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Abstract

Simple polytopes play important role in applications of algebraic geometry to physics. They are also main objects in toric topology.

There is a commutative associative ring $\mathscr{P}$ generated by simple polytopes. The ring $\mathscr{P}$ possesses a natural derivation d, which comes from the boundary operator. We shall describe a ring homomorphism from the ring $\mathscr{P}$ to the ring of polynomials $\mathbb{Z}[t, \alpha]$ transforming the operator d to the partial derivative $\partial/\partial t$.

This result opens way to a relation between polytopes and differential equations. As it has turned out, certain important series of polytopes (including some recently discovered) lead to fundamental non-linear differential equations in partial derivatives.

Definition. A polytope P^n of dimension n is said to be *simple* if every vertex of P is the intersection of exactly n facets, i.e. faces of dimension $n-1$.

Definition. Two polytopes P_1 and P_2 of the same dimension are said to be *combinatorially equivalent* if there is a bijection between their sets of faces that preserves the inclusion relation.

Definition. A combinatorial polytope is a class of combinatorial equivalent geometrical polytopes.

The collection of all n-dimensional combinatorial simple polytopes is denoted by $\mathscr{P}_n$.

An Abelian group structure on $\mathscr{P}_n$ is induced by the disjoint union of polytopes.

The zero element of the group $\mathscr{P}_n$ is the empty set.

The weak direct sum

$$\mathscr{P} = \sum_{n \geqslant 0} \mathscr{P}_n$$

yields a *graded* commutative associative ring.
The product $P_1^n P_2^m$ of homogeneous elements P_1^n and P_2^m is given by the direct product $P_1^n \times P_2^m$.
The unit element is a single point.

Remarks:

1. The direct product $P_1^n \times P_2^m$ of simple polytopes P_1^n and P_2^m is a simple polytope as well.
2. Each face of a simple polytope is again a simple polytope.

Let $P^n \in \mathscr{P}_n$ be a simple polytope. Denote by $dP^n \in \mathscr{P}_{n-1}$ the disjoint union of all its facets.

Lemma. We have a linear operator of degree -1

$$d : \mathscr{P} \longrightarrow \mathscr{P},$$

such that

$$d(P_1^n P_2^m) = (dP_1^n)P_2^m + P_1^n(dP_2^m).$$

Examples:

$$d\Delta^n = (n+1)\Delta^{n-1},$$

$$dI^n = n(dI)I^{n-1} = 2nI^{n-1},$$

where Δ^n is the standard n-simplex and $I^n = I \times \cdots \times I$ is the standard n-cube.

Face-polynomial.

Consider the linear map

$$F\colon \mathscr{P} \longrightarrow \mathbb{Z}[t,\alpha],$$

which send a simple polytope P^n to the homogeneous *face-polynomial*

$$F(P^n) = \alpha^n + f_{n-1,1}\alpha^{n-1}t + \cdots + f_{1,n-1}\alpha t^{n-1} + f_{0,n}t^n,$$

where $f_{k,n-k} = f_{k,n-k}(P^n)$ is the number of its k-dimensional faces. Thus, $f_{n-1,1}$ is the number of facets and $f_{0,n}$ is the number of vertex.

Note that $f_{k,n-k} = f_{n-k-1}$, where $f(P^n) = (f_0,\ldots,f_{n-1})$ is f-vector of P^n.

Theorem The mapping F is a ring homomorphism such that

$$F(dP^n) = \frac{\partial}{\partial t}F(P^n).$$

Corollary.

$$F(I^n) = (\alpha + 2t)^n,$$

$$F(\Delta^n) = \frac{(\alpha + t)^{n+1} - t^{n+1}}{\alpha}.$$

Set

$$U(t, x; \alpha, I) = \sum_{n \geqslant 0} F(I^n) x^{n+1}.$$

Lemma. The function $U(t, x; \alpha, I)$ is the solution of the equation

$$\frac{\partial}{\partial t} U(t, x) = 2x^2 \frac{\partial}{\partial x} U(t, x)$$

with the initial condition $U(0, x) = \frac{x}{1-\alpha x}$.

