clear from the contest what is the manifold N we are referring to. Let us denote $G = |g|^{\frac{1}{2}}g^{-1}$, where g is the matrix $\{g_{ij}\}_{i,\,j=1}^n$ and g^{-1} is its inverse $\{g^{ij}\}_{i,\,j=1}^n$. The following theorem provides the construction of singular solutions obtained in [A], [AG], on a geodesic ball of a Riemannian manifold.

THEOREM 3.1. (Singular solutions on manifolds). *If Δ_g is the Laplace Beltrami operator satisfying (3.1)-(3.3), for any $m = 0, 1, 2, \ldots$ there exists $u \in W_{loc}^{2,\,p}(B_r \setminus \{\bar{x}\}) \cap W^{1,\,2}(N)$ solution to*

$$\Delta_g u = 0, \qquad in \quad N, \tag{3.4}$$

such that there exist coordinates $(x^i)_{i=1}^n$ on N with

$$u(x) = |J(x-\bar{x})|^{2-n-m}\,S_m\left(\frac{J(x-\bar{x})}{|x-\bar{x}|}\right) + w(x), \quad in \quad B_r \setminus \{\bar{x}\}, \tag{3.5}$$

where S_m is a spherical harmonic of degree m, $J = \sqrt{G^{-1}(\bar{x})}$ and w satisfies

$$|\,w(x)| + |\,x\,|\,|Dw(x)| \leq C\,|\,x\,|^{2-n-m+\alpha}, \quad in \quad B_r \setminus \{\bar{x}\}, \tag{3.6}$$

$$\left(\int_{s<|x|<2s} |D^2 w|^p\right)^{\frac{1}{p}} \leq C\,s^{-n-m+\alpha+\frac{n}{p}}, \quad for\ every \quad s,\ 0 < s < r/2. \tag{3.7}$$

Here α is any number such that $0 < \alpha < 1 - \frac{n}{p}$, and C is a constant depending only on α, n, p, r, λ, and E. Furthermore

$$\| \, \mathrm{d}u \, \|_{g_0} \leq C \, d(x, \, \bar{x})^{1-n-m}, \quad \text{for every} \quad x \in B_r(\bar{x}) \setminus \{\bar{x}\} \qquad (3.8)$$

$$\| \, \mathrm{d}u \, \|_{g_0} > \frac{1}{2} \, d(x, \, \bar{x})^{1-n-m}, \quad \text{for every} \quad x \in B_{r_0}(\bar{x}) \setminus \{\bar{x}\}, \qquad (3.9)$$

where r_0 is a positive constant which depends only on λ, E, p, m and the diameter of N, $\mathrm{diam}(N)$.

Proof of Theorem 3.1 By [AG, Theorem 3.4] and by choosing normal coordinates on $B_r(\bar{x})$ we can construct u_m solution to

$$\Delta_g u_m = 0, \qquad \text{in} \quad x \in B_r(\bar{x}) \setminus \{\bar{x}\} \qquad (3.10)$$

and u_m satisfies (3.5)-(3.7). By expressing g_0 in normal coordinates we obtain

$$(g_0)_{ij}(x) = \delta_{ij} + O\big(d(x, \, \bar{x})^2\big), \qquad (3.11)$$

for any $x \in B_r(\bar{x})$, where the geodesic distance d induced by g_0 satisfies $d(x, \, \bar{x}) = |x - \bar{x}|$ on B_r. Therefore

$$
\begin{aligned}
\| \, du \, \|_{g_0}^2 \; &= \; g_0^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \\
&= \; \Big(\delta^{ij} + O\big(d(x, \, \bar{x})^2\big)^{ij} \Big) \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \\
&\leq \; |Du|^2 + O\big(d(x, \, \bar{x})^2\big)|Du|^2 \\
&\leq \; C|Du|^2 & (3.12) \\
&\leq \; C \, d(x, \, \bar{x})^{2-2(n+m)}, & (3.13)
\end{aligned}
$$

