Variations on a theme of Timmesfeld: a finite group-theoretic analogue of the classification of groups of finite Morley rank and even type

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Variations on a theme of Timmesfeld: A finite group-theoretic analogue of the classification of groups of finite Morley rank and even type

A Discussion Document

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Abstract. The paper addresses a question whether there is a reasonable self-contained theory of *finite simple groups of even type* which is closely parallel to the theory of groups and finite Morley rank.

1. Groups of finite Morley rank and even type

The underlying methodology of the classification theory of groups of finite Morley rank and even type is the systematic use of ideas from the Classification of Finite Simple Groups (CFSG), both from the original (first generation) papers and from the later revisionism, especially from the Third Generation Proof. One specific feature of our theory, however, is the systematic use of definable connected subgroups. Almost all subgroups appearing in the proofs are definable. Moreover, we are trying to ignore the finite factor groups H/H° wherever possible. In particular, we work with 2-Sylow$^\circ$ subgroups, that is, connected components of 2-Sylow subgroups, and many standard operators, such as $N_G(X)$, $C_G(X)$ are systematically used in their connected versions: $N^\circ_G(X)$, $C^\circ_G(X)$, etc.

The aim of these notes is to outline some problems arising from an attempt to reverse the transfer of ideas and formulate a fragment of CFSG closest to the classification of groups of finite Morley rank and even type.

Indeed, let G be a simple group of finite Morley rank and even type. Take A a minimal infinite definable connected 2-subgroup in G. Then A has the following properties:

- For all elements $g \in G$, $A \cap A^g$ is either finite or coincides with A.
- If $g \in N_G(A)$, then either $C_A(g)$ is finite[1] or $C_A(g) = A$.
- If $Q > 1$ is a definable connected 2-group then $Z(Q)$ contains a minimal infinite definable connected 2-subgroup.

When looking for a finite group analogue, the idea to use elementary abelian TI-subgroups as substitutes for minimal connected subgroups appears to be natural, especially in the light of Franz Timmesfeld's fundamental works [**31, 32**]. Indeed, in the finite groups context, it is so natural to interpret the word 'finite' in the statements above as 'trivial'.

This is a very raw and not carefully read draft; essentially, it is a stream of conscience. Blunders and howlers are more than possible. I have not checked the existing literature yet, and apologise for any omissions. Any comments and criticism are most welcome.

2. Back to finite groups

2.1. Atomic subgroups. Therefore let us consider a finite group G with a normal set $\mathcal{A}$ of elementary abelian 2-subgroups of order ≥ 4, called *atomic* subgroups. We assume the validity of the following axioms for all $A, B \in \mathcal{A}$.

A1. For any $g \in G$, $A \cap A^g = 1$ or A. We say in this situation that A is a *TI-subgroup*, "TI" meaning "trivial intersection".[2]

A2. $C_G(A) = C_G(a)$ for all $a \in A^{\#}$.

A3. If $A, B \in \mathcal{A}$ and $[A, B] \neq 1$ then $[A, B]$ contains a subgroup from $\mathcal{A}$. [3]

If $[A, B] = 1$ for all $A, B \in \mathcal{A}$ such that $\langle A, B\rangle$ is a 2-group, then we shall say that $\mathcal{A}$ is *degenerate*.

A4. $G = \langle \mathcal{A} \rangle$.

Notice that it means, in particular, that A is of *root-type* in the sense of [**30**]. Recall that an elementary abelian p-subgroup A of the finite group G is of *root-type* if the following two conditions are satisfied.

(1) A is a TI-subgroup, and
(2) for all $g \in G$, if $N_A(A^g) \neq 1$, then $[A, A^g] = 1$.

[1] Moreover, I believe Frecon has proven in that case that $C_A(g) = 1$.

[2] Michael Aschbacher suggested to replace A1 by a stronger axiom: If $A, B \in \mathcal{A}$ and $A \neq B$ then $A \cap B = 1$.

[3] This axiom has been significantly strengthened on advice from Franz Timmesfeld. The previous version read: if $\langle A, B\rangle$ is a 2-group then either $[A, B] = 1$ or $Z(\langle A, B\rangle) \cap [A, B]$ contains a subgroup from $\mathcal{A}$.

2.2. Abstract root subgroups. It is instructive to compare the definition of atomic subgroups with that of abstract root subgroups, due to Timmesfeld [**32**].