We have

$$U(t, x; \alpha, I) = \frac{x}{1 - (\alpha + 2t)x}.$$

Set

$$U(t,x;\alpha,\Delta)=\sum_{n\geqslant 0}F(\Delta^n)x^{n+2}.$$

Lemma. The function $U(t,x;\alpha,\Delta)$ is the solution of the equation

$$\frac{\partial}{\partial t}U(t,x)=x^2\frac{\partial}{\partial x}U(t,x)$$

with the initial condition $U(0,x)=\frac{x^2}{1-\alpha x}$.

We have

$$U(t,x;\alpha,\Delta)=\frac{x^2}{(1-tx)(1-(\alpha+t)x)}.$$

Consider the series of Stasheff polytopes
(the associahedra)

$$As = \{As^n = K_{n+2}, n \geqslant 0\}.$$

Each facet of As^n is $As^i \times As^j,\ i \geqslant 0,\ i+j = n-1$, where embedding $\mu_k\colon As^i \times As^j \to \partial As^n,\ 1 \leqslant k \leqslant i+2$, correspondes to the pairing

$$(a_1 \cdots a_{i+2}) \times (b_1 \cdots b_{j+2}) \longrightarrow$$
$$\longrightarrow a_1 \cdots a_{k-1}(b_1 \cdots b_{j+2})a_{k+1} \cdots a_{i+2}.$$

Lemma.

$$dAs^n = \sum_{i+j=n-1} \sum_{k=1}^{i+2} \mu_k(As^i \times As^j) = \sum_{i+j=n-1} (i+2)(As^i \times As^j).$$

Corollary.

$$\frac{\partial}{\partial t} F(As^n) = \sum_{i+j=n-1} (i+2) F(As^i) F(As^j).$$

Set

$$U(t,x;\alpha,\mathcal{A}s)=\sum_{n\geqslant 0}F(\mathcal{A}s^{n})x^{n+2}.$$

Theorem. The function $U(t,x;\alpha,\mathcal{A}s)$ is the solution of the Hopf equation

$$\frac{\partial}{\partial t}U(t,x)=U(t,x)\frac{\partial}{\partial x}U(t,x)$$

with the initial condition $U(0,x)=\frac{x^2}{1-\alpha x}$.
The function $U(t,x;\alpha,\mathcal{A}s)$ satisfies the equation

$$t(\alpha+t)U^2-(1-(\alpha+2t)x)U+x^2=0.$$

Quasilinear Burgers–Hopf Equation

The Hopf equation (Eberhard F.Hopf, 1902–1983)
is the equation

$$U_t + f(U)U_x = 0.$$

The Hopf equation with $f(U) = U$ is a limit case of the following equations:

$U_t + UU_x = \mu U_{xx}$ (the Burgers equation),

$U_t + UU_x = \varepsilon U_{xxx}$ (the Korteweg–de Vries equation).

The Burgers equation (Johannes M.Burgers, 1895–1981) occurs in various areas of applied mathematics (fluid and gas dynamics, acoustics, traffic flow). It used for describing of wave processes with velocity u and viscosity coefficient μ. The case $\mu = 0$ is a prototype of equations whose solution can develop discontinuities (shock waves).

K-d-V equation (Diederik J.Korteweg, 1848–1941 and Hugo M. de Vries, 1848–1935) was introduced as equation for the long waves over water (in 1895). It appears also in plasma physics. Today K-d-V equation is a most famous equation in soliton theory.