where $C > 0$ is a constant depending on n, $\mathrm{diam}(N)$ and Du is the gradient of u in $\mathbb{R}^n$. By combining (3.11) with [AG, Lemma 3.5],

$$\| \, du \, \|_{g_0}^2 > \frac{1}{4} \, d(x, \, \bar{x})^{2-2(n-m)}, \quad \text{on} \quad B_r(\bar{x}) \qquad (3.14)$$

and this concludes the proof. $\blacksquare$

4 Proofs of main results.

We will only give the proofs of Theorems 2.3, 2.4 as proofs of Theorem 2.5, and Corollary 2.6 follow the same line of these proofs and arguments in [A] and [AG].

Since the boundary ∂M is Lipschitz, the normal unit vector field might not be defined on ∂M. Therefore, we consider the vector field $\tilde{\nu}$ introduced in Definition 2.3, instead. $\tilde{\nu}$ is locally defined near ∂M, it is C^{∞} smooth and it is not tangential to ∂M. With the same arguments used in [AG, Section 3] we can state the following

LEMMA 4.1. *For any $x^0 \in \partial M$, let $C_{r,\,h}$ be the cylinder introduced in Definition 2.3, such that $x^0 \in C_{r,\,h}$, then the point*

$$z_\sigma = \rho_{x^0,\,\tilde{\nu}}(\sigma)$$

satisfies

$$C\,\tau \leq d(z_\tau,\,\partial M) \leq \tau, \quad \text{for any} \quad \tau,\, 0 \leq \tau \leq \tau^0, \tag{4.1}$$

where τ^0 and C depend only on L, r, h.

Proof. The proof follows by rephrasing arguments of [AG, Lemma 3.3] and by substituting the Euclidean distance with the geodesic one. ∎

LEMMA 4.2. *If $g \in \mathcal{H}$ and a is a function satisfying conditions* (2.12), (2.13), *we have*

$$|g(\cdot,\,a(\cdot))|^{\frac{1}{2}} g^{-1}(\cdot,\,a(\cdot)) \in W^{1,\,p}\big(M,\,T_0^2(M)\big). \tag{4.2}$$

Proof. The proof is a straight forward consequence of [AG, Lemma 3.6] and conditions (2.9)-(2.11) of Remark 2.2. ∎

Proof of Theorem 2.3. We start by recalling the identity (see [A, (b), p. 253], [G1, (6.35), p.99])

$$\langle (\Lambda_{g(x,\,a(x))} - \Lambda_{g(x,\,b(x))})u,\,v \rangle = (-1)^{n-1} \int_M \big(*_{g(x,\,a)} - *_{g(x,\,b)}\big) \mathrm{d}u \wedge \mathrm{d}v, \tag{4.3}$$

which holds for any u, v solutions to the Laplace-Beltrami equations

$$\Delta_{g(x,\,a)}\,u = \Delta_{g(x,\,b)}\,v = 0, \quad \text{in} \quad M. \tag{4.4}$$

With no loss of generality we suppose that $(-1)^{n-1} = 1$, the case when $(-1)^{n-1} = -1$ can be treated in a very similar way. Let $x^0 \in \partial M$ be a point such that

$$(a - b)(x^0) = \| a - b \|_{L^\infty(\partial M)} .$$

Let $0 < \tau \le \left\{ \tau_0, \frac{r_0}{4} \right\}$, where τ_0 is the number fixed in (4.1) and r_0 is the number appearing in (3.9). We consider $z_\tau = \rho_{x^0, \tilde{\nu}}(\tau)$, where $\tilde{\nu}$ is the outer unit vector field at the boundary ∂M introduced in Definition 2.3 and $\rho_{x^0, \tilde{\nu}}(\tau)$ is the geodesic introduced in Definition 2.1. Any point in the geodesic ball $B_\eta(z_\tau)$, with $\eta = r_0$ and r_0 small enough so that there are no cut points in $B_\eta(z_\tau)$, is uniquely connected with the center z_τ by the unique shortest geodesic. By fixing m, let u_m, v_m be the two singular solutions of

