On group actions on free Lie algebras

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On group actions on free Lie algebras

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Contents

Abstract

We first study the module structure of the free Lie algebra $L(V)$ in characteristic zero under the action of the general linear group. Here we give a new, purely combinatorial, proof of Klyachko's celebrated theorem on Lie representations using the Kraśkiewicz-Weyman theorem.

We then give a new factorisation of the Dynkin-Specht-Wever idempotent and use this to prove that $L_2(L_k(V))$ is a KG-module direct summand of $L_{2k}(V)$, for G an arbitrary group, K a field of characteristic $p \nmid k$ and V a KG-module. For finite-dimensional modules V, this follows immediately from the Decomposition Theorem of Bryant and Schocker. We consider a small example of this theorem, namely the sixth Lie power over a field of characteristic 3. Here we show explicitly that $L_6(V)$ decomposes into a direct sum of the modules $L_3(L_2(V))$ and $L_2(V) \otimes S_2(V) \otimes S_2(V)$, where $S_2(V)$ denotes the symmetric square of V. We give a description, up to isomorphism, of the modules $B_{p^m k}$ occurring in the Decomposition Theorem.

Finally, we apply our knowledge of Lie powers to a group theoretic problem. We show that the torsion subgroup t_c of the quotient $\gamma_c R/[\gamma_c R, F]$ is bounded as follows, for $c = 2p^m$ or $c = 3p^m$, where p is an arbitrary prime, $m \geq 0$:

$$2t_{2p^m} = 0 \quad \text{provided } G = F/R \text{ has no 2-torsion and no p-torsion,}$$

$$3t_{3p^m} = 0 \quad \text{provided } G = F/R \text{ has no 3-torsion and no p-torsion.}$$

Thus, we have that $\gamma_6 R/[\gamma_6 R, F]$ is torsion-free, provided that $G = F/R$ has no elements of order 2 or 3.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

We consider the problem of describing the module structure of the free Lie algebra $L(V)$ under the action of an arbitrary group. We use combinatorial methods to obtain information about the module structure of the free Lie algebra. As an application of our results we then use homological methods to translate this information into the language of free central extensions. We shall begin by formulating the decomposition problem and by giving a brief survey of the literature.

Let G be a group, K a field and V a KG-module. Let $L(V)$ denote the free Lie algebra on V over K, freely generated by any K-basis of V. Then the action of G extends uniquely to the whole of $L(V)$, where each element acts as an algebra automorphism. Moreover, since the action of G respects the degree of each homogeneous element, we have that each homogeneous component, $L_n(V)$, is also a KG-module. We want to be able to describe the module structure of $L(V)$ and its homogeneous components $L_n(V)$ - the so-called Lie powers of V. For the classical case of representations in characteristic zero, this problem is well understood via the pioneering works of R. M. Thrall [57], A. J. Brandt [4] and F. Wever [58]. In his 1942 paper, Thrall considered the case where G is the general linear group, K is a field of characteristic zero and V is the natural module. Each homogeneous component $L_n(V)$ is embedded in the nth tensor power $V^{\otimes n}$ and it is well known [49] that, in this case, $V^{\otimes n}$ can be written up to isomorphism as the KG-module direct sum

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} t_\lambda[\lambda],$$

where the sum runs over all partitions λ of n into at most $\dim V$ parts. Here $[\lambda]$ denotes the irreducible KG-module corresponding to λ and the multiplicity t_λ is equal to the number of standard tableaux of shape λ. Thrall asked what could be said about the nth Lie power $L_n(V)$. Since $V^{\otimes n}$ is a polynomial module (see [21] for further reference), it is semisimple and we must have that $L_n(V)$ is isomorphic to a KG-module direct summand of $V^{\otimes n}$. Hence

$$L_n(V) \cong \bigoplus_{\lambda \vdash n} l_\lambda[\lambda],$$

for some multiplicity l_λ, with $0 \leq l_\lambda \leq t_\lambda$. Thrall was able to calculate the multiplicities l_λ for all partitions λ of $n \leq 10$, with a correction in the case $n = 10$ later given by Brandt [4]. Brandt also gave a formula for the character of the nth Lie representation, which can be used, together with standard orthogonality relations, to calculate the multiplicities of the irreducible KG-module summands of $L_n(V)$, for G a finite group over a field K of characteristic zero. In 1949 Wever [58] gave a formula for the multiplicities l_λ in terms of characters of the symmetric group of degree n. Whilst Wever's multiplicity formula may be used to calculate the l_λ for particular values of n, it is rather cumbersome to use. For instance, it is not easy to see when $l_\lambda > 0$, that is, which irreducible modules actually occur in the Lie representations. In 1974 Alexander Klyachko [33] was able to answer this question. In the case where G is the general linear group, K is a field of characteristic zero and V is the natural module, he proved that for $n > 6$ almost every irreducible KG-module occurs in the Lie power $L_n(V)$, the exceptions being the irreducible modules corresponding to the partitions $\lambda = (n)$ and $\lambda = (1^n)$. Finally, in 1987 Kraśkiewicz and Weyman [36] gave a beautiful combinatorial description of the multiplicities l_λ. They proved that l_λ is equal to the number of standard tableaux of shape λ with major index congruent to a modulo n, for any fixed positive integer a which is coprime to n.

Since its publication in 1974, Klyachko's theorem has attracted the attention of a number of authors, most recently Schocker [47] and Kovács and Stöhr [35]. Whilst Klyachko's original proof was based on character theory, the arguments in [47] and [35] are mainly combinatorial. It has been a longstanding challenge to deduce Klyachko's theorem from the Kraśkiewicz-Weyman theorem. One of the main results of this thesis (and the topic of a recent paper of the author [31]) achieves exactly that. The proof is purely combinatorial and requires only a few basic definitions, along with one key observation. This result will be the main subject of Chapter 3. We also look at the relationship between the multiplicities l_λ and $l_{\lambda'}$, where λ' denotes the partition conjugate to λ. We show that these two multiplicities are equal whenever n is odd or divisible by 4.

For K a field of characteristic $p > 0$ the situation becomes more complicated, as our modules are no longer necessarily semisimple and, hence, the Lie power is not always isomorphic to a module direct summand of the tensor power. In fact, for n not divisible by p, it turns out that $L_n(V)$ is isomorphic to a direct summand of $V^{\otimes n}$. In this case we can exploit our knowledge of the tensor power, as we did in characteristic zero. In 1998 Donkin and Erdmann [17] described the indecomposable summands of $L_n(V)$ (up to isomorphism) for K an infinite field of characteristic p, G the general linear group, $G = GL(r, K)$, and V the natural r-dimensional KG-module, for all n not divisible by p. In this situation $V^{\otimes n}$ is a tilting module (see [16] and [17] for further reference) and its indecomposable summands are the (partial, polynomial) tilting modules $T(\lambda)$ corresponding to the p-regular partitions λ of n with at most r parts. Moreover, if $n \leq r$ the multiplicity of $T(\lambda)$ in $V^{\otimes n}$ is simply equal to $\beta_\lambda(1)$, where β_λ is the Brauer character of the irreducible $K\mathrm{Sym}(n)$-module corresponding to λ. Analogously to the classical characteristic zero case, Donkin and Erdmann give a formula for the multiplicity of $T(\lambda)$ inside $L_n(V)$ in terms of Brauer characters of the symmetric group of degree n. In addition to this description of $L_n(V)$ in terms of tilting modules, Bryant [5] has shown that, for $p \nmid n$, $L_n(V)$ can be described in terms of Adams operations on the Green ring, R_{KG}, for all K, G and V.

The case of degrees divisible by p largely remained a mystery for several years, with only a few results given for specific finite groups G with heavy restrictions upon V. Much work concentrated on the case where G is cyclic of order p over a field K of characteristic p. This problem was studied in a number of specific cases. Bryant and Stöhr [12], [13] considered the case where $p = 2$ and V is the regular module and also the case where p is an arbitrary prime and V is a free module. Guilfoyle and Stöhr [22] and Michos [44] looked at the case $p = 2$ with V an arbitrary finite-dimensional module. Finally Bryant, Kovács and Stöhr [9] considered the more general situation where V is an arbitrary finite-dimensional module and p is an arbitrary prime. They gave a recursive method for calculating the multiplicities of the indecomposable summands. A closed formula could only be obtained in certain cases. In 2004, Bryant [6] gave a general formula for these multiplicities. A number of other papers [8], [34], focused upon the case where G is taken to be a specific symmetric or general linear group.