Let us consider the Burgers equation

$$U_t = UU_x - \mu U_{xx}.$$

Set $U = U_0 + \sum\limits_{k\geqslant 1} \mu^k U_k$. Then

$$U_{0,t} + \sum_{k\geqslant 1} \mu^k U_{k,t} = \left(U_0 + \sum_{k\geqslant 1} \mu^k U_k\right)\left(U_{0,x} + \sum_{k\geqslant 1} \mu^k U_{k,x}\right) - \\ - \mu U_{0,xx} - \sum_{k\geqslant 1} \mu^{k+1} U_{k,xx}.$$

Thus we obtain:

$$\begin{aligned} U_{0,t} &= U_0 U_{0,x}, \\ U_{1,t} &= (U_0 U_1)_x - U_{0,xx}. \end{aligned}$$

For simple polytopes, the formula for the Euler characteristic admits a generalization in the form of Dehn–Sommerville relations. In terms of the f-vector of an n-dimensional polytope P, they can be written as follows:

$$f_{k-1} = \sum_{j=k}^{n} (-1)^{n-j} \binom{j}{k} f_{j-1}, \qquad k = 0, 1, \ldots, n.$$

Consider the ring homomorphism

$$T\colon \mathbb{Z}[t, \alpha] \longrightarrow \mathbb{Z}[t, \alpha],$$

$$T\, p(t, \alpha) = p(t + \alpha, -\alpha).$$

Theorem. The Dehn–Sommerville relations are equivalent to the formula

$$T\, F(P^n) = F(P^n).$$

Consider the ring homomorphism

$$\lambda\colon \mathbb{Z}[t,\alpha] \longrightarrow \mathbb{Z}[z,\alpha] \ : \ \lambda(t) = \frac{1}{2}(z-\alpha),\ \lambda(\alpha) = \alpha,$$

and

$$\widehat{T}\colon \mathbb{Z}[z,\alpha] \longrightarrow \mathbb{Z}[z,\alpha] \ : \ \widehat{T}(z) = z,\ \widehat{T}(\alpha) = -\alpha.$$

Lemma. $\widehat{T}\lambda p(t,\alpha) = \lambda T p(t,\alpha)$

Corollary. For any $P^n \in \mathscr{P}_n$ the polynomial

$$p(z,\alpha) = \lambda F(P^n)$$

is such that $p(z,\alpha) = p(z,-\alpha)$.

Examples. Set additionally $\lambda(x) = x$. Then

1. $\lambda U(t,x;\alpha,I) = \frac{x}{1-zx}$.

2. $\lambda U(t,x;\alpha,\Delta) = \dfrac{x^2}{\left(1-\frac{1}{2}(z-\alpha)x\right)\left(1-\frac{1}{2}(z+\alpha)x\right)}$.

3. Set $U = U(t,x;\alpha,\mathit{As})$. The function $\widehat{U} = \lambda U$ satisfies the equation

$$(z-\alpha)(z+\alpha)\widehat{U}^2 - 4(1-zx)\widehat{U} + 4x^2 = 0.$$

The solution of this quadratic equation with the initial condition $\widehat{U}(0,x) = \frac{x^2}{1-\alpha x}$ gives

$$(z^2 - \alpha^2)\widehat{U} = 2\Big[(1 - zx) - (1 - 2zx + \alpha^2 x^2)^{1/2}\Big].$$

Consider two vectors r, r' such that

$$|r| = 1, \ |r'| = \alpha x, \ \langle r, r' \rangle = zx.$$

Then $|r||r'|\cos(r,r') = \alpha x \cos(r,r') = zx$.

Thus, $z = \alpha\cos(r,r'), \quad z^2 - \alpha^2 = -\alpha^2 \sin^2(r,r')$,

$1 - zx = |r|^2 - \langle r, r' \rangle = \langle r, r - r' \rangle$,

$(1 - 2zx + \alpha^2 x^2)^{1/2} = |r - r'|$.