$$\Delta_{g(x, a)} u_m = \Delta_{g(x, b)} v_m = 0, \qquad \text{in} \quad B_\eta(z_\tau) \setminus \{z_\tau\}.$$

obtained in Theorem 3.1. The manifold M can be enlarged by introducing

$$M_{\tau/2} := \left\{ x \in N \,|\, d(x, \partial M) < \tau/2) \right\}.$$

$M \subset M_{\tau/2}$ and $z_\tau \in N \setminus M_{\tau/2}$, for any $0 < \tau \le \left\{ \tau_0, \frac{r_0}{4} \right\}$. Let $\chi_{\eta/2} \in C^\infty(N)$ be the cut-off function defined by

$$\chi_{\eta/2} = \begin{cases} 1 & \text{on} \quad B_{\eta/2}(z_\tau), \\ 0 & \text{on} \quad N \setminus B_\eta(z_\tau) \end{cases}$$

and consider

$$u = \chi_{\eta/2} \, u_m + \tilde{w}, \tag{4.5}$$

where $\tilde{w}$ solve the problem

$$\begin{cases} \Delta_{g(x, a)}\tilde{w} = -\Delta_{g(x, a)}\big(\chi_{\eta/2} \, u_m\big) & \text{in} \quad M_{\tau/2}, \\ \tilde{w} = 0 & \text{on} \quad \partial M_{\tau/2}. \end{cases}$$

Therefore

$$\begin{aligned} \Delta_{g(x, a)} \, u &= 0, & \text{in} \quad M \\ u &= u_m + \tilde{w} & \text{in} \quad B_{\eta/2(z_\tau)} \cap M \\ u &= \tilde{w} & \text{in} \quad M \setminus B_{\eta(z_\tau)}, \end{aligned}$$

where

$$\| \tilde{w} \|_{W^{1,\,2}(M)} \leq C \tag{4.6}$$

and $C > 0$ is a constant which depends on n, m, L, r and h only. The same argument can be applied to the singular solution v_m and by setting $m = 0$, (4.3) leads to

$$
\langle (\Lambda_{g(x,\,a(x))} - \Lambda_{g(x,\,b(x))})\, u,\, v \rangle = \int_{M \cap B_\eta(z_\tau)} \left(*_{g(x,\,a)} - *_{g(x,\,b)} \right) \mathrm{d}u \wedge \mathrm{d}v
$$
$$
+ \int_{M \setminus B_\eta(z_\tau)} \left(*_{g(x,\,a)} - *_{g(x,\,b)} \right) \mathrm{d}u \wedge \mathrm{d}v, \tag{4.7}
$$

where u and v are the solutions (4.5) of (4.4) for $m = 0$. By possibly reducing η

$$u = u_0 + \tilde{w}, \qquad v = v_0 + \tilde{w}, \qquad \text{in} \quad B_\eta(z_\tau), \tag{4.8}$$

where $\tilde{w}$ satisfies (4.6). (4.7) leads in any coordinate system to

$$
\| \Lambda_{g(x,\,a(x))} - \Lambda_{g(x,\,b(x))} \|_* \| u \|_{H^{\frac{1}{2}}(\partial M)} \| v \|_{H^{\frac{1}{2}}(\partial M)}
$$
$$
\geq \left| \int_{M \cap B_\eta(z_\tau)} \left(|g(x,\,a)|^{\frac{1}{2}} g^{ij}(x,\,a) - |g(x,\,b)|^{\frac{1}{2}} g^{ij}(x,\,b) \right) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx \right|
$$
$$
- \; C_1, \tag{4.9}
$$

where C_1 is a positive constant depending on n, m, L, r, h and $\mathrm{diam}(M)$ only. By choosing normal coordinates centered in z_τ on $B_\eta(z_\tau)$ and by combining (4.9) with (3.5), we obtain