A set of *abstract root subgroups* of a group G (not necessarily finite!) is a normal set (that is, invariant under conjugation) Σ of nontrivial abelian subgroups of G such that $G = \langle \Sigma \rangle$ and, for any two $A, B \in \Sigma$, one of the following holds:

(1) $[A, B] = 1$;
(2) $\langle A, B \rangle$ is a rank one group;
(3) $[A, B] = [a, B] = [A, b] \in \Sigma$ for any $a \in A^{\#}$ and $b \in B^{\#}$ and $[A, B] \leq Z(\langle A, B \rangle)$.

Here, a *rank one group* is a group X generated by two different nilpotent subgroups A and B such that for each $a \in A^{\#}$ there exists $b \in B$ satisfying $A^b = B^a$. Equivalently, a rank one group is a group with a split BN-pair of rank 1. When case (3) never occurs, Σ is said to be *degenerate*.

As a simplified test bench problem one might consider the one arising from a hybrid condition which picks the best features of atomic and abstract root subgroups.

2.3. A simplified hybrid version of the theory. Here, we assume that a finite group G is generated by a normal system $\mathcal{H}$ of elementary abelian 2-subgroups of order ≥ 4 such that, for every $A, B \in \mathcal{H}$, the following holds:

H1. A is a TI-subgroup
H2. $C_G(a) = C_G(A)$ for all $a \in A^{\#}$.
H3. If $\langle A, B \rangle$ is a *proper subgroup* of G then one of the following holds:
  (1) $[A, B] = 1$;
  (2) $\langle A, B \rangle$ is a rank one group;
  (3) $[A, B] = [a, B] = [A, b] \in \mathcal{H}$ for any $a \in A^{\#}$ and $b \in B^{\#}$ and $[A, B] \leq Z(\langle A, B \rangle)$.

Franz Timmesfeld has kindly informed me that the hybrid condition groups can be classified. I quote his e-mail of 21 August 2003:

> *If $\mathcal{H}$ is a normal set of TI-subgroups of a group G satisfying your conditions 2.3 and $O_2(G) = 1$, it follows immediately that $D(\mathcal{H})$ (in the sense of p. 68 of my book* [**32**]*) is a set of root-involutions of G.*

3. Atomic group theory?

Now I shall try to outline the theory of finite groups G generated by a system of atomic subgroups as it emerges from the theory of groups of finite

Morley rank. In many aspects, it appears to be an alternative approach to Timmesfeld's theory of root subgroups.

For a subgroup $H < G$, denote by H° the subgroup generated in H by atomic subgroups contained in H, together with all elements of odd order in H. We shall call H° the *connected component* of H. A subgroup H is *connected* if $H = H^\circ$. Connected 2-subgroups will be called *unipotent.*

We use also another version of the connected subgroups: for a subgroup H, we define $B(H) = \langle \mathcal{A} \cap H \rangle$ and call it the *bounded part* of H. A subgroup H is *bounded* if $H = B(H)$.

Let T be a 2-Sylow subgroup of G. We shall work mostly with $S = T^\circ$, despite encountering the disturbing fact of the existence of simple groups with disconnected 2-Sylows, say, $\mathrm{SU}_3(2^{2n})$ and $\mathrm{Sz}(2^{2n+1})$. We shall call S a 2-*Sylow*$^\circ$ subgroup. We use usual conventions for symbols like $N^\circ_G(X) = (N_G(X))^\circ$, etc.

The following problems are word-for-word reformulations of theorems from the theory of groups of finite Morley rank and even type. We shall be happy to prove them even under the additional assumption that G is a K*-group. However, an absolute proof, one which does not use the K*-assumption, will be of some interest. Indeed, it would make the theory of atomic groups a self-contained chapter of finite group theory.

3.1. First obstacle: quadratic modules.

As Franz Timmesfeld pointed out, we have, unfortunately, a pretty nasty behaviour of $O_2(G)$. The first piece of news is actually quite good.

LEMMA 3.1 (Timmesfeld). *If A is root-type 2-subgroup in a finite group G then either $A \leqslant O_2(G)$ or $A \cap O_2(G) = 1$. In particular, this dichotomy holds for atomic subgroups.*

However, we may encounter a situation when $O_2(G)$ contains no unipotent subgroups, $\mathcal{A} \cap O_2(G) = \emptyset$. If this is the case and $A \in \mathcal{A}$ then it follows from Axiom A3 that $[A, A^x] = 1$ for every $x \in O_2(G)$. It follows that $[O_2(G), A, A] = 1$. Assume that $G = \langle \mathcal{A} \rangle$; then the composition factors of G in $O_2(G)$ are *quadratic modules* for the group $G/O_2(G)$. Modulo the CFSG, quadratic actions of almost quasisimple finite groups are described [**13, 22, 23, 28**]. A pretty powerful form of Axiom A3 allows to reduce the classification of $G/O_2(G)$ to the case of the quasisimple group $G/O_2(G)$. However, it is really unclear how to handle the situation without the CFSG.