Lemma. The function $\widehat{U}$ satisfies the equation

$$\alpha^2 \sin^2(r,r')\widehat{U} = 2\Big[|r - r'| - \langle r, r - r' \rangle\Big].$$

We have

$$\frac{d}{dz}\left((z^2-\alpha^2)\widehat{U}\right) = 2\left(-x+\frac{x}{|r-r'|}\right) =$$

$$= 2x\sum_{n\geqslant 1}^{\infty}\alpha^n L_n\left(\frac{z}{\alpha}\right)x^n,$$

where $L_n(\cdot)$ are Legendre polynomials.
We have

$$L_n\left(\frac{z}{\alpha}\right) = \frac{1}{n(n+1)}\frac{d}{dz}\left((z^2-\alpha^2)\frac{d}{dz}L_n\left(\frac{z}{\alpha}\right)\right).$$

Thus,

$$\widehat{U} = 2\frac{\partial}{\partial z}\left(\sum_{n\geqslant 1}\frac{\alpha^n}{n(n+1)}L_n\left(\frac{z}{\alpha}\right)x^{n+1}\right),$$

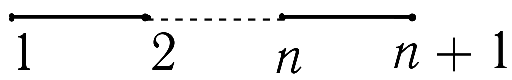
$$\frac{\partial^2\widehat{U}}{\partial x^2} = 2\frac{\partial}{\partial z}\left(\sum_{n\geqslant 1}\alpha^n L_n\left(\frac{z}{\alpha}\right)x^{n-1}\right).$$

Corollary. $x\frac{\partial^2}{\partial x^2}U = \frac{\partial}{\partial t}\frac{1}{|r-r'|}.$

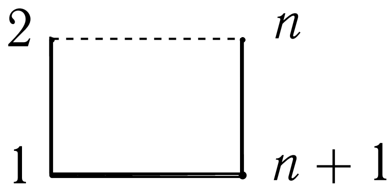
Graph-associahedra.

Given a finite graph Γ. The graph-associahedron $P(\Gamma)$ is a simple polytope whose poset is based on the connected subgraph of Γ. When Γ is:

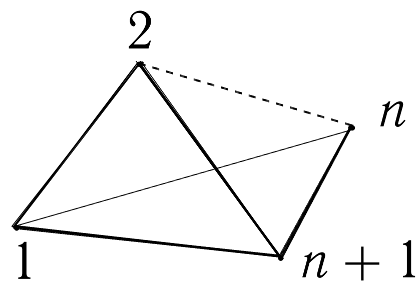
a path



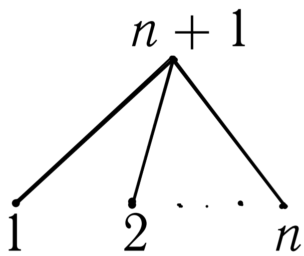
a cycle



a complete graph



an n-star graph



the polytope $P(\Gamma)$ results in the:

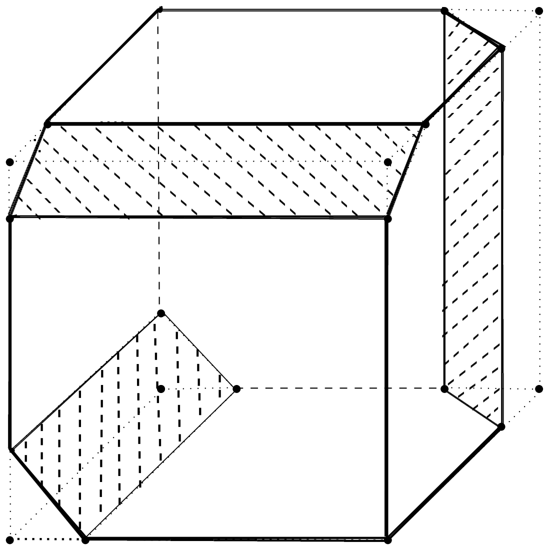
- associahedron (Stasheff polytope) As^n,
- cyclohedron (Bott–Taubes polytope) Cy^n,
- permutohedron Pe^n,
- stellohedron St^n,

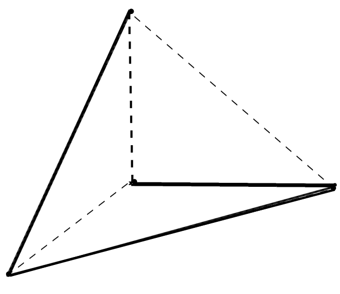
respectively.