$$
\left| \int_{M \cap B_\eta(z_\tau)} \frac{J_b^2 \left(|g(x,\,a)|^{\frac{1}{2}} g^{-1}(x,\,a) - |g(x,\,b)|^{\frac{1}{2}} g^{-1}(x,\,b) \right) J_a^2\, (x - z_\tau) \cdot (x - z_\tau)}{|J_a(x - z_\tau)|^n\, |J_b(x - z_\tau)|^n}\, dx \right|
$$
$$
\leq \int_{M \cap B_\eta(z_\tau)} \left| \left(|g(x,\,a)|^{\frac{1}{2}} g^{ij}(x,\,a) - |g(x,\,b)|^{\frac{1}{2}} g^{ij}(x,\,b) \right) \frac{\partial \tilde{w}}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} \right|\, dx
$$
$$
+ C \int_{M \cap B_\eta(z_\tau)} |x - z_\tau|^{2 - 2n + \alpha} dx
$$
$$
+ C_1 + \| \Lambda_{g(x,\,a(x))} - \Lambda_{g(x,\,b(x))} \|_* \| u \|_{H^{\frac{1}{2}}(\partial M)} \| v \|_{H^{\frac{1}{2}}(\partial M)}.
$$

By recalling that $|g(x,\,a)|^{\frac{1}{2}} g^{-1}(x,\,a)$ is Hölder continuous (see Lemma 4.2 and [AG]),

$$\int_{M \cap B_\eta(z_\tau)} \frac{J_b^2 \left(|g(x^0, a)|^{\frac{1}{2}} g^{-1}(x^0, a) - |g(x^0, b)|^{\frac{1}{2}} g^{-1}(x^0, b) \right) J_a^2 (x - z_\tau) \cdot (x - z_\tau)}{|J_a(x - z_\tau)|^n |J_b(x - z_\tau)|^n} dx$$

$$\leq C \Bigg\{ \int_{M \cap B_\eta(z_\tau)} |x - z_\tau|^{2-2n+\alpha} dx$$

$$+ \int_{M \cap B_\eta(z_\tau)} |x - z_\tau|^{2-2n} |x - x^0|^\beta dx \Bigg\}$$

$$+ C_1 + C_2 + \| \Lambda_{g(x, a(x))} - \Lambda_{g(x, b(x))} \|_* \| u \|_{H^{\frac{1}{2}}(\partial M)} \| v \|_{H^{\frac{1}{2}}(\partial M)} .$$

By recalling that $J_a^2 = \frac{g(z_\tau, a)}{|g(z_\tau, a)|^{\frac{1}{2}}}$ and similarly $J_b^2 = \frac{g(z_\tau, b)}{|g(z_\tau, b)|^{\frac{1}{2}}}$, we get (see [AG])

$$J_b^2 \left(|g(x^0, a)|^{\frac{1}{2}} g^{-1}(x^0, a) - |g(x^0, b)|^{\frac{1}{2}} g^{-1}(x^0, b) \right) J_a^2$$

$$\geq \left(\frac{g(x^0, b)}{|g(x^0, b)|^{\frac{1}{2}}} - \frac{g(x^0, a)}{|g(x^0, a)|^{\frac{1}{2}}} \right) (x - z_\tau) \cdot (x - z_\tau)$$

$$- C\tau^\beta (a - b)(x^0) |x - z_\tau|^2. \tag{4.10}$$

The function $t \longrightarrow \frac{g(x^0, t)}{|g(x^0, t)|^{\frac{1}{2}}}$ is absolutely continuous (see [Mo, Lemma 3.1.1]) and by combining it with (2.5),

$$\left(\frac{g_{ij}(x^0, b)}{|g(x^0, b)|^{\frac{1}{2}}} - \frac{g_{ij}(x^0, a)}{|g(x^0, a)|^{\frac{1}{2}}} \right) (x - z_\tau)^i (x - z_\tau)^j$$