And here is an example, provided by Franz Timmesfeld: If we take for G the semidirect product of the natural module $O_2(G)$ for $\mathrm{SL}_2(2^n)$ with $\mathrm{SL}_2(2^n)$, and for $\mathcal{A}$ the set of all elementary abelian subgroups of order 2^n in G which intersects trivially with $O_2(G)$ and are invariant under the action of some cyclic subgroup of order $2^n - 1$ from G, then Axioms A1–A4 are obviously satisfied, but $O_2(G)$ contains no atomic subgroups.

QUESTION 3.1. *Given a group G generated by a class of atomic subgroups, $G = \langle \mathcal{A} \rangle$ and such that*

- *$\mathcal{A} \cap O_2(G) = \emptyset$ and*
- *$G/O_2(G)$ is a simple group,*

list possibilities for $G/O_2(G)$ without the use of the CFSG.

In finite group theory, such configurations appeared in works on failure of factorisation in 2-constrained groups [**7, 8**]. Some of them have been pinpointed and formalised by Ron Solomon [**25**] under the name of 2-*blocks*.

By definition, a 2-block X of a finite group G is a subgroup X subnormal in some 2-local subgroup M of G and satisfying the following conditions:

- $X = O^2(X)$ and $X/O_2(X)$ is quasisimple;
- X has a unique non-central 2-chief factor and $[O_2(X), X] \leqslant \Omega_1(Z(O_2(X)))$.

A block is said to be of $\mathrm{SL}_2(2^n)$-*type* if $X/O_2(X) \simeq \mathrm{SL}_2(2^n)$ and $V = [O_2(X), X]$ is such that $V/C_V(X)$ is isomorphic to the standard $2n$-dimensional (over $\mathbb{F}(2)$) irreducible $\mathrm{SL}_2(2^n)$-module. To give some flavour of the theory, we quote one of the results by Ron Solomon.

THEOREM 3.2. ***[Solomon*** [**25**, Theorem 1.8]*] Assume that*

- *G is a simple group of characteristic 2-type;*
- *M is a maximal 2-local subgroups of G;*
- *X is a block of M of $\mathrm{SL}_2(2^n)$-type, $n \geqslant 2$; and*
- *M is a unique 2-local subgroup of G containing X.*

Then either

- $X/O_2(X) \simeq \mathrm{SL}_2(4)$ and $G \simeq M_{22}, M_{23}$ or J_3;
- $X/O_2(X) \simeq \mathrm{SL}_2(2^n)$ and $G \simeq \mathrm{PSL}_3(2^n)$ or $\mathrm{PSp}_4(2^n)$.

It still has to be seen whether the study of blocks involving quadratic modules can be avoided in a proof of a CGT-Theorem (see Section 3.6); it is a context where they naturally appear in the classical CFGG.

3.2. Second obstacle: the axioms are not inductive. If G is a group generated by a family of atomic subgroups, it is not clear why this condition can be transferred to factorgroups G/N, even if the normal subgroup N is connected or unipotent.

There is strong temptation to resolve this problem by adding an extra axiom,

A5. If N is a connected subgroup of G, $\overline{G} = G/N$ and

$$\overline{\mathcal{A}} = \{\overline{A} \mid A \in \mathcal{A},\ A \cap N = 1\},$$

then $\overline{\mathcal{A}}$ is a family of atomic subgroups in $\overline{G}$.

Unfortunately, it comes too close to cheating: this condition is very unnatural in the context of finite group theory. However, one may wish to prove it in the special case when subgroups from $\mathcal{A}$ intersect with N trivially.

Still, it might happen that the only feasible approach is to never work with factorgroups, trying just ignore $O_2(G)$, especially in the situation when $O_2(G)$ contains no atomic subgroups.

3.3. Weakly embedded subgroups. However, despite the difficulties outlined, let us try to say something positive.

Let S be a 2-Sylow° subgroup of G.

We call a proper subgroup $M < G$ *weakly embedded* in G if $S \leqslant M$ and, for every non-trivial unipotent subgroup $U \leqslant M$,

$$B(N_G(U)) \leqslant M.$$

Some observations help to clarify the nature of this concept.