GRAPH-ASSOCIAHEDRON

Associahedron As^3

The Stasheff polytope K_5.

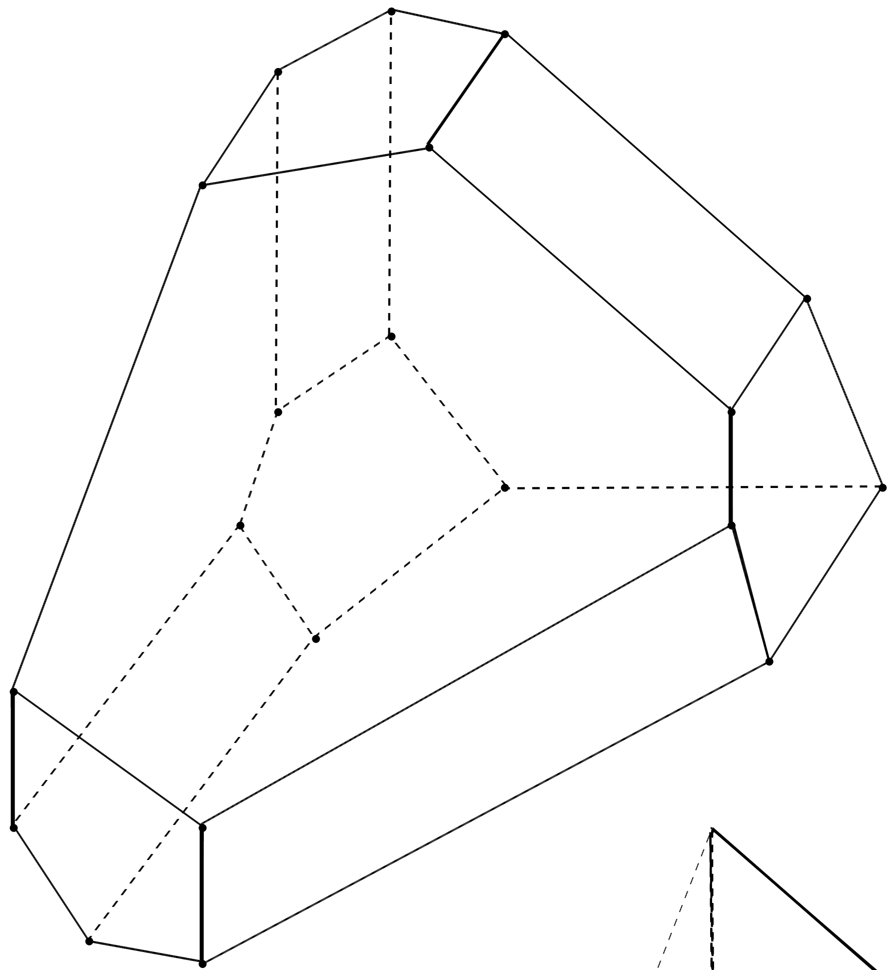


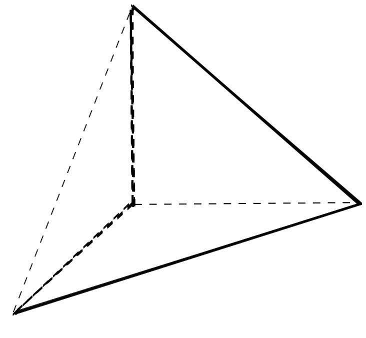


GRAPH-ASSOCIAHEDRON

Cyclohedron C^3

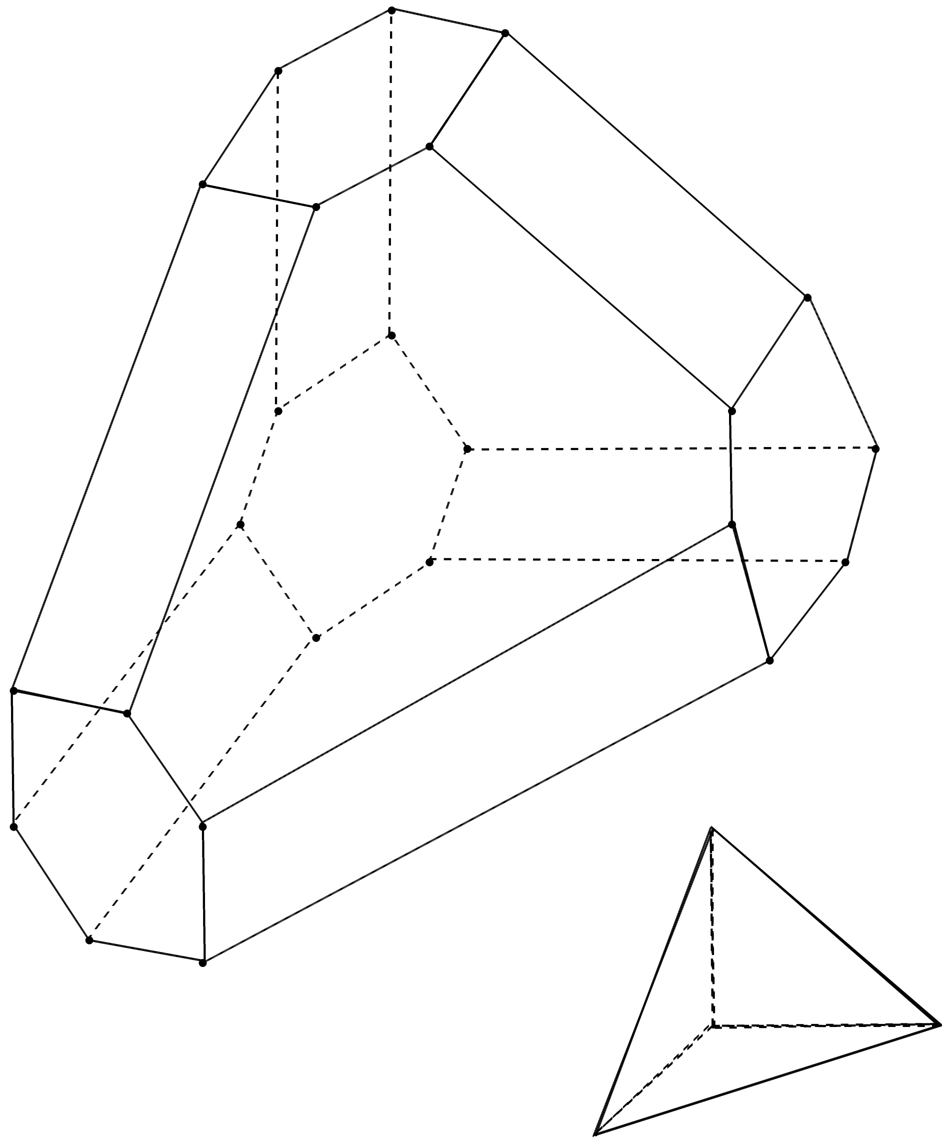
Bott–Taubes polytope





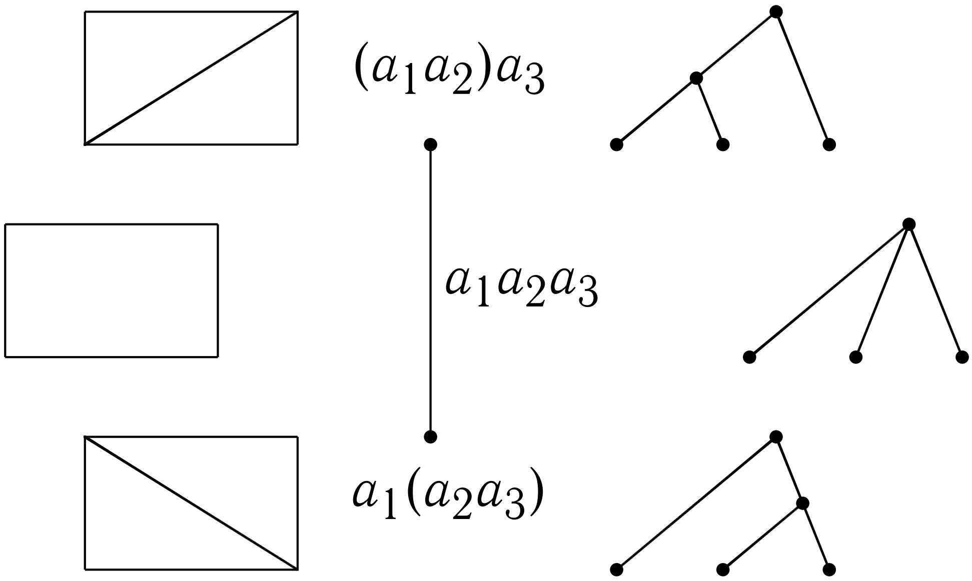
GRAPH-ASSOCIAHEDRON

Permutoedron Π^3.



The connection between bracketing and plane trees was known to A. Cayley (see [*])

The Stasheff polytope K_3