$$= \int_{a(x^0)}^{b(x^0)} \left(D_t \frac{g(x^0, t)}{|g(x^0, t)|^{\frac{1}{2}}} \right) (x - z_\tau)^i (x - z_\tau)^j dt$$

$$= \int_{a(x^0)}^{b(x^0)} -|g(x^0, t)|^{-1} g_{il}(x^0, t) D_t \left(g^{lk}(x^0, t) |g(x^0, t)|^{\frac{1}{2}} \right) \cdot$$

$$\cdot g_{kj}(x^0, t)(x - z_\tau)^i (x - z_\tau)^j \, dt$$

$$= \int_{b(x^0)}^{a(x^0)} |g(x^0, t)|^{-1} D_t \left(g^{lk}(x^0, t) |g(x^0, t)|^{\frac{1}{2}} \right) \cdot$$

$$\cdot \left(g_{ik}(x^0, t)(x - z_\tau)^k \right) \left(g_{jl}(x^0, t)(x - z_\tau)^l \right) dt$$

$$= \int_{b(x^0)}^{a(x^0)} |g(x^0, t)|^{-1} D_t \left(*_{g(x^0, t)} \right) \theta \wedge \theta dt$$

$$\geq \mathcal{E}^{-1} E^{-1} \int_{b(x^0)}^{a(x^0)} \| \theta \|_{g_0}^2 \, dt, \tag{4.11}$$

where $\theta = \theta(x^0, z_\tau, x, t)$ and $\theta = \theta_i(x^0, z_\tau, x, t) \, dx^i \in \Omega^1(M)$. If we recall that in normal coordinates we have

$$\begin{aligned}
\| \theta \|_{g_0}^2 &= |\theta|^2 + \left(d(x, z_\tau)^2 \right)_{ij} \theta_i \theta_j \\
&> \frac{1}{2} |\theta|^2
\end{aligned} \tag{4.12}$$

and we combine together (4.11), (4.12) with (2.4), we obtain

$$\begin{aligned}
&\left(\frac{g_{ij}(x^0, b)}{|g(x^0, b)|^{\frac{1}{2}}} - \frac{g_{ij}(x^0, a)}{|g(x^0, a)|^{\frac{1}{2}}} \right) (x - z_\tau)^i (x - z_\tau)^j \\
&\geq \frac{1}{2} E^{-1} \lambda^{-2} \mathcal{E}^{-3} (a - b)(x^0) |x - z_\tau|^2.
\end{aligned} \tag{4.13}$$

Hence, we have

$$\begin{aligned}
&J_b^2 \left(|g(x^0, a)|^{\frac{1}{2}} g^{-1}(x^0, a) - |g(x^0, b)|^{\frac{1}{2}} g^{-1}(x^0, b) \right) J_a^2 \\
&\geq \left(\frac{1}{2} E^{-1} \lambda^{-2} \mathcal{E}^{-3} - C\tau^\beta \right) (a - b)(x^0) |x - z_\tau|^2
\end{aligned}$$

and, choosing

$$\tau \leq \left(\frac{1}{4C} E^{-1} \lambda^{-2} \mathcal{E}^{-2} \right)^{\frac{1}{\beta}},$$

we obtain

$$\begin{aligned}
&J_b^2 \left(|g(x^0, a)|^{\frac{1}{2}} g^{-1}(x^0, a) - |g(x^0, b)|^{\frac{1}{2}} g^{-1}(x^0, b) \right) J_a^2 \\
&\geq C(a - b)(x^0) |x - z_\tau|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\| a - b \|_{L^\infty(\partial M)} \int_{M \cap B_\eta(z_\tau)} |x - z_\tau|^{2-2n} \, dx \\
&\leq C \Bigg\{ \int_{M \cap B_\eta(z_\tau)} |x - z_\tau|^{2-2n+\alpha} \, dx \\
&\quad + \int_{M \cap B_\eta(z_\tau)} |x - z_\tau|^{2-2n} |x - x^0|^\beta \, dx \\
&\quad + C_1 + C_2 + \| \Lambda_{g(x, a(x))} - \Lambda_{g(x, b(x))} \|_* \| u \|_{H^{\frac{1}{2}}(\partial M)} \| v \|_{H^{\frac{1}{2}}(\partial M)} \Bigg\}
\end{aligned}$$