Recently their have been major developments in this area, with papers by Bryant and Stöhr [14], Erdmann and Schocker [19] and Bryant and Schocker [10], [11]. From now on let K be a field of characteristic $p > 0$, let G be a group and let V be a finite-dimensional KG-module. The first case of difficulty, namely $n = p$, was dealt with in [14] by Bryant and Stöhr via an analysis of the structure of $V^{\otimes p}$. They proved that $L_p(V) \cong B_p(V) \oplus M_p(V)$, where $B_p(V) = L''(V) \cap L_p(V)$ and $M_p(V)$ is the pth metabelian Lie power of V. In the case where K is an infinite field, G is the general linear group and V is the natural KG-module, they calculated the indecomposable summands of $B_p(V)$, along with their multiplicities. Since $M_p(V)$ is also known to be indecomposable, this gives full information for $L_p(V)$ in this case. Additionally, Bryant and Stöhr give a description of $L_p(V)$ in terms of Adams operations on R_{KG}, for all K, G and V.

Erdmann and Schocker [19] next considered the case $n = pk$, where $p \nmid k$, and were able to prove that the study of $L_{pk}(V)$ can be reduced to the study of $L_p(L_k(V))$. Their methods make use of the Solomon descent algebra - a certain subring of the symmetric group ring (see [48] for further reference). Finally, Bryant and Schocker [10] were able to develop this idea much further, proving that the study of arbitrary Lie powers can, in some sense, be reduced to the study of the Lie powers of p-power degree. Their main result, the 'Decomposition Theorem', states that for all $k > 0$, with $p \nmid k$, and for all $m \geq 0$, there exists a KG-module direct sum decomposition

$$L_{p^m k}(V) = L_{p^m}(B_k) \oplus L_{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L_1(B_{p^m k}),$$

for some submodules $B_{p^i k}$ of $L_{p^i k}(V)$, which are in turn isomorphic to KG-module direct summands of $V^{\otimes p^i k}$. Whilst the Decomposition Theorem represents a huge step forwards in our knowledge of modular Lie powers, the theorem offers no description of the mysterious modules $B_{p^i k}$. A further paper of Bryant and Schocker [11] gives a recursive description of these modules up to isomorphism.

In Chapter 5 we concentrate on the case $n = pk$, where $p \nmid k$, for certain values of p and k. For $p = 2$, k odd, we prove that $L_2(L_k(V))$ is a direct summand of $L_{2k}(V)$. Our proof is completely independent of the methods employed in [10]; we give an explicit projection of $L_{2k}(V)$ onto $L_2(L_k(V))$. The proof is combinatorial in nature and utilises an alternative factorisation of the Dynkin-Specht-Wever idempotent, as described in Chapter 4. In particular, our proof shows that $L_2(L_k(V))$ is a direct summand of $L_{2k}(V)$ for arbitrary KG-modules V - not just finite-dimensional modules. In fact, Bryant has recently generalised the arguments in [10] and [11] to give a decomposition theorem and recursive description (up to isomorphism) of the modules $B_{p^i k}$ occurring in this theorem for arbitrary modules V. We reproduce an early draft of his manuscript for reference purposes in Appendix B, with kind permission.

For G a group, K a field of characteristic 3 and V an arbitrary KG-module, we give an explicit decomposition

$$L_6(V) \cong L_3(L_2(V)) \oplus [L_2(V) \otimes S_2(V) \otimes S_2(V)],$$

where $S_2(V)$ denotes the second symmetric power of V. We describe a projection of $L_6(V)$ onto $L_3(L_2(V))$ and use this to give a description of a complement, $B_6^{(3)}$ (here the superscript is to remind us of the characteristic of K). We then describe a surjection of $L_6(V)$ onto $L_2(V) \otimes S_2(V) \otimes S_2(V)$, with kernel $L_3(L_2(V))$, thus giving that $B_6^{(3)} \cong L_2(V) \otimes S_2(V) \otimes S_2(V)$. This isomorphism gives us a description of the isomorphism type of $B_6^{(3)}$, and it is easy to see from this description that $B_6^{(3)}$ occurs as a direct summand of $V^{\otimes 6}$. We not only show that the decomposition holds, but find embeddings and projections to prove the decomposition explicitly.

In a similar manner, for G a group, K a field of characteristic 2 and V an arbitrary KG-module, we give an explicit projection of $L_6(V)$ onto $L_2(L_3(V))$ and use this to give a description of a complement, $B_6^{(2)}$. We also conjecture that $B_6^{(2)}$ is isomorphic to the tensor product $\text{Ł}_3(V) \otimes A_3(V)$, where $A_3(V)$ is a certain KG-module direct summand of the third tensor power.

We now return to the situation where V is finite-dimensional. In Chapter 6 we address the problem of describing the modules B_{p^ik}, up to isomorphism. The majority of this chapter represents joint work with R. M. Bryant [7]. We prove that each module B_{p^ik} can be written, up to isomorphism, as a direct sum of tensor products of certain direct summands $U_{k,d}$ of $V^{\otimes k}$, indexed by the divisors d of k. We also prove a sharper result; the modules B_{p^ik} can be written, up to isomorphism, as a direct sum of tensor products of a smaller collection of direct summands $W_{k,c}$ of $V^{\otimes k}$, indexed by the divisors c of k with c coprime to k/c. Our proofs make use of a number of results on Witt vectors (see [50] for reference). We consider a few examples of our result, in particular, the examples of Chapter 5, in the case where V is a finite-dimensional KG-module. The main results of Chapter 6 can in fact be generalised to the case where V is a module of arbitrary dimension. This has recently been shown by R. M. Bryant and we reproduce his arguments with kind permission in Appendix B. We conclude Chapter 6 with a discussion of these developments.

Finally, we apply our knowledge of Lie powers in characteristic p to give further insight into another problem, involving torsion in free central extensions of groups. In 1973 Kanta Gupta [23] discovered elements of finite order in free centre–by–metabelian groups. More precisely, she proved that the relatively free group $F/[F'', F]$ (where F is a free group of rank $d \geq 4$) contains an elementary abelian 2-group of rank $\binom{d}{4}$ in its centre. This was the first result to remark upon an, at the time, surprising phenomenon: the appearance of torsion in free central extensions of certain torsion-free groups. Gupta's proof was purely group-theoretical and consists of several pages of intricate commutator calculations. In his pioneering 1977 paper [37], Yu.V. Kuz'min introduced homological methods into the study of Gupta's torsion elements, and, subsequently, was able to identify the torsion subgroup of $F/[F'', F]$ with $H_4(F/F') \otimes \mathbb{Z}_2$, the fourth homology group of the free abelian group F/F' reduced modulo 2. The unexpected presence of torsion and the connection with group homology created considerable interest in such groups.

More generally, for G a group given by free presentation $G = F/R$, we have that the quotient $F/[\gamma_c R, F]$, where $c \geq 2$ and $\gamma_c R$ denotes the cth term of the lower central series of R, is a free central extension of $F/\gamma_c R$. The quotient $F/\gamma_c R$ is in turn an extension of $G = F/R$ with free nilpotent kernel $R/\gamma_c R$. Whilst $F/\gamma_c R$ is always torsion free [52], Gupta's result shows that elements of finite order can occur in the centre of $F/[\gamma_c R, F]$, that is, in the quotient $\gamma_c R/[\gamma_c R, F]$. The problem is then to determine the torsion subgroup of $\gamma_c R/[\gamma_c R, F]$. We shall see later that this problem is connected to Lie powers via the following isomorphism,

$$\gamma_c R/[\gamma_c R, F] \cong L_c(R_{ab}) \otimes_G \mathbb{Z},$$

where $L_c(R_{ab})$ denotes the cth homogeneous component of the free Lie ring $L(R_{ab})$ (that is, the free Lie algebra over $\mathbb{Z}$) and where R_{ab} denotes the relation module R/R' stemming from our original free presentation.

In Chapter 7, we shall use our results on Lie powers to approach the problem of determining the torsion subgroup t_c of $\gamma_c R/[\gamma_c R, F]$. Kuz'min [38] ($c = 2$) and Stöhr [54] ($c \geq 2$) have shown that the exponent of the torsion subgroup t_c is bounded as follows:

$$4t_2 = 0 \text{ and } ct_c = 0, \text{ for } c \geq 3.$$

In the case $c = 2$, this bound cannot be improved - Kuz'min [38] gives an example of a normal subgroup R such that the quotient $R'/[R', F]$ contains an element of order 4.