First of all, I very much hope that the following **normaliser condition** is true (it should be an easy corollary of Axiom A3).

Lemma 3.3. *If $W < U$ are unipotent subgroups then $N^\circ_U(W) > W$*

Next, let $\mathcal{U}$ be the graph whose vertices are all non-trivial unipotent subgroups of G, with two vertices U, V being connected by an edge iff UV is a group (in which case it is, of course, UV is unipotent). If $\mathcal{U}$ is disconnected and $\mathcal{W}$ its connected component, then its setwise stabiliser M in G (with respect to action by conjugation) is a weakly embedded subgroup. Vice versa, if M is a weakly embedded subgroup then $\mathcal{U} \cap M$ is a union of connected components of $\mathcal{U}$. After replacing M by $N_G(\mathcal{U} \cap M)$, we come to the following result.

Lemma 3.4. *If G contains a weakly embedded subgroup then G contains a weakly embedded subgroup M such that $N_G(U) \leqslant M$ for every non-trivial unipotent group $U \leqslant M$.*

Question 3.2. *If a finite group G contains a weakly embedded subgroup M, then is it true that $G/O_2(G)$ is one of the groups* $\mathrm{PSL}_2(2^n)$, $\mathrm{SU}_3(2^{2n})$ *or* $\mathrm{Sz}(2^{2n+1})$*?*

Richard Lyons pointed out that this question is likely to lead to a strongly embedded subgroup or a standard subgroup configuration: for some involution $t \in M$, $B(C_G(t)) \simeq \mathrm{PSL}_2(2^n)$, $\mathrm{SU}_3(2^{2n})$ or $\mathrm{Sz}(2^{2n+1})$; but this is exactly what has happened in the finite Morley rank context.

Franz Timmesfeld suggests a geometric way of getting a positive answer to Question 3.2: one has to consider the graph $\mathcal{D}$ whose vertices are involutions in atomic subgroups of G, and edges connect commuting involutions. It is easy to see that, if G contains a weakly embedded subgroup, $\mathcal{D}$ is disconnected. To prove that G actually contains a strongly embedded subgroup and therefore is known by Bender's Theorem [**10**], it suffice to check that $\mathcal{D}$ is closed under *commuting products*, that is, if $a, b \in \mathcal{D}$ and $ab = ba$ then $ab \in \mathcal{D}$; then the normaliser of a connected component of $\mathcal{D}$ is strongly embedded in G by Aschbacher [**4**].

3.4. Corollaries from the Weakly Embedded Subgroup Theorem. The following is likely to be very easy:

QUESTION 3.3. *If so, can the previous result be used to prove that*

$$O(N^\circ_G(U)) = 1$$

for every connected 2*-subgroup* U*?*

Notice that we have an easy to prove but extremely useful fact:

> if a connected 2-group U normalises a $2'$-group R then $[U, R] = 1$,

which lends to a nice signaliser functor theory.

3.5. Special cases of the Weakly Embedded Subgroup Theorem. We encounter, as an important special case of the weakly embedded subgroup problem (and, which is very likely, crucially important), the classical configuration of a TI-subgroup weakly closed in its centraliser [**29**]. It is worth quoting [**29**, Corollary B]:

THEOREM 3.5. ***(Timmesfeld 1975*** [**29**, Corollary B]*) Let* G *be a finite group and* A *an elementary abelian* TI 2*-subgroup of* G*. Assume, in addition, that, for any* $g \in G$*,* $[A, A^g] = 1$ *implies* $A = A^g$*. Set* $G^* = \langle A^G \rangle$*. Then either*

(a) G^* *is solvable, or*
(b) G^* *contains a normal elementary abelian* 2*-subgroup* N *such that* G^*/N *is isomorphic to a covering group of* $\mathrm{PSL}_n(2^m)$*,* $\mathrm{Sz}(2^{2m+1})$*,* $\mathrm{SU}_3(2^{2m})$*,* Alt_6*,* Alt_7*,* Alt_8*,* Alt_9*,* M_{22}*,* M_{23}*, or* M_{24}*.*

I have not checked the details but it is unlikely that groups Alt_6*,* Alt_7*,* Alt_8*,* Alt_9*,* M_{22}*,* M_{23}*, or* M_{24} *are generated by atomic subgroups.*

Another special case of Question 3.2 is on simple groups with a *degenerate* system of atomic subgroups, which is equivalent to the following question.