and by estimating the above integrals and the $H^{\frac{1}{2}}(\partial M)$ norms of u and v (see [A], [AG]) we finally obtain

$$
\| a - b \|_{L^\infty(\partial M)} \, \tau^{2-n} \leq \; C \; \left\{ \tau^{2-n+\alpha} + \tau^{2-n+\beta} + C_1 + C_2 \right.
$$
$$
\left. + \; \| \Lambda_{g(x,\,a)} - \Lambda_{g(x,\,b)} \| \, \tau^{n-2} \right\}.
$$

If we let $\tau \to 0$ we obtain the following inequality

$$
\| a - b \|_{L^\infty(\partial M)} \leq C \, \| \Lambda_{g(x,\,a)} - \Lambda_{g(x,\,b)} \|_\star . \qquad (4.14)
$$

Recalling that, for almost every $x \in \Omega$, the function

$$
t \longrightarrow g(x,\, t)
$$

is absolutely continuous on $[\lambda^{-1},\, \lambda]$ we may write

$$
|g(x,\, a(x)) - g(x,\, b(x))| \; = \; \left| \int_{b(x)}^{a(x)} D_t g(x,\, t) \, \mathrm{d}t \right|
$$
$$
\leq \int_{b(x)}^{a(x)} \mathrm{Sup}_{\,t,\,x} | \, D_t g(x,\, t) \, | \, \mathrm{d}t
$$
$$
\leq C \, | \, (a(x) - b(x)) \, |,
$$

for every $x \in M$. Taking the L^∞-norm on both sides, we obtain

$$
\| g(x,\, a) - g(x,\, b) \|_{L^\infty(\partial M)} \leq C \, \| a - b \|_{L^\infty(\partial M)} . \qquad (4.15)
$$

By combining (4.14) and (4.15) we conclude the proof.   ∎

Proof of Theorem 2.4. Let $\tilde{\nu}$ be the vector field introduced in Definition 2.3 and let us define some coordinate system on a neighborhood of the boundary ∂M.

DEFINITION 4.1. *For any point x in a neighborhood of ∂M we consider the unique point $y \in \partial M$ such that $x = \rho_{y,\,\tilde{\nu}}(s)$ for some s. If $\{\tilde{x}^i\}_{i=1}^{n-1}$ is a coordinate system around x on ∂M, we define*

$$
x^i \; = \; \tilde{x}^i, \qquad \text{for} \quad i = 1, \ldots, n - 1,
$$
$$
x^n \; = \; s.
$$

$\{x^i\}_{i=1}^n$ is a coordinate system around x on M and we call it boundary quasi-normal coordinate system (in accordance to the well-known boundary normal coordinates for the case where $\tilde{\nu}$ is the C^∞ smooth normal unit vector field at the boundary ∂M).

By following the same line of [AG, proof of Theorem 2.2] and arguments of the proof of Theorem 2.3, we obtain

$$\| \frac{\partial^j}{\partial \tilde{\nu}^j}(a-b) \|_{L^\infty(\partial M \cap \bar{W})} \leq C \| \Lambda_{g(x,a)} - \Lambda_{g(x,b)} \|_\star^{\delta_j} \quad \text{for} \quad \text{any } j \leq k,$$

(4.16)

in boundary quasi-normal coordinates, where $\delta_j = \prod_{i=0}^j \frac{\alpha}{\alpha+i}$. By recalling the interpolation inequality

$$\| Df \|_{L^\infty(\partial M \cap \bar{W})} \leq C \Big\{ \| \frac{\partial}{\partial \tilde{\nu}} f \|_{L^\infty(\partial M)} + \| f \|_{L^\infty(\partial M \cap \bar{W})}^{\frac{\alpha}{1-\alpha}} \| f \|_{C^{1+\alpha}(\bar{W})}^{\frac{1}{1+\alpha}} \Big\},$$