Stöhr [54], [55] was also able to give precise identification of the torsion subgroup for $c = p$, a prime, and $c = 4$ under mild conditions on G. If G has no elements of order $d|c$, for $d > 1$, then we have

$$t_c = \begin{cases} H_4(G, \mathbb{Z}_p), & \text{for } c = p, \text{ a prime,} \\ H_6(G, \mathbb{Z}_2), & \text{for } c = 4. \end{cases}$$

As far as the concrete identification of the torsion subgroup is concerned, this was the last word on the subject for some time. The methods available at the time were exhausted. It has been made possible to resume progress on this problem by recent developments in the theory of modular Lie powers, in particular, the Decomposition Theorem of Bryant and Schocker. The main result of Chapter 7 is that for $G = F/R$ with no elements of order 2 or 3, the quotient $\gamma_6 R/[\gamma_6 R, F]$ is torsion-free. In fact, we prove shall that

$$2t_{2p^m} = 0 \quad \text{for all } m \geq 0, \text{ provided } G \text{ has no 2-torsion and no p-torsion,}$$

$$3t_{3p^m} = 0 \quad \text{for all } m \geq 0, \text{ provided } G \text{ has no 3-torsion and no p-torsion.}$$

It is easy to see that these two results coincide when $c = 6$, hence giving the fact that $t_6 = 0$, that is, $\gamma_6 R/[\gamma_6 R, F]$ is torsion-free. Hence, if G has no elements of order $d|c$, for $d > 1$, we have

$$t_c = \begin{cases} H_4(G, \mathbb{Z}_p), & \text{for } c = p, \text{ a prime,} \\ H_6(G, \mathbb{Z}_2), & \text{for } c = 4, \\ 0, & \text{for } c = 6. \end{cases}$$

The thesis is organised as follows: in Chapter 2 we give the preliminary definitions required to proceed. In Chapter 3 we consider the case where K is a field of characteristic zero, G is the general linear group and V is the natural module. There we present some combinatorial proofs of well-known results. In Chapter 4 we give an alternative factorisation of the Dynkin-Specht-Wever idempotent. This factorisation is then used in Chapter 5 to prove that $L_2(L_k(V))$ is a direct summand of $L_{2k}(V)$ for all odd k, where K is a field of characteristic 2. We use certain module homomorphisms to prove some of the decompositions asserted by the Decomposition Theorem and to give new information about the modules B_{pk}, for certain values of p and k. In Chapter 6 we describe, up to isomorphism, the modules $B_{p^m k}$ of the Decomposition Theorem. Here we do not need to construct explicit maps, but instead perform calculations in the Green ring and use techniques developed for dealing with Witt vectors. Finally, in Chapter 7, we apply our knowledge of the modular Lie powers to a group theoretic problem to obtain some surprising results.

Chapter 2

Preliminaries

2.1 Free Lie algebras

Let K be a commutative ring with identity element and let L be a K-module. We say that L is a K-algebra if there is a given bilinear map $L \times L \to L$, called the multiplication of L. A Lie algebra L over K is a K-algebra with multiplication denoted by $(u, v) \mapsto [u, v]$, for all $u, v \in L$, satisfying

(i) $[u, u] = 0, \quad \forall u \in L,$

(ii) $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0, \quad \forall u, v, w \in L.$

Condition (i) implies that $[u, v] = -[v, u]$ for all $u, v \in L$ (anticommutativity). The second condition is called the Jacobi identity. We call a multiplication satisfying both (i) and (ii) a Lie multiplication.

Let L be a Lie algebra over K and let X be a subset of L. We say that L is free on X if every map from X into another Lie algebra L' over K extends uniquely to a Lie algebra homomorphism of L to L'. We note that if L is free on X then L is generated by X. For every set X there exists a free Lie algebra L which is free on X and this algebra is unique up to isomorphism. Thus, we shall speak of the free Lie algebra on X and denote this algebra by $L(X)$.

We define the elements of X to be Lie monomials and define a product of the form $[u, v] \in L(X)$ to be a Lie monomial if both u and v are Lie monomials. We shall use the left-normed convention in writing Lie products as follows:

$$[u_1, u_2, u_3, \dots, u_n] = [\dots [[u_1, u_2], u_3], \dots, u_n],$$

for all $u_1, \dots, u_n \in L(X)$. If each u_i is in fact an element of X we say that $[u_1, \dots, u_n]$ is a left-normed monomial. It is easy to see that $L(X)$ is spanned by the left-normed monomials. Indeed, by the Jacobi identity, we have that

$$[u_1, \dots, [u_i, u_{i+1}], \dots, u_n] = [u_1, \dots, u_i, u_{i+1}, \dots, u_n] - [u_1, \dots, u_{i+1}, u_i, \dots, u_n],$$

for all $u_1, \dots, u_n \in L(X)$, $1 < i < n$. However, the left-normed monomials are not linearly independent. It is a curious fact that there is no known basis of $L(X)$ consisting entirely of left-normed monomials.

A basis of $L(X)$ can be constructed in the following manner. We define the standard monomials of degree 1 to be the elements of X. Suppose that we have defined the standard monomials of degrees $1, \dots, n$ and that they are totally ordered in such a way that for any two standard monomials u and v, with $\deg(u) > \deg(v)$, we have $u > v$. We say that $[u, v]$ is a standard monomial of degree $n + 1$ if we have (i) $\deg(u) + \deg(v) = n + 1$, (ii) u and v are standard with $u > v$ and (iii) if u can be written as a product $u = [z, w]$, where z and w are standard, then $v \geq w$. The standard monomials form a basis of $L(X)$ called a Hall basis [24]. Thus, $L(X)$ is a free module over K.

If $|X| = r$, we say that $L(X)$ is the free Lie algebra of rank r. In this case each homogeneous component $L_n(X)$ is a free K-module of finite rank, given by Witt's formula:

$$\text{rank } L_n(X) = \frac{1}{n} \sum_{d|n} \mu(d) r^{n/d},$$

where μ denotes the Möbius function,

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1 \\ 0 & \text{if } p^2 | d \text{ , for some prime } p, \\ (-1)^k & \text{otherwise, where } k \text{ is the number of distinct prime divisors of d.} \end{cases}$$

Further reference on Lie algebras can be found in [29] and [51] - a section on free Lie algebras can be found in each [29, Chapter V, §4], [51, Chapter IV]. For further information about free Lie algebras we recommend [45] and also [1, Chapter 2]. (These also serve as a good reference for the decomposition problem for $L(V)$; see [45, Chapter 8] and [1, Chapter 3]).)

Let $W(X)$ denote the free monoid on X. Then the elements of $W(X)$ are simply strings of the form $x_1x_2\cdots x_n$, where $x_i \in X$ for $i = 1, \ldots, n$, called words on X. Let $A(X)$ be the free K-module with basis $W(X)$ (the elements of $A(X)$ are K-linear combinations of words from $W(X)$). We may extend the multiplication of $W(X)$ to the whole of $A(X)$ distributively. Then $A(X)$ is the free associative algebra on X.

We may turn any associative algebra A into a Lie algebra by defining a new multiplication on A as follows:

$$(u, v) \mapsto [u, v] = uv - vu,$$

where uv denotes the usual associative product of u and v in A. It is easy to verify that this is a Lie multiplication. Now, let us turn $A(X)$ into a Lie algebra in this way. Let L be the Lie subalgebra of $A(X)$ generated by X. Then L is isomorphic to the free Lie algebra on X, that is, L is generated by X and any map from X into a Lie algebra L' over K extends uniquely to a homomorphism. We therefore consider the free Lie algebra $L(X)$ to be embedded in $A(X)$. In fact, $A(X)$ is the universal enveloping algebra of $L(X)$ (see [51] for further reference). When $K = \mathbb{Z}$, we call $L(X)$ the free Lie ring on X and $A(X)$ the free associative ring on X.

The algebras $L(X)$ and $A(X)$ are graded. For $n \geq 1$ let $L_n(X)$ be the nth homogeneous component of $L(X)$, that is, the K-submodule of $L(X)$ spanned by all left-normed monomials of length n. Then

$$L(X) = \bigoplus_{n \geq 1} L_n(X).$$

If $u \in L_n(X)$ we say that u is homogeneous of degree n and write $\deg(u) = n$. Moreover, we can refine this grading further by considering the multiplicity with which each $x \in X$ occurs within a monomial of $L(X)$. Let $\alpha : X \to \mathbb{N}_0$ and define $L_\alpha(X)$ to be the submodule of $L(X)$ spanned by all left-normed monomials $u \in L(X)$ such that x appears in u with multiplicity $\alpha(x)$ for all $x \in X$. Then

$$L(X) = \bigoplus L_\alpha(X),$$

where the sum ranges over all functions $\alpha : X \to \mathbb{N}_0$. We call $L_\alpha(X)$ the fine homogeneous component of $L(X)$ of multidegree α and if $u \in L_\alpha(X)$ we say that u is fine homogeneous of multidegree α.

Similarly, for $n \geq 0$ let $A_n(X)$ denote the nth homogeneous component of $A(X)$, that is, the K-submodule of $A(X)$ spanned by all words in $W(X)$ of length n. Then

$$A(X) = \bigoplus_{n \geq 0} A_n(X).$$

If $u \in A_n(X)$ we say that u is homogeneous of degree n and write $\deg(u) = n$. We also define $A_\alpha(X)$ to be the submodule of $A(X)$ spanned by all words $w \in W(X)$ such that x appears in w with multiplicity $\alpha(x)$ for all $x \in X$. Then

$$A(X) = \bigoplus A_\alpha(X),$$

where the sum ranges over all functions $\alpha : X \to \mathbb{N}_0$. We call $A_\alpha(X)$ the fine homogeneous component of $A(X)$ of multidegree α and if $u \in A_\alpha(X)$ we say that u is fine homogeneous of multidegree α.