QUESTION 3.4. *Assume that a* 2*-Sylow*$^\circ$ *subgroup* S *of* G *is abelian. Is it true that* $G/O_2(G)$ *is one of the groups* $\mathrm{PSL}_2(2^n)$*,* $\mathrm{SU}_3(2^{2n})$ *or* $\mathrm{Sz}(2^{2n+1})$*?*

Observe that, under the assumptions of Question 3.4, $N_G(S)$ is weakly embedded in G.

This is the point of the development of the theory where we expect, by analogy with [**1**], the so-called *pseudoreflection* subgroups of $N^\circ_G(B)/C_G(B)$ to come into play (see Question 4.3 in Section 4).

Question 3.4 would play in our theory the role of classical Goldschmidt's theorem on groups with strongly closed abelian subgroup [**14**]. However, it is unlikely that we can find a shorter alternative proof of a version of [**14**] specialised and adapted for our context.

3.6. Continuously characteristic subgroups and the Global CGT-Theorem. Let $Q \leq P$ be connected 2-subgroups. We say that Q is *continuously characteristic* in P if Q is invariant under the action of $N^\circ_G(P)$. If T is a 2-Sylow$^\circ$ subgroup of G, we denote

$$C(G,T) = \langle N^\circ_G(Q) \mid Q \text{ is continuously characteristic in } T\rangle$$

and

$$BC(G,T) = \langle B(N_G(Q)) \mid Q \text{ is continuously characteristic in } T\rangle$$

QUESTION 3.5. *Is it true that if $C(G,T) < G$ then G contains a proper weakly embedded subgroup?*

Is it true that if $BC(G,T) < G$ then G contains a proper weakly embedded subgroup?

This would make a finite group theoretic analogue of [**2**].

However this is a context where we may have complications related to the so-called "failure of factorisation" and where we might need a study of "quadratic module 2-blocks", see Section 3.1.

3.7. Pushing Up. We shall need an analogue of another result from [**2**].

For a finite group H, denote by either of two symbols $O_{\mathcal{A}}(H) = R_u(H)$ the maximal normal connected 2-subgroup of H.

QUESTION 3.6. *Let Q be a connected 2-group of G such that $Q = R_u(N^\circ_G(Q))$ and $B(N^\circ_G(Q))/Q \simeq \mathrm{SL}_2(2^n)$, $\mathrm{SU}_3(2^{2n})$ or $\mathrm{Sz}(2^{2n+1})$. Is it true that $N^\circ_G(Q)$ contains a 2-Sylow$^\circ$ subgroup of G?*

Here, $B(H) = \langle \mathcal{A} \cap H\rangle$.

3.8. Parabolic subgroups. Let S be a 2-Sylow$^\circ$ subgroup of G. A connected subgroup P is *parabolic* if it has the form $P = N^\circ_G(R)$ where R is a B-subgroup containing S as a proper subgroup.[4]

[4]This definition has been adjusted after Richard Lyons pointed out that more traditional definition, "P is a proper connected subgroup containing $N^\circ_G(S)$", leads to some difficult configurations of the kind present in D_4 extended by the triality automorphism.

QUESTION 3.7. *For a parabolic subgroup P, prove that*

- $R_u(P) \neq 1$ *and*
- $C^\circ_G(R_u(P)) \leq R_u(P)$.

It is here that we expect the use of the strongly closed abelian subgroup theorem and some rudimentary component analysis.

3.9. Minimal parabolic subgroups and the final identification. Let $\mathcal{M}$ be the set of minimal parabolic subgroups properly containing a 2-Sylow$^\circ$ subgroup S.

QUESTION 3.8. *Prove that if $|\mathcal{M}| \leq 1$ then G has a proper weakly embedded subgroup.*

QUESTION 3.9. *If $|\mathcal{M}| = 2$ then $G/O_2(G)$ is a Lie rank 2 group over a field of characteristic 2.*

This would be an analogue of the result on groups of finite Morley rank [**3**], which, in its turn, is an analogue (or, possibly, even a direct consequence) of well-known papers by Delgado-Stellmacher [**12**] and Stellmacher [**27**].

QUESTION 3.10. *If $|\mathcal{M}| \geq 3$ then $G/O_2(G)$ has a BN-pair of rank at least 3 and thus is a group of Lie type over a field of characteristic 2.*

This is an analogue of [**11**], which, in its turn, is a direct analogue of Niles [**24**].