(4.17)

for any $f \in C^{1,\alpha}(M)$ (see [A, Lemma 3.2], [AG, estimate (3.38)]) and combining it with (4.16), we obtain

$$\| D^k(a-b) \|_{L^\infty(\partial M \cap \bar{W})} \leq C \| \Lambda_{g(x,a)} - \Lambda_{g(x,b)} \|_\star^{\delta_k},$$

(4.18)

where D^k denotes the gradient in boundary quasi-normal coordinates. If we observe that

$$D^\beta g(x, a(x)) = \sum_{\gamma+\delta \leq \beta} P_{\gamma\delta}\big(a(x), \ldots, D^{|\delta|}a(x)\big) \cdot D_x^\gamma D_t^{|\delta|} g(x, a(x)),$$

where β is any multiindex and $P_{\gamma\delta}$ is a polynomial in the variables $p = (p_\eta)$, $|\eta| \leq |\delta|$, in any coordinate system (see [AG, equality (3.40)]), we obtain

$$\| D^k\big(g(x, a) - g(x, b)\big) \|_{L^\infty(\partial M \cap \bar{W})} \leq C \| a - b \|_{C^k(\partial M \cap \bar{W})}^\alpha .$$

(4.19)

(4.19) and (4.18) leads to

$$\| D^k\big(g(x, a) - g(x, b)\big) \|_{L^\infty(\partial M \cap \bar{W})} \leq C \| \Lambda_{g(x,a)} - \Lambda_{g(x,b)} \|_\star^{\delta_k} .$$

(4.20)

Let Z be a smooth vector field on M, with $\| Z \|_{L^\infty(M)} = 1$ and let $\{X^i\}_{i=1}^n$ be a local basis of vector fields on $\bar{W}$ such that

$$\nabla_Z X^i = 0 \qquad \text{for} \quad \text{any } i = 1, \dots, n, . \qquad (4.21)$$

With this choice of basis we have

$$
\begin{aligned}
\nabla_Z\big(g(x,\, a)\big) &= Z\big(g^{ij}(x,\, a)\big) X^i(x) \otimes X^j(x) \\
&= Z^k \frac{\partial}{\partial x^k}\big(g^{ij}(x,\, a)\big) X^i(x) \otimes X^j(x)
\end{aligned}
$$

and similarly for $g(x,\, b)$, where $Z = Z^i \frac{\partial}{\partial x^i}$, $\{x^i\}_{i=1}^n$ are boundary quasi-normal coordinates on $\bar{W}$ and the Einstein convention on indices summation has been applied. Therefore

$$\nabla_Z\big(g(x,\, a) - g(x,\, b)\big) = Z^k \frac{\partial}{\partial x^k}\big(g(x,\, a) - g(x,\, b)\big). \qquad (4.22)$$

(4.22) together with (4.20) leads to (2.17), which concludes the proof. ∎

5 Conclusions.

In this study we improve the results obtained in [AG] in the following aspects.

i) We give a geometric formulation of the inverse conductivity problem considered in [AG], in dimension $n > 2$, where it is well known that the conductivity σ of a manifold uniquely determines a metric g such that $\sigma = *_g$, where $*_g$ is the *Hodge star* operator mapping 1-forms into $(n-1)$-forms (see [G1], [L], [LU]).
We prove results of uniqueness and stability at the boundary similar to [AG, Theorems 2.1-2.3] and in the interior as in [AG, Theorem 2.4], in the case where the body in question is a compact manifold with Lipschitz boundary embedded in an open C^∞ smooth Riemannian manifold N (Theorems 2.3-2.5 and Corollary 2.6 respectively);

ii) the so-called monotonicity assumption of [AG, p.255] is here stated in terms of the Riemannian metric g_0 on N. The case of a manifold with a flat metric g_0 will be still more general than the one treated in [AG]. The case when $(g_0)_{ij} = \delta_{ij}$ is the Euclidean metric on $\mathbb{R}^n$ will lead to the monotonicity assumption given in [AG].