Let $X = \{x_1, \ldots, x_r, \ldots\}$. It is easy to see that each of the homogeneous components $L_n(X)$ and $A_n(X)$ can be written as a direct sum of fine homogeneous components, $L_\alpha(X)$ and $A_\alpha(X)$, respectively:

$$\begin{aligned} L_n(X) &= \bigoplus L_\alpha(X), \\ A_n(X) &= \bigoplus A_\alpha(X), \end{aligned}$$

where each sum ranges over all functions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r, \ldots) : X \to \mathbb{N}_0$ given by $\alpha : x_i \mapsto \alpha_i$, with each $\alpha_i \geq 0$ and $\sum_{i\geq 1} \alpha_i = n$.

2.2 Lie, tensor and symmetric powers

Let $\langle X \rangle$ denote the K-span of the set X. Thus, $\langle X \rangle$ is a free K-module. We first remark that if B is a basis of $\langle X \rangle$, then B is a free generating set for $L(X)$. Now, let V be a free K-module. From now on we shall write, with a slight abuse of notation, the free Lie algebra on V or $L(V)$ to mean the free Lie algebra which has free generating set given by any basis of V. We shall also write $L_n(V)$ to denote the nth homogeneous component of this free Lie algebra.

Similarly, we may write the 'free associative algebra on V', however, it is useful to make the following identifications:

$$A_n(V) = V^{\otimes n} \quad \text{for } n \geq 1,$$

where each product $v_1 \cdots v_n \in A(V)$, with $v_1, \ldots, v_n \in V$, is identified with the element $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$. Hence, we shall think of the free Lie algebra $L(V)$ as embedded inside the tensor algebra $T(V) = \bigoplus_{n\geq 0} T_n(V)$, where $T_0 = K$ and $T_n(V) = V^{\otimes n}$ for $n \geq 1$.

Let $G \leq \mathrm{GL}(V)$ be a subgroup of the automorphism group of V. Then V is a right KG-module. The action of G on V may be extended uniquely to $L(V)$ and $T(V)$, where each element $g \in G$ acts by algebra automorphisms. Thus, $L(V)$ is isomorphic to a KG-submodule of $T(V)$, via the above embedding. Moreover, it is easy to see that each homogeneous component $L_n(V)$ is a KG-submodule of $L(V)$ and each $V^{\otimes n}$

is a KG-submodule of $T(V)$. We call the modules $L_n(V)$ the Lie powers of V and the modules $V^{\otimes n}$, the tensor powers of V.

Let $S(V)$ denote the symmetric algebra on V over K. The elements of $S(V)$ may be thought of as polynomials over K in indeterminates from any K-basis of V. Then $S(V)$ is a KG-module, where each element $g \in G$ acts as an algebra automorphism. Moreover, $S(V)$ is a graded algebra,

$$S(V) = \bigoplus_{n \geq 0} S_n(V),$$

where each $S_n(V)$ is the K-submodule of $S(V)$ spanned by the monomials of degree n. Each $S_n(V)$ is a KG-submodule of $S(V)$, called the nth symmetric power of V. For a free K-module V of finite rank r, each symmetric power $S_n(V)$ is of finite rank $\binom{r+n+1}{n}$.

Chapter 3

Standard tableaux and Lie powers

In this chapter we show that for all but two partitions λ of $n > 6$ there exists a standard tableau of shape λ with major index coprime to n. This result appears as Theorem 1 in a recent paper of the author [31].

3.1 Introduction

The main result of this chapter is the following theorem.

Theorem 3.1.1. *Let $n \geq 3$ and let $\lambda \vdash n$. There exists a standard tableau of shape λ with major index coprime to n if and only if $\lambda \neq (1^n), (n), (2^2)$ or (2^3).*

The significance of the theorem lies in the fact that when combined with a deep result of Kraśkiewicz and Weyman, it implies a celebrated theorem of Klyachko on Lie representations of the general linear group. Let V be a finite-dimensional vector space over a field K of characteristic zero. Each tensor power $V^{\otimes n}$ is a semisimple module for the general linear group $\mathrm{GL}(V)$, with the isomorphism types of the irreducible submodules of $V^{\otimes n}$ parameterised by the partitions of n with at most $\dim(V)$ parts. Since each Lie power $L_n(V)$ is isomorphic to a $K\mathrm{GL}(V)$-submodule of $V^{\otimes n}$, we have that the isomorphism types of the irreducible $K\mathrm{GL}(V)$-submodules of $L_n(V)$ form a subset of those occurring in $V^{\otimes n}$. Klyachko's theorem tells us that almost all of the irreducible modules occurring in $V^{\otimes n}$ also appear (up to isomorphism) in the nth Lie representation, $L_n(V)$.

Theorem 3.1.2. (Klyachko [33]). *Let $n \geq 3$ and let $\lambda \vdash n$. There exists an irreducible $K\mathrm{GL}(V)$-submodule of $L_n(V)$ with isomorphism type corresponding to λ if and only if λ has no more than* $\dim(V)$ *parts and $\lambda \neq (1^n), (n), (2^2)$ or (2^3).*

(We follow the convention that the $K\mathrm{GL}(V)$-module corresponding to a partition with more than $\dim V$ parts is 0, and hence there is no such irreducible module.)

Since publication in 1974, Klyachko's theorem has attracted the attention of a number of authors. Recently new proofs have been given by Schocker [47] and Kovács and Stöhr [35]. While Klyachko's original proof was based on character theory, the arguments in [35] and [47] are mainly combinatorial.

In 1987, Kraśkiewicz and Weyman gave a beautiful combinatorial interpretation of the multiplicities of the irreducible $K\mathrm{GL}(V)$-modules occurring in $L_n(V)$.

Theorem 3.1.3. (Kraśkiewicz–Weyman [36]). *Let $a, n \in \mathbb{N}$ be fixed coprime numbers and let λ be a partition of n with at most* $\dim(V)$ *parts. The irreducible $K\mathrm{GL}(V)$-module corresponding to λ occurs in $L_n(V)$ with multiplicity equal to the number of standard tableaux of shape λ with major index congruent to a modulo n.*

New proofs of this result have been given by Garsia [20] and Jöllenbeck and Schocker [32] (for another account of Garsia's proof see [45, Theorems 8.8 and 8.9]). It has been a longstanding challenge to deduce Klyachko's theorem directly from Theorem 3.1.3. As Schocker [47] remarks "it seems to be rather difficult to give a combinatorial proof ... by some analysis of the tableaux numbers ... and the Kraśkiewicz-Weyman theorem only". Exactly that, however, is achieved in this chapter, as our Theorem 3.1.1 and Theorem 3.1.3 clearly imply Theorem 3.1.2.

3.2 Standard tableaux, descents and major index

Let n and l be positive integers with $l \leq n$. We say that a finite sequence of positive integers λ, of the form $\lambda = (\lambda_1, \ldots, \lambda_l)$, is a composition of n ($\lambda \vDash n$) if $\lambda_1 + \cdots + \lambda_l = n$. We say that λ is a partition of n ($\lambda \vdash n$) if, additionally, $\lambda_1 \geq \cdots \geq \lambda_l > 0$. We call the components λ_i the parts of λ and write $|\lambda|$ to denote the sum of all parts, $\lambda_1 + \cdots + \lambda_l = n$. For a partition λ of n we define the Young diagram of shape λ to be a collection of n boxes arranged in left-justified rows, with λ_i boxes in the ith row.

Let λ be a partition of n ($\lambda \vdash n$). A tableau of shape λ is a numbering of the Young diagram of λ with the numbers from $\{1, \ldots, k\}$, where $k \leq n$, such that the entries weakly increase along each row and strictly increase down each column. We say that a tableau is standard if each number in $\{1, \ldots, n\}$ occurs exactly once. An entry i in a standard tableau is called a descent if $i + 1$ occurs in any row below i. For a standard tableau T we denote the set of all descents in T by $D(T)$ and define the sum of all descents to be the major index of T, $\mathrm{maj}(T)$.

For example, let $\lambda = (5, 3, 2, 1)$. Shown below is the Young diagram of shape λ, together with a tableau of shape λ and a standard tableau of shape λ.

A Young diagram

1	1	1	2	4
2	2	3		
3	4			
5				

A tableau

1	**2**	**4**	8	**9**
3	**5**	11		
6	10			
7				

A standard tableau, T

For the standard tableau T given above notice that the descents are shown in bold font. Hence, we have

$$\begin{aligned} D(T) &= \{2, 4, 5, 6, 9\}, \\ \text{and } \mathrm{maj}(T) &= 2 + 4 + 5 + 6 + 9 = 26. \end{aligned}$$

For partitions with many equal parts it is often convenient to use the following notation. Let $(a_1^{\alpha_1}, \ldots, a_m^{\alpha_m})$ denote the partition $(\underbrace{a_1, \ldots, a_1}_{\alpha_1}, \ldots, \underbrace{a_m, \ldots, a_m}_{\alpha_m})$ consisting of α_i parts of length a_i, for $1 \leq i \leq m$, with $a_1 < \cdots < a_m$.