QUESTION 3.11. ***In this plan, are there any shortcuts via the theory of groups generated by abstract root subgroups?***

4. Questions on finite linear groups

I list a few questions about finite linear groups which arise in 2-local analysis of finite groups generated by atomic subgroups. Some of these questions are probably easy and relatively well known. In any case, I would appreciate any information on their status.

In all cases the setting is the same: X is a finite group which acts irreducibly on an elementary abelian group V of order p^n, p a prime. Questions are formulated in greater generality than needed for the theory of groups generated by atomic 2-subgroups. In particular, we actually need only the case $p = 2$.

4.1. TI-submodules.

QUESTION 4.1. *Assume that V contains a subgroup $W < V$ of order $q > p$ such that, for all $x \in X$, $W^x = W$ or $W^x \cap W = 1$.*

Under the additional assumption that V is primitive and tensor indecomposable module, does this imply that V is actually a $GF(q)X$-module?

As Bob Guralnick pointed out, without the additional assumption about the action of X the answer is "no". Indeed, if X acts imprimitively (i.e. the module is induced), then a counterexample is provided by $\mathrm{GL}_m(2) \wr \mathrm{Sym}_r$ acting on a space of dimension $n = mr$. Similarly, if G preserves a tensor product on V, e.g. $\mathrm{GL}_a(2) \times \mathrm{GL}_b(2)$ acting on $V_a \otimes V_b$, and take $W = V_a \otimes v$ for a fixed vector v, then its conjugates are precisely $V_a \otimes v'$ ranging over v' and these are all disjoint.

Question 4.2. *We again assume that V contains a subgroup $W < V$ of order $q > p$ such that, for all $x \in X$, $W^x = W$ or $W^x \cap W = 1$. This time we are not making the assumption that the action of X on V is primitive and tensor indecomposable. Can V be made into a $GF(q)X$-module if we add, in addition, the requirement that $C_X(w) = C_X(W)$ for all non-zero elements $w \in W$? This is, of course, the case when W is an atomic subgroup of $V \rtimes X$.*

Bob Guralnick suggested a possible approach to Question 4.2. If you have such a W, consider the lattice of subspaces generated by its conjugates—it seems plausible that this is a proper sublattice (and is clearly X-invariant). If this is the case, then it is shown in [**16**] that X either preserves a tensor structure, or is imprimitive, or preserves a field extension structure or a subfield structure, or $\dim W = n/2$. In the latter case, Bob points out there are lots of such examples, but not one immediately over $\mathbb{F}_2$ and certainly not with the $C_X(w) = C_X(W)$ assumption.

4.2. Pseudoreflection groups. Assume that X is generated by a conjugacy class of abelian subgroups K of order coprime to p and such that K acts irreducibly on $W = [V, K]$ and $W < V$. We may also assume that W is much smaller than V, by demanding that $\dim W \leq \frac{1}{2} \dim V$. Let $q = |W|$. A prominent example is, of course, $X = GL(m, q)$, with K being the group consisting of diagonal matrices of the form $\mathrm{diag}(\lambda, 1, \ldots, 1)$. We call K a *pseudoreflection group.*

Question 4.3. *Can groups generated by pseudoreflections be easily classified? If we assume that the composition factors of X are known simple groups (that is, assume CFSG), what one can say about X?*

There are still too many examples of groups generated by pseudoreflections. Question 4.3 is possibly easier if we assume, in addition, that W is a TI-subgroup.

Question 4.4. *Assume, in addition, that W is a TI-subgroup, $W^x = W$ or $W^x \cap W = 1$ for all $x \in X$. Classify pairs (X, V).*

The case when $K \simeq Z_{q-1}$ is of special interest.

Question 4.5. *A very small special case is $|K| = 3$ and $|W| = 4$. In that case for any $x \in X$ the group $\langle K, K^x \rangle$ is isomorphic to $Z_3 \times Z_3$, Alt_4 or $\mathrm{SL}_2(4) \simeq \mathrm{Alt}_5$.*

Aschbacher and Hall [**5**] and Stellmacher [**26**] classified abstract finite groups with a conjugacy class of subgroups of order 3 with these properties.

THEOREM 4.1 (Aschbacher and Hall). *If the finite group G with no non-trivial soluble normal subgroups is generated by a conjugacy class D of subgroups of order* 3*, and any two non-commuting elements of D generate a subgroup isomorphic to* $\mathrm{SL}_2(3)$ *or to* $\mathrm{PSL}_2(3)$*, then G is isomorphic to* $\mathrm{Sp}_n(3)$*,* $\mathrm{U}_n(3)$ *or* $\mathrm{PGU}_n(2)$*, for some n.*

THEOREM 4.2 (Stellmacher). *Let G be a finite group with the following properties.*

(1) *G is generated by a conjugacy class D of elements of order* 3*. Any two non-commuting elements in D generate a subgroup isomorphic to* Alt_4*,* Alt_5*, or* $\mathrm{SL}(2,3)$*, and* Alt_5 *occurs for some pair.*
(2) $O_2(G) = Z(G) = 1$.