References

[A]   G. ALESSANDRINI, *Singular Solutions of Elliptic Equations and the Determination of Conductivity by Boundary Measurements*, J. Differential Equations, Vol.**84**, No.**2** (1990), 252-272.

[AG]   G. ALESSANDRINI AND R. GABURRO, *Determining Conductivity with Special Anisotropy by Boundary Measurements*, SIAM J. MATH. ANAL., Vol.**33** (2001), 153-171.

[Ar]   N. ARONSZAJN, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl., Vol.**36**, (1957), 235-249.

[G]   R. GABURRO, *Sul Problema Inverso della Tomografia da Impedenza Elettrica nel Caso di Conduttivitá Anisotropa*, Tesi di Laurea in Matematica, Universitá degli Studi di Trieste, 1999.

[G1]   R. GABURRO, *Anisotropic Conductivity. Inverse Boundary Value Problems*, PhD Thesis, University of Manchester Institute of Science and Technology (UMIST), 2003.

[GT]   D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 ed., Springer, Berlin-Heidelberg-New York- Barcelona-Hong Kong-London-Milan-Paris-Singapore-Tokyo, 2001.

[K]   Y. KURYLEV, *Inverse boundary problems on Riemannian manifolds*, In Geometry of the spectrum (Seattle, WA, 1993), Vol.**173** of Contemp. Math., 181-192. Amer. Math. Soc., Providence, RI, 1994.

[KKL]   A. KATSUDA, Y. KURYLEV AND M. LASSAS, *Stability of boundary distance representation and reconstruction of Riemannian manifolds*, Inverse Prob. Imagine, Vol.**1** (2007), No. **1**, 181-192.

[KV1]   R. KOHN AND M. VOGELIUS, *Identification of an Unknown Conductivity by Means of Measurements at the Boundary*, SIAM-AMS Proc., Vol.**14** (1984), 113-123.

[KV2]   R. KOHN AND M. VOGELIUS, *Determining Conductivity by Boundary Measurements II. Interior Results*, Comm. Pure Appl. Math., Vol.**38** (1985), 643-667.

[L]   W. R. B. LIONHEART, *Conformal Uniqueness Results in Anisotropic Electrical Impedance Imaging*, Inverse Problems Vol.**13** (1997), 125-134.

[LaU]   M. LASSAS AND G. UHLMANN, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, Ann. Sci. École Norm. Sup., Vol. **(4) 34** (2001), No. **5**, 771-787.

[LSU]   M. LASSAS, V. SHARAFUTDINOV AND G. UHLMANN, *Semiglobal boundary rigidity for Riemannian metrics*, Math. Ann., Vol. **325** (2003), 767-793.

[LaTU] M. LASSAS, M. TAYLOR AND G. UHLMANN, *The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary*, Comm. Anal. Geom., Vol. **11**, (2003), No. **2**, 207-221.

[LU]   J. M. LEE AND G. UHLMANN, *Determining anisotropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math., Vol.**42** (1989), 1097-112.

[Mo]   C. B. MORREY, *Multiple Integrals in the Calculus of Variations*, Springer, Berlin, 1966.

[N]   A. NACHMAN, *Global Uniqueness for a two Dimensional Inverse Boundary Value Problem*, Ann. Math., Vol.**142** (1995), 71-96.

[S]   J. SYLVESTER, *An anisotropic inverse boundary value problem*, Comm. Pure. Appl. Math., Vol.**43** (1990), 201-232.

[SU]   J. SYLVESTER AND G. UHLMANN, *A Global Uniqueness Theorem for an Inverse Boundary Valued Problem*, Ann. of Math., Vol.**125** (1987), 153-169.