From now on in our examples descents shall appear in bold font. We refer to [42] and [45] for further reference on partitions and tableaux and to [30] for further reference on the representation theory of the symmetric and general linear groups.

3.3 A combinatorial proof of Klyachko's theorem

We now come to a proof of Theorem 3.1.1. It is easy to see that for $n \geq 3$ there are no standard tableaux of shape $\lambda = (1^n)$, (n), (2^2) or (2^3) with major index coprime to $|\lambda|$. Indeed, if $\lambda = (1^n)$, the only standard tableau T of shape λ is given by

$$T = \begin{array}{|c|}\hline \mathbf{1} \\ \hline \mathbf{2} \\ \hline \vdots \\ \hline n \\ \hline \end{array}$$

where $\mathrm{maj}(T) = 1 + \cdots + (n-1) = \frac{n(n-1)}{2}$. Hence, we have that

$$\mathrm{maj}(T) \equiv \begin{cases} 0 \bmod n, & \text{if } n \text{ is odd,} \\ n/2 \bmod n, & \text{if } n \text{ is even.} \end{cases}$$

Thus we see that the major index of T is not coprime to n, for all $n \geq 3$. Similarly, if $\lambda = (n)$, the only standard tableau T of shape λ is given by

$$T = \begin{array}{|c|c|c|}\hline 1 & \cdots & n \\ \hline \end{array},$$

where $\mathrm{maj}(T) = 0$, which is not coprime to n. The two standard tableaux of shape (2^2),

$$\begin{array}{|c|c|}\hline 1 & \mathbf{2} \\ \hline 3 & 4 \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{3} \\ \hline 2 & 4 \\ \hline\end{array},$$

have major indices 2 and 4 respectively. Finally, the five standard tableaux of shape (2^3),

$$\begin{array}{|c|c|}\hline 1 & \mathbf{2} \\ \hline 3 & \mathbf{4} \\ \hline 5 & 6 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & \mathbf{2} \\ \hline \mathbf{3} & \mathbf{5} \\ \hline 4 & 6 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{3} \\ \hline 2 & \mathbf{5} \\ \hline 4 & 6 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{3} \\ \hline 2 & \mathbf{4} \\ \hline 5 & 6 \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{4} \\ \hline \mathbf{2} & \mathbf{5} \\ \hline 3 & 6 \\ \hline\end{array},$$

have major indices 6, 10, 9, 8 and 12 respectively - none of which are coprime to 6.

For all other partitions λ of $n \geq 3$ we shall show that there exists a standard tableau of shape λ with major index coprime to n. We first consider partitions λ of n into two parts. Next we look at rectangular partitions with more than two parts. Finally, we prove that any non-rectangular partition of n into more than two parts has a standard tableau with major index coprime to n.

3.3.1 Partitions with two parts

Let $n \geq 3$. For each two part partition $\lambda = (n-s, s)$ of n, where $1 \leq s \leq [n/2]$ and $\lambda \neq (2^2)$, we define a standard tableau $T(\lambda)$ with major index coprime to n.

For n odd, $n = 2m+1$ and $1 \leq s \leq m$, we define

$$T(n-s,s) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & \cdots & s & \cdots & \mathbf{m} & m+s+1 & \cdots & 2m+1 \\ \hline m+1 & \cdots & m+s \\ \cline{1-3}\end{array}$$

Then $\mathrm{maj}(T(n-s,s)) = m$ which is coprime to n.

Now let n be even, $n = 2m$ and $1 \leq s \leq m$. For $s = 1$ we define

$$T(n-1,1) = \begin{array}{|c|c|c|c|}\hline \mathbf{1} & 3 & \cdots & 2m \\ \hline 2 \\ \cline{1-1}\end{array}$$

and for $1 < s < m-1$ we define $T(n-s, s)$ to be the standard tableau given by

1	2	$\cdots$	s	$\cdots$	$\mathbf{m-1}$	$m+1$	$\mathbf{m+2}$	$m+s+2$	$\cdots$	$2m$
m	$m+3$	$\cdots$	$m+s+1$							

When $s = m-1$ we put

$T(m+1, m-1)=$

1	2	$\cdots$	$\mathbf{m-1}$	$m+1$	$\mathbf{m+2}$
m	$m+3$	$\cdots$	$2m$		

and, finally, for $s = m$, where $m > 2$, we set

$T(m^2)=$

1	2	3	$\cdots$	$\mathbf{m-1}$	$\mathbf{m+2}$
m	$m+1$	$m+3$	$\cdots$	$2m-1$	$2m$

Hence, we have

$$\operatorname{maj}(T(n-s, s)) = \begin{cases} 1 & \text{if } s = 1, \\ 2m+1 & \text{if } 1 < s \leq m, \end{cases}$$

which is coprime to n for all values of s.

3.3.2 Rectangular partitions with more than two parts

We consider rectangular partitions $\lambda = (m^k) \vdash n = mk$, where $k > 2$, $m > 1$ and $\lambda \neq (2^3)$. First suppose that $m = 2$ and let

$T(2^k; j) =$

1	$\mathbf{2}$
$\vdots$	$\vdots$
$2j-1$	$\mathbf{2j}$
$\mathbf{2j+1}$	$\mathbf{2j+3}$
$2j+2$	$\mathbf{2j+5}$
$2j+4$	$\mathbf{2j+6}$
$\vdots$	$\vdots$
$2k-1$	$2k$

where $1 \leq j \leq k-3$ and $k > 3$. Then $T(2^k; j)$ is a standard tableau with descent set given by

$$\begin{array}{lll} & \{2, \ldots, 2j, 2j+1, 2j+3, 2j+5, 2j+6, \ldots, 2k-2\}, & \text{for } 1 \leq j < k-3, \\ \text{and} & \{2, \ldots, 2k-6, 2k-5, 2k-3, 2k-1\}, & \text{for } j = k-3, \end{array}$$

and it is easy to see that

$$\operatorname{maj}(T(2^k; j)) \;=\; 2j + 3 + 2\sum_{i=1}^{k-1} i = 2j + 3 + k(k-1),$$

for all $1 \leq j \leq k-3$.

If k is odd then $k-1$ is even and, since $k > 3$, we have that $(k-1)/2$ is an integer in the range $1 < (k-1)/2 \leq k-3$. Hence

$$\operatorname{maj}(T(2^k; \frac{k-1}{2})) \;=\; k + 2 + k(k-1) \equiv k + 2 \mod n,$$

since $n = 2k$ and $k-1$ is even. Thus, we have that $\operatorname{maj}(T(2^k; (k-1)/2)$ is coprime to n.

Similarly, if k is even then, since $k > 3$, we have that $(k-2)/2$ is an integer in the range $1 \leq (k-2)/2 \leq k-3$. Hence,

$$\operatorname{maj}(T(2^k; \frac{k-2}{2})) \;=\; k + 1 + k(k-1) = k^2 + 1 \equiv 1 \mod n,$$

giving that $\operatorname{maj}(T(2^k; (k-2)/2)$ is coprime to n. This proves Theorem 3.1.1 for all partitions $\lambda = (2^k)$, where $k > 3$.

Next suppose that $\lambda = (m^k)$ where $m, k > 2$ and let $T(m^k; i, s)$ denote the standard tableau given by

<table>
<tr><td>1</td><td colspan="5">$\cdots$</td><td>$\mathbf{m}$</td></tr>
<tr><td>$\vdots$</td><td colspan="5"></td><td>$\vdots$</td></tr>
<tr><td>$(i-1)m+1$</td><td colspan="5">$\cdots$</td><td>$\mathbf{im}$</td></tr>
<tr><td>$im+1$</td><td>$im+2$</td><td>$\cdots$</td><td>$\mathbf{im+s}$</td><td>$im+s+2$</td><td>$\cdots$</td><td>$\mathbf{(i+1)m+1}$</td></tr>
<tr><td>$im+s+1$</td><td>$(i+1)m+2$</td><td colspan="4">$\cdots$</td><td>$\mathbf{(i+2)m}$</td></tr>
<tr><td>$\vdots$</td><td colspan="5"></td><td>$\vdots$</td></tr>
<tr><td>$(k-1)m+1$</td><td colspan="5">$\cdots$</td><td>km</td></tr>
</table>

where $0 \leq i \leq k-2$, $1 \leq s \leq m-1$. Then $T(m^k; i, s)$ has descent set given by

$$\begin{array}{lll} & \{s, m+1, 2m, \ldots, (k-1)m\}, & \text{for } i = 0, \\ & \{m, \ldots, im, im+s, (i+1)m+1, (i+2)m, \ldots, (k-1)m\}, & \text{for } 0 < i < k-2, \\ \text{and} & \{m, \ldots, (k-2)m, (k-2)m+s, (k-1)m+1\}, & \text{for } i = k-2, \end{array}$$

which gives

$$\begin{aligned} \operatorname{maj}(T(m^k; i, s)) &= m \sum_{j=1}^{k-1} j + im + s + 1 \\ &= \frac{mk(k-1)}{2} + im + s + 1, \end{aligned}$$

for all $0 \leq i \leq k-2$.