Then G is isomorphic to a symplectic group $\mathrm{Sp}(2n,2)$*, $n \geq 3$, an orthogonal group* $O^{\varepsilon}(2n,2)$*, $\varepsilon = \pm 1$, $n \geq 3$, an alternating group* Alt_n*, $n \geq 5$, the Chevalley group* $\mathrm{G}_2(4)$*, or one of the sporadic groups* HJ*,* Sz*,* Co (.1)*.*

Question 4.3 is in a sense dual to the setting of the paper by Guralnick et al. [**15**] who have recently classified linear groups with elements whose orders are large primitive prime divisors.

Although in the context of the present paper we are primarily concerned with the case $p = 2$, the odd characteristic version of pseudoreflection groups also have analogues in the theory of groups of finite Morley rank. Assume now that $p > 2$and that $K \simeq Z_{q-1}$, so that K contains an involution. A natural approach to Question 4.3 and 4.4 in that case is to study subgroups $H = \langle K, K^x \rangle$ for $x \in X$ and to try to show that $H \simeq \mathrm{GL}_2(q)$ and acts on $V/C_V(K) \cap C_V(K^x)$ as on its natural 2-dimensional (over $\mathbb{F}_q$) module. Take $J = [H, H] \simeq \mathrm{SL}_2(q)$ and let z be the involution from $Z(J)$. Then one can see that $J \lhd C_X(z)$ and therefor z is a classical involution in X in the sense of Aschbacher [**6**], which identifies X with a Chevalley group over a field of characteristic p. This approach would mimic Ho's treatment of quadratic pairs in odd characteristic [**19, 20, 21**].

5. Technical lemmas

Notice that the concept of atomic subgroup fairs well when we pass from the ambient group G to a subgroup H such that $\mathcal{A} \cap H \neq \emptyset$, and not so well when we pass to factor groups G/K. This has to be kept in mind when discussing the inductive argument.

A lot of minor results will be needed. Some of them are fairly obvious, but some analogues of very useful results about groups of finite Morley rank are no longer true. For example, if a $2'$-element α acts on a connected 2-group Q, it is not any longer true that the subgroups $[Q,\alpha]$ and $C_Q(\alpha)$ are connected. A counter example is provided by the group

$$Q = A_1 \times A_2 \times A_3$$

with $\mathcal{A} \cap Q = \{A_1, A_2, A_3\}$ and an element α of order 3 which cyclically permutes A_i.

However, I am relatively optimistic about the following questions.

Let H be a subgroup of G generated by atomic subgroups, $H = B(H)$.

QUESTION 5.1. *Prove that if H is solvable then H is a 2-group.*

QUESTION 5.2. *Prove that, for a connected 2-group Q, $Z^\circ(Q) \neq 1$.*

QUESTION 5.3. *Prove that for connected 2-groups $Q < P$,*

$$\mathcal{A} \cap Q \neq \mathcal{A} \cap N_P(Q).$$

[This is an analogue of the fact on 2-groups of finite Morley rank: $|N_P(Q)/Q|$ is infinite.]

QUESTION 5.4. *Describe K*-groups generated by atomic subgroups.*

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References

[1] T. Altınel, A. V. Borovik and G. Cherlin, *Groups of finite Morley rank and even type with strongly closed abelian subgroup*, J. Algebra 232 (2000), 420–461

[2] T. Altınel, A. V. Borovik and G. Cherlin, *Pushing up and a global C(G,T)-theorem for groups of finite Morley rank*, J. Algebra 247 (2002), 541–576.

[3] T. Altınel, A. V. Borovik and G. Cherlin, *Classification of simple K*-groups of finite Morley rank and even type: geometric aspects*, in "Groups, Combinatorics and Geometry: Durham 2001," (A. Ivanov, M. Liebeck and J. Saxl, eds.), World Scientific 2003, pp. 1–12.

[4] M. Aschbacher, *A condition for the existence of a strongly embedded subgroup*, Proc. Amer. Math. Soc. 38 (1973), 509–511.