The languages: diagonals, brackets and plane trees.

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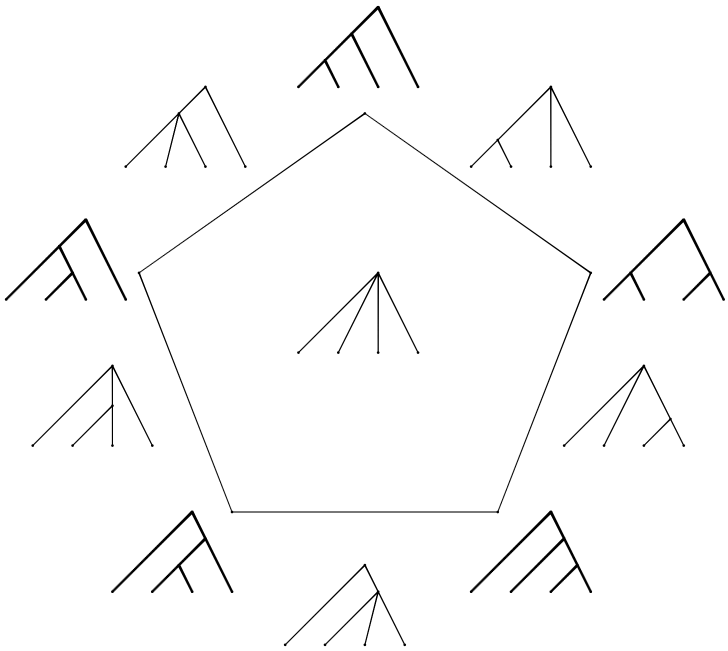
*A.Cayley, On the analytical form called trees, Part II, Philos. Mag. (4) 18,1859,374–378.

The Stasheff polytope K_4.



The language of plane trees.

Consider the series of Bott–Taubes polytopes (the cyclohedra)

$$Cy = \{Cy^n \, : \, n \geqslant 0\}.$$

Lemma. (A.Fenn)

$$dCy^n = (n+1) \sum_{i+j=n-1} Cy^i \times As^j.$$

Set

$$U(t,x;\alpha,Cy) = \sum_{n\geqslant 0} F(Cy^n)x^n.$$

Theorem. The function $U(t,x;\alpha,Cy)$ is the solution of the equation

$$\frac{\partial}{\partial t} U_1 = \frac{\partial}{\partial x}(U_0 U_1)$$

with the initial condition $U_{1,0}(0,x) = \frac{1}{1-\alpha x}$, where U_0 is the solution of the Hopf equation

$$\frac{\partial}{\partial t} U_0 = U_0 \frac{\partial}{\partial x} U_0$$

with the initial condition $U_0(0,x) = \frac{x^2}{1-\alpha x}$.

Complex cobordism.