If n is odd, then k must be odd and it follows that

$$\operatorname{maj}(T(m^k; i, s)) \equiv im + s + 1 \bmod n.$$

Hence, choosing $i = 0$, $s = 1$ gives $\operatorname{maj}(T(m^k; 0, 1)) \equiv 2 \bmod n$, which is coprime to n.

If n is even, we consider two cases:

Case (i). If k is odd then $\operatorname{maj}(T(m^k; i, s)) \equiv im + s + 1 \bmod n$ and, since n is even, we have that m must be even.

If $4|m$, choosing $i = (k-1)/2$ and $s = m/2$ gives

$$\begin{aligned} \operatorname{maj}\left(T\left(m^k; \frac{k-1}{2}, \frac{m}{2}\right)\right) &\equiv \frac{k-1}{2}m + \frac{m}{2} + 1 \mod n \\ &= \frac{mk}{2} + 1 \mod n \\ &= \frac{n}{2} + 1 \mod n, \end{aligned}$$

which is coprime to n.

If $4 \nmid m$, choosing $i = (k-1)/2$ and $s = (m/2)+1$ gives

$$\begin{aligned} \operatorname{maj}\left(T\left(m^k; \frac{k-1}{2}, \frac{m}{2}+1\right)\right) &\equiv \frac{k-1}{2}m + \left(\frac{m}{2}+1\right) + 1 \mod n \\ &= \frac{mk}{2} + 2 \mod n \\ &= \frac{n}{2} + 2 \mod n, \end{aligned}$$

which is coprime to n.

Case (ii). If k is even then

$$\operatorname{maj}\left(T\left(m^k; i, s\right)\right) \equiv \frac{mk}{2} + im + s + 1 \mod n.$$

Hence choosing $i = (k/2) - 1$ and $s = m - 2$ gives

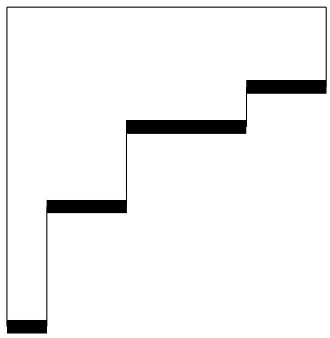
$$\begin{aligned} \operatorname{maj}\left(T\left(m^k; \frac{k}{2} - 1, m-2\right)\right) &\equiv \frac{mk}{2} + \left(\frac{k}{2} - 1\right)m + (m-2) + 1 \mod n \\ &= n - 1 \mod n, \end{aligned}$$

which is coprime to n.

Thus we have shown that for almost every rectangular partition, $\lambda = (m^k)$, of $n = mk$ into k parts, where $2 < k < n$, there exists a standard tableau of shape λ with major index coprime to n; the exception being the rectangle $\lambda = (2^3)$.

3.3.3 Non-rectangular partitions with more than two parts

We begin with a few definitions and a technical lemma which are required for the final part of the proof. Let $\mu = (\mu_1, \ldots, \mu_k)$ be a composition of n ($\mu \vDash n$). We say that a tableau T of shape $\lambda \vdash n$ has weight μ if i occurs μ_i times in T. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n. We define the lower rim of λ to be the union of all boxes B in the Young diagram of λ such that there is no box directly below B.



The lower rim of λ

In other words, each box in the lower rim lies at the end of a column. Hence, the number of boxes in the lower rim is equal to the number of columns in λ, which in turn is equal to λ_1. Notice that if we remove a box from the end of a row in the lower rim of λ, then the shape which remains is a Young diagram corresponding to a partition of $n-1$. It is easy to see that the remaining boxes in the lower rim of λ all lie in the lower rim of the new diagram. Hence, we can successively remove up to λ_1 boxes from the lower rim of λ in such a way that the shape which remains is a Young diagram.

Lemma 3.3.1. *Let $n = mk + r$ where $m, k > 0$ and $0 \leq r < k$. Furthermore, let λ be a partition of n into k parts and $\mu = (\mu_1, \dots, \mu_k)$ be a composition of n with each $\mu_i \in \{m, m+1\}$. Then there exists a tableau of shape λ with weight μ and first column with entries $1, \dots, k$.*

Proof. We prove the result by induction on k. When $k = 1$ we have that $n = m$ and $\lambda = \mu = (m)$. So the tableau consisting of a single row of ones satisfies the conditions of the lemma. For $k > 1$ there are two cases.

Case(i). $\mu_k = m$. Since λ is a partition of $n = mk + r$ into k parts, we have that $\lambda_1 \geq m$ and $\lambda_k \leq m$. In other words, there are at least m boxes in the lower rim of λ and at most m boxes in the kth row. Hence, we can remove m boxes from the lower rim of λ in such a way that all boxes are removed from the kth row and that the shape which remains is a Young diagram. Let $\tilde{\lambda}$ be the partition corresponding to this diagram. Then $\tilde{\lambda}$ is a partition of $\tilde{n} = \tilde{m}(k-1) + \tilde{r}$ into $k-1$ parts, where $\tilde{m} = m$, $\tilde{r} = r$ for $0 \leq r < k-1$ and $\tilde{m} = m+1$, $\tilde{r} = 0$ for $r = k-1$.

Notice that if $r = k-1$ we must have that $\mu_i = m+1$ for all $i < k$, since $n = mk + k - 1 = (m+1)(k-1) + m$ and $\mu_k = m$. Applying induction to $\tilde{\lambda}$ and $\tilde{\mu} = (\mu_1, \ldots, \mu_{k-1})$ gives a tableau $\tilde{T}$ of shape $\tilde{\lambda}$ with weight $\tilde{\mu}$ and entries $1, \ldots, k-1$ in the first column. Finally, by returning the m boxes which we removed and entering a k into each of these, we obtain the desired tableau.

Case(ii). $\mu_k = m+1$. Since μ is a composition of $n = mk + r$ into k parts with each $\mu_i \in \{m, m+1\}$, we have that $r \geq 1$. Moreover, since λ is a partition of $n = mk + r$ into k parts, we have that $\lambda_1 \geq m+1$ and $\lambda_k \leq m$. In other words, there are at least $m+1$ boxes in the lower rim of λ and at most m boxes in the kth row. Hence, we can remove $m+1$ boxes from the lower rim of λ in such a way that all boxes are removed from the kth row and that the shape which remains is a Young diagram. Let $\tilde{\lambda}$ be the partition corresponding to this diagram. Then $\tilde{\lambda}$ is a partition of $\tilde{n} = \tilde{m}(k-1) + \tilde{r}$ into $k-1$ parts, where $\tilde{m} = m$, $\tilde{r} = r-1$, for $0 < r < k$.

Applying induction to $\tilde{\lambda}$ and $\tilde{\mu} = (\mu_1, \ldots, \mu_{k-1})$ gives a tableau $\tilde{T}$ of shape $\tilde{\lambda}$ with weight $\tilde{\mu}$ and entries $1, \ldots, k-1$ in the first column. Again, by returning the $m+1$ boxes which we removed and entering a k into each of these, we obtain the desired tableau. $\square$

We are now in position to complete the proof of Theorem 3.1.1. Consider the non-rectangular partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n = mk + r$ into k parts, where $m > 0$, $k > 2$ and $0 \leq r < k < n$. We claim that there exists a standard tableau T of shape λ, with major index coprime to n and descent set of the form

$$D(T) \quad = \quad \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_{k-1}\},$$

where (i) $\mu_i \in \{m, m+1\}$ if $k \nmid n$ and (ii) $\mu_i \in \{m-1, m\}$ if $k | n$.