[5] M. Aschbacher and M. Hall, Jr. *Groups generated by a class of elements of order* 3. J. Algebra 24 (1973), 591–612.

[6] *A characterization of Chevalley groups over fields of odd order.* I, II, Ann. of Math. 106 (1977), 353-468; Correction, Ann. of Math. 111 (1980), 411414.

[7] M. Aschbacher, *On the failure of the Thompson factorization in* 2*-constrained groups*, Proc. London Math. Soc. (3) 43 (1981), 425–449.
[8] M. Aschbacher, *A factorisation theorem for* 2*-constrained groups*, Proc. London Math. Soc. (3) 43 (1981), 450–477.
[9] B. Baumann and C. Ho, *Linear groups generated by a pair of quadratic action subgroups*, Arch. Math. (Basel) 44 (1985), no. 1, 15–19.
[10] H. Bender, *Transitive Gruppen gerader Ordnung, in dene jede Involution genau einen Punkt festlässt*, J. Algebra 17 (1971), 527–554.
[11] A. Berkman and A. V. Borovik, *An identification theorem for groups of finite Morley rank and even type,* J. Algebra 266 (2003), 375–381.
[12] A. Delgado and B. Stellmacher, *Weak* (B, N)*-pairs of rank 2*, in "Groups and Graphs: New Results and Methods", by A. Delgado, D. Goldschmidt, and B. Stellmacher, Birkhaüser, DMV Seminar 6, Basel, 1985.
[13] T. Englund, *Quadratic representations for groups of Lie type over fields of characteristic two*, J. Algebra 268 (2003), 118–155.
[14] D. M. Goldschmidt, 2*-fusion in finite groups*, Annals Math. 111 (1980), 70–117.
[15] R. Guralnick, T. Penttila, C. E. Praeger and J. Saxl, *Linear groups having certain large prime divisors*, Proc. London Math. Soc. (3) 78 (1999), 167–214.
[16] R. Guralnick and J. Shareshian, *Subgroups of* $\mathrm{GL}_n(q)$ *which fix sublattices of the subspace lattice*, J. Algebra 259 (2003), 147–176.
[17] C. Hering, *Transitive linear groups and linear groups which contain irreducible subgroups of prime order*, Geom. Dedicat. 2 (1974) 425–460.
[18] C. Hering, *Transitive linear groups and linear groups which contain irreducible subgroups of prime order, II*, J. Algebra 93 (1985) 151–164.
[19] C.-Y. Ho, *On the quadratic pairs.* J. Algebra 43 (1976), no. 1, 338–358.
[20] C.-Y. Ho, *Quadratic pairs for odd primes.* Bull. Amer. Math. Soc. 82 (1976), no. 6, 941–943.
[21] C.-Y. Ho, *Chevalley groups of odd characteristic as quadratic pairs.* J. Algebra 41 (1976), no. 1, 202–211.
[22] U. Meierfrankenfeld and G. Stroth, *Quadratic* $GF(2)$*-modules for sporadic groups and alternating groups*, Comm. Alg. 18 (1990), 2099–2140.
[23] U. Meierfrankenfeld and G. Stroth, *On quadratic* $GF(2)$*-modules for Chevalley groups over fields of odd order*, Arch. Math. 55 (1990), 105–110.
[24] R. Niles, BN*-pairs and finite groups with parabolic-type subgroups*, J. Algebra 75 (1982), 484–494.
[25] R. Solomon and S. K. Wong, *On* $L_2(2^n)$*-blocks*, Proc. London Math. Soc. (3) 43 (1981), 499-519.
[26] B. Stellmacher, *Einfache Gruppen, die von einer Konjugiertenklasse von Elementen der Ordnung drei erzeugt werden.* J. Algebra 30 (1974), 320–354.
[27] B. Stellmacher, *On graphs with edge-transitive automorphism groups*, Illinois J. Math. 28 (1984), 211–266.
[28] G. Stroth, *Strong quadratic modules*, Israel J. Math. 79 (1992), 257–279.
[29] F. Timmesfeld, *Groups with weakly closed TI-subgroups*, Math. Z. 143 (1975), 243–278.
[30] F. Timmesfeld, *On elementary abelian TI-groups*, J. Algebra 44 (1977), 457–476.
[31] F. Timmesfeld, *Groups generated by k-transvections*, Invent. Math. 100 (1990), 167–206.
[32] F. Timmesfeld, "Abstract root subgroups and simple groups of Lie type." Birkhäuser Verlag, Basel, 2001. xiv+389 pp. (ISBN 3-7643-6532-3)

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