Consider the complex cobordism ring

$$\Omega_U = \mathbb{Z}[z_n \;:\; n \geqslant 1],\ \deg z_n = -2n.$$

We have $\Omega_U \otimes Q = Q\big[\,[\mathbb{C}P^n] \;:\; n \geqslant 1\big]$.
The ring Ω_U is a module over Landweber–Novikov algebra S, which is a Hopf algebra over $\mathbb{Z}$.

There are primitive elements $s_n \in S,\ n \geqslant 1,$ and they generate a Lie algebra:

$$[s_n, s_m] = (m - n)s_{m+n}.$$

The operations s_n are derivations of the ring Ω_U.

One can describe **the two-parameter Todd genus**

$$Td_{a,b}\colon \Omega_U \longrightarrow \mathbb{Z}[a,b]$$

as exponential of the formal group law:

$$f(u,v) = \frac{u+v-auv}{1-buv}, \ \deg a = -2, \ \deg b = -4.$$

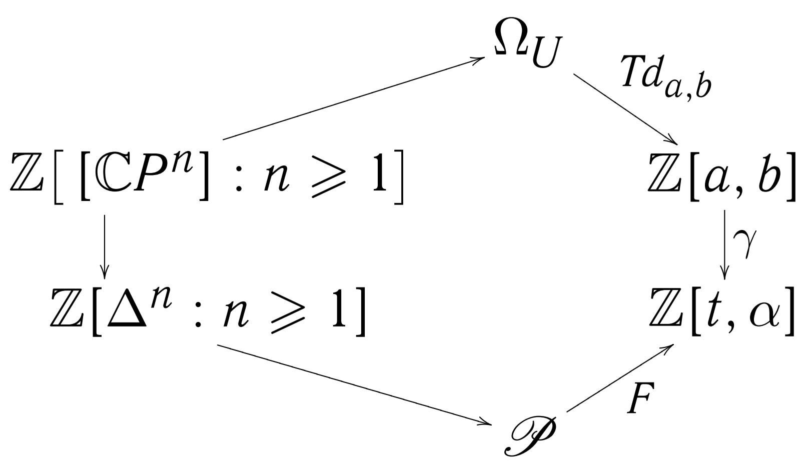
Consider the ring homomorphism

$$\gamma\colon \mathbb{Z}[a,b] \longrightarrow \mathbb{Z}[t,\alpha] \ : \ \gamma(a) = \alpha + 2t, \ \gamma(b) = \alpha t + t^2,$$

and $T_{t,\alpha} = \gamma Td_{a,b}$.

Lemma. $T_{t,\alpha}(s_1[M^{2n}]) = \frac{\partial}{\partial t} T_{t,\alpha}([M^{2n}])$.

The sending $[\mathbb{C}P^n]$ to Δ^n gives the commutative diagram



Let M^{2n} be a smooth symplectic manifold with an effective hamiltonian actions of a compact torus T^n and $\Phi(M) \subset \mathbb{R}^n$ be a convex polytope, where $\Phi\colon M^{2n} \to \mathbb{R}^n$ is a moment map.

Theorem. $T_{t,\alpha}[M^{2n}] = \gamma Td_{a,b}[M^{2n}] = F(\Phi(M^{2n}))$.

Corollary. $T_{t,\alpha}(S_1[M^{2n}]) = \frac{\partial}{\partial t}F(\Phi(M^{2n}))$.

The genus $T_{t,\alpha}[M^{2n}]$ is:
the n-th Chern number $c_n(M^{2n})$ for $\alpha = 0$,
the Todd genus $Td(M^{2n})$ for $t = 0$,
the L-genus (the signature) $\sigma(M^{2n})$ for $z = \alpha + 2t = 0$,
respectively.

Corollary.

$$c_n(M^{2n}) = f_{0,n}t^n,$$
$$Td(M^{2n}) = \alpha^n,$$
$$\sigma(M^{2n}) = (-1)^n[2^n - 2^{n-1}f_{n-1,1} + \cdots \cdots + (-1)^{n-1}2f_{1,n-1} + (-1)^n f_{0,n}]t^n.$$

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