Case (i). If $k \nmid n$ then, by Lemma 3.3.1, it is clear that for all such $\lambda \vdash n$ and for all $\mu = (\mu_1, \ldots, \mu_k) \vDash n$, where $\mu_i \in \{m, m+1\}$, there exists a standard tableau T of shape λ with descent set as above. We simply replace the μ_i occurrences of i in the tableau of Lemma 3.3.1 by the entries

$$\sum_{j=1}^{i-1}\mu_j + 1, \sum_{j=1}^{i-1}\mu_j + 2, \ldots, \sum_{j=1}^{i}\mu_j,$$

from left to right, to obtain a standard tableau T of shape λ. Let $\mathfrak{T}$ denote the set of all standard tableaux obtained in this way (by Lemma 3.3.1 we see that there is at least one for every choice of μ) and let $T \in \mathfrak{T}$ be one such standard tableau corresponding to the choice $\mu = (\mu_1, \ldots, \mu_k)$. Since $\mu \vDash n$ we have that exactly r of the μ_i's are equal to $m+1$; $\mu_{i_1}, \ldots, \mu_{i_r}$, say. Hence, the major index of T is given by

$$\begin{aligned} \operatorname{maj}(T) &= (k-1)\mu_1 + (k-2)\mu_2 + \cdots + \mu_{k-1} \\ &= \sum_{i=1}^{k-1}(k-i)\mu_i \\ &= \sum_{i=1}^{k-1}(k-i)m + \sum_{j=1}^{r}(k-i_j) \\ &= \frac{mk(k-1)}{2} + rk - \sum_{j=1}^{r} i_j. \end{aligned}$$

The sum of the i_j ranges from $1+2+\cdots+(r-1)+r = r(r+1)/2$ at the smallest to $(k-r+1)+(k-r+2)+\cdots+(k-1)+k = rk - r(r-1)/2$ at the greatest, giving $r(k-r)+1$ consecutive values of $\operatorname{maj}(T)$ for $T \in \mathfrak{T}$. Hence, for each integer z such that

$$\frac{n(k-1)}{2} - \frac{r(k-r)}{2} \;\le\; z \;\le\; \frac{n(k-1)}{2} + \frac{r(k-r)}{2},$$

there exists a standard tableau $T \in \mathfrak{T}$ with $\operatorname{maj}(T) = z$. If k is odd then for all $T \in \mathfrak{T}$ we have $\operatorname{maj}(T) \equiv a \mod n$, where

$$-\frac{r(k-r)}{2} \;\le\; a \;\le\; \frac{r(k-r)}{2},$$

and, since $0 < r < k$ and $k > 2$, we must have that $\operatorname{maj}(T) \equiv 1$ for some standard tableau $T \in \mathfrak{T}$.

If k is even then for all $T \in \mathfrak{T}$ we have $\mathrm{maj}(T) \equiv a \mod n$ where

$$\frac{n}{2} - \frac{r(k-r)}{2} \;\leq\; a \;\leq\; \frac{n}{2} + \frac{r(k-r)}{2}.$$

One of these values is coprime to n. Indeed, since $0 < r < k$ we must have that $r(k-r) \geq k-1$, giving that $(r(k-r))/2 \geq 2$, for $k > 4$. For $k = 4$ it is easy to check that $((mk+r) + r(k-r))/2 = [n/2] + 2$. Hence, for $r > 0$, k even and greater than 2 we have that there exist standard tableaux in $\mathfrak{T}$ with major indices congruent to $[n/2] + 1$ and $[n/2] + 2$ modulo n, one of which is coprime to n.

Case (ii). If $k|n$ then we have that $r = 0$ and $n = mk$. Since λ is non-rectangular, we must have that $\lambda_1 \geq m+1$ and $\lambda_k \leq m-1$. Recall that λ_1 is equal to the number of boxes in the lower rim of λ and λ_k is the number of boxes in the kth row of λ. Hence we can remove $m+1$ boxes from the lower rim of λ, including all λ_k boxes from the kth row, in such a way that the shape that remains is a Young diagram. Let $\tilde{\lambda}$ denote the partition corresponding to this Young diagram. Then $\tilde{\lambda}$ is a partition of $\tilde{n} = mk - (m+1) = (m-1)(k-1) + (k-2)$, into $k-1$ parts.

Since $k > 2$, we have that $(k-1) \nmid \tilde{n}$ and, applying the reasoning of Case (i), we see that for every $\mu = (\mu_1, \ldots, \mu_{k-1}) \vDash \tilde{n}$ with each $\mu_i \in \{m-1, m\}$, there exists a standard tableau of shape $\tilde{\lambda}$ with descent set given by

$$D(T) \;=\; \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_{k-2}\}.$$

Notice that $k-2$ of the μ_i are in fact equal to m, with just one of the μ_i equal to $m-1$. Let $\tilde{\mathfrak{T}}$ denote the set of all such standard tableaux. Recall that there are $k-1$ consecutive values for the major indices of the tableaux in $\tilde{\mathfrak{T}}$. Indeed, for each integer z such that

$$\frac{(k-1)(k-2)m}{2} - (k-2) \leq z \leq \frac{(k-1)(k-2)m}{2},$$

there exists a standard tableau $\tilde{T} \in \tilde{\mathfrak{T}}$ with $\mathrm{maj}(\tilde{T}) = z$.

For each $\tilde{T} \in \tilde{\mathfrak{T}}$, replacing the $m+1$ boxes which we removed earlier and entering the numbers $mk-m, \ldots, mk$ from left to right gives rise to a standard tableau T of shape λ with

$$\begin{aligned} D(T) &= D(\tilde{T}) \cup \{mk - m - 1\} \\ &= \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_{k-2} + \mu_{k-1}\}, \end{aligned}$$

where $\mu_i \in \{m-1, m\}$, as above. Hence,

$$\mathrm{maj}(T) = \mathrm{maj}(\tilde{T}) + (mk - m - 1)$$

and for every integer z in the range

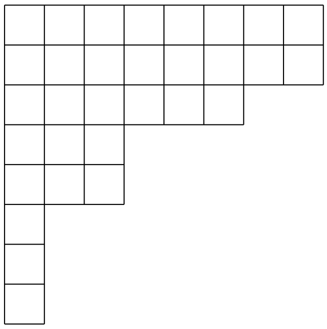
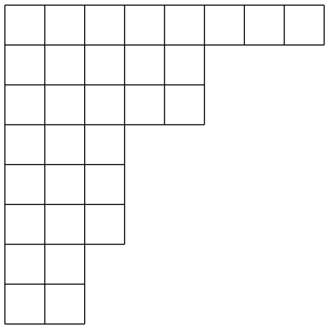
$$\frac{m(k-1)k}{2} - k + 1 \leq z \leq \frac{m(k-1)k}{2} - 1$$

we can construct a standard tableau T of shape λ with $\mathrm{maj}(T) = z$. We show that one of these $k-1$ values is coprime to n modulo n.

If k is odd, then we have that for each positive integer a with $n - k + 1 \leq a \leq n - 1$, there exists a standard tableau T_a of shape λ, with major index congruent to a modulo n, giving $\mathrm{maj}(T_{n-1})$ coprime to n, for example. Finally, if k is even then we have that for each positive integer a with $\frac{n}{2} - k + 1 \leq a \leq \frac{n}{2} - 1$, there exists a standard tableau T_a of shape λ, with major index congruent to a modulo n. Since one of $\mathrm{maj}(T_{(n/2)-1})$ and $\mathrm{maj}(T_{(n/2)-2})$ is coprime to n, this completes the proof of Theorem 3.1.1.

3.4 Conjugacy

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n into k parts. We denote by λ' the partition obtained from λ by reflecting along the main diagonal. For example, let $\lambda = (8^2, 6, 3^2, 1^3)$. Then we have that $\lambda' = (8, 5^2, 3^3, 2^2)$. We say that λ' is the conjugate of λ.

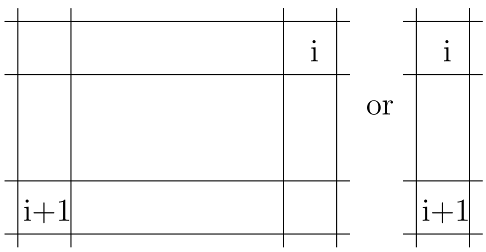
$\lambda =$  $\lambda' =$ 

Let λ be a partition of n and let T be a standard tableau of shape λ. Consider the tableau T' of shape λ' obtained from T by reflecting along the main diagonal. Then T' is standard. Indeed, since T is standard, each number from $1, \ldots, n$ occurs exactly once and entries increase along rows and down columns. Now, T' contains the same entries as T, so each number from $1, \ldots, n$ occurs exactly once. Moreover, since the rows of T' are equal to the columns of T, we have that the entries increase along each row. Similarly, the columns of T' are equal to the rows of T, hence the entries increase down each column. Thus, T' is standard.

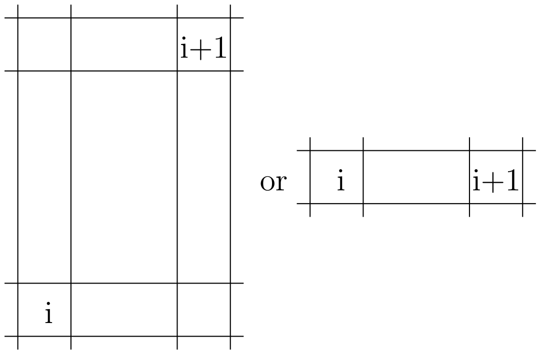
Recall that $i \in \{1, \ldots, n-1\}$ is called a descent in T if $i+1$ occurs in any row below i in T. We shall say that $i \in \{1, \ldots, n-1\}$ is a non-descent in T if $i+1$ does not occur in any row below i in T.

Proposition 3.4.1. *Conjugation interchanges descents and non-descents.*

Proof. Let $i \in \{1, \ldots, n-1\}$. If i is a descent in T then the part of the tableau containing i must look like one of



since T is standard. Hence T' contains one of



and, in either case, i is a non-descent. The proof in the opposite direction is similar.

□

Theorem 3.4.2. *Let λ be a partition of $n > 0$ and let T be a standard tableau of shape λ. If n is odd or divisible by 4 then the major index of T is coprime to n if and only if the major index of T' is coprime to n.*

Proof. This follows easily from Proposition 3.4.1. First of all, notice that

$$\sum \text{descents} + \sum \text{non-descents} = \frac{n(n-1)}{2}.$$

By Proposition 3.4.1, it follows that

$$\text{maj}(T) + \text{maj}(T') = \frac{n(n-1)}{2}.$$

Now, if n is odd, $n = 2k+1$ say, then

$$\text{maj}(T) + \text{maj}(T') = kn.$$

Suppose that one of $\text{maj}(T)$ or $\text{maj}(T')$ is coprime to n. Then we must have that they are both coprime to n. Indeed, suppose that $\text{maj}(T')$ is coprime to n and suppose, for contradiction, that there exists $a > 1$ such that $a|\,\text{maj}(T)$ and $a|n$. Then, since $\text{maj}(T)+\text{maj}(T') = kn$, we must have that $a|\,\text{maj}(T')$, contradicting $\text{maj}(T')$ coprime to n.

Similarly, if n is divisible by 4, $n = 4k$ say, then

$$\text{maj}(T) + \text{maj}(T') = 8k^2 - 2k.$$

Again, suppose that one of $\mathrm{maj}(T)$ or $\mathrm{maj}(T')$ is coprime to n. Then we must have that they are both coprime to n. Indeed, suppose that $\mathrm{maj}(T')$ is coprime to $n = 4k$ and suppose, for contradiction that there exists $a > 1$ such that $a|\,\mathrm{maj}(T')$ and $a|n = 4k$. If $a|k$ then, since $\mathrm{maj}(T) + \mathrm{maj}(T') = 8k^2 - 2k$, we have that $a|\,\mathrm{maj}(T')$, contradicting $\mathrm{maj}(T')$ coprime to n. Hence we may assume that a is even. However, since $a|\,\mathrm{maj}(T)$, we have that $\mathrm{maj}(T)$ is even and hence, $\mathrm{maj}(T')$ must be even, contradicting $\mathrm{maj}(T')$ coprime to $4k$. □

There are a wealth of examples showing that the above result does not necessarily hold when $n \equiv 2 \bmod 4$. Indeed, we have already seen that this is the case for the partitions $\lambda = (2^3)$ and $\lambda' = (3^2)$.

Example. Let $n = 30$ and let

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline \mathbf{1} & \mathbf{3} & 5 & \mathbf{6} & \mathbf{14} & 18 & \mathbf{19} \\ \hline 2 & 4 & \mathbf{8} & \mathbf{10} & 17 & \mathbf{28} \\ \cline{1-6} 7 & 9 & \mathbf{11} & \mathbf{15} & \mathbf{20} & \mathbf{29} \\ \cline{1-6} 12 & 13 & 16 & \mathbf{21} & 27 \\ \cline{1-5} 22 & \mathbf{23} & \mathbf{25} \\ \cline{1-3} 24 & 26 \\ \cline{1-2} 30 \\ \cline{1-1} \end{array} \qquad T' = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & \mathbf{2} & \mathbf{7} & \mathbf{12} & \mathbf{22} & \mathbf{24} & 30 \\ \hline 3 & \mathbf{4} & \mathbf{9} & \mathbf{13} & 23 & \mathbf{26} \\ \cline{1-6} \mathbf{5} & 8 & 11 & \mathbf{16} & 25 \\ \cline{1-5} 6 & 10 & 15 & 21 \\ \cline{1-4} 14 & \mathbf{17} & 20 & \mathbf{27} \\ \cline{1-4} \mathbf{18} & 28 & 29 \\ \cline{1-3} 19 \\ \cline{1-1} \end{array}$$

then $\mathrm{maj}(T) = 233$ and $\mathrm{maj}(T') = 202$, where $(233, 30) = 1$, but $(202, 30) = 2$.

The following corollary is an easy consequence of Theorem 3.4.2 and Theorem 3.1.3.

Corollary 3.4.3. *Let n be odd or divisible by 4 and let λ be a partition of n with at most $\dim V$ parts and greatest part less than or equal to $\dim V$. Then the irreducible $K\mathrm{GL}(V)$-modules corresponding to λ and λ' occur in $L_n(V)$ with equal multiplicity.*

Proof. Let $a, n \in \mathbb{N}$ be fixed coprime numbers and let $\lambda \vdash n$ be as above. By the Kraśkiewicz-Weyman theorem (Theorem 3.1.3), we have that the irreducible $K\mathrm{GL}(V)$-module corresponding to λ occurs with multiplicity l_λ equal to the number of standard tableaux of shape λ with major index congruent to a modulo n. Let

$0 < a_1 < \cdots < a_j < n$ be a complete set of positive integers coprime to n. Let $l_\lambda(b)$ denote the number of standard tableaux of shape λ with major index congruent to b modulo n. Then we have that

$$l_\lambda = l_\lambda(a) = l_\lambda(a_1) = \cdots = l_\lambda(a_j)$$

and

$$jl_\lambda = \sum_{i=1}^{j} l_\lambda(a_i) = c_\lambda,$$

where c_λ denotes the number of standard tableaux of shape λ with major index coprime to n.

Similarly, for the conjugate partition λ', by the Kraśkiewicz-Weyman theorem (Theorem 3.1.3), we have that the irreducible $K\mathrm{GL}(V)$-module corresponding to λ' occurs with multiplicity $l_{\lambda'}$ equal to the number of standard tableaux of shape λ' with major index congruent to a modulo n. Hence, we have that

$$l_{\lambda'} = l_{\lambda'}(a) = l_{\lambda'}(a_1) = \cdots = l_{\lambda'}(a_j)$$

and

$$jl_{\lambda'} = \sum_{i=1}^{j} l_{\lambda'}(a_i) = c_{\lambda'}.$$

Now, since n is odd or divisible by 4, Theorem 3.4.2 tells us that $c_\lambda = c_{\lambda'}$. Thus,

$$l_\lambda = \frac{c_\lambda}{j} = \frac{c_{\lambda'}}{j} = l_{\lambda'}.$$

□

In fact, Zhuravlev [61] has proven the following stronger result using character theoretic methods.

Theorem 3.4.4. [61, Proposition 4.1]. *Let λ be a partition of n with at most* $\dim V$ *parts and greatest part less than or equal to* $\dim V$.

(i) If n is odd or divisible by 4 then the irreducible $K\mathrm{GL}(V)$-modules corresponding to λ and λ' occur in $L_n(V)$ with equal multiplicity.

(ii) Otherwise, the multiplicity of the irreducible $K\mathrm{GL}(V)$*-module corresponding to* λ' *in* $L_n(V)$ *is equal to the sum of the multiplicity of* λ *in* $L_n(V)$ *and*

$$\frac{2}{n} \sum_{d|(n/2)} \mu(d) \chi^{\lambda}(\tau^{n/2d}),$$

where τ *is a cycle of length* n *and* χ^{λ} *denotes the character of the irreducible* $K\mathrm{Sym}(n)$*-module corresponding to* λ*.*

Chapter 4

The Dynkin-Specht-Wever idempotent

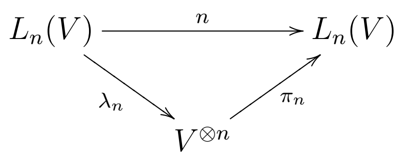
4.1 Introduction

Let K be a commutative ring with identity element, G a group and V a KG-module. Let λ denote the canonical embedding of the free Lie algebra on V into the tensor algebra on V, $\lambda : L(V) \to T(V)$. Since λ respects the degree of monomials, we have that the restriction of λ to the homogeneous component $L_n(V)$ maps into $V^{\otimes n}$. We shall call the restriction of λ to $L_n(V)$, λ_n. An element of $T(V)$ that is the image of some element of $L(V)$ under λ is called a Lie element of the tensor algebra.

Next let $\pi_n : V^{\otimes n} \to L_n(V)$ be the projection of $V^{\otimes n}$ onto $L_n(V)$ defined by

$$\pi_n : v_1 \otimes \cdots \otimes v_n \mapsto [v_1, \ldots, v_n], \tag{4.1.1}$$

for all $v_1, \ldots, v_n \in V$ and extended to the whole of $V^{\otimes n}$ by linearity. We see that π_n is a surjection since $L_n(V)$ is spanned by the left-normed monomials. The composition $\lambda_n \pi_n$ is simply multiplication by n. That is, the following diagram commutes,



where n denotes the map given by multiplication by n.

Let $\mathrm{Sym}(n)$ denote the symmetric group of degree n. We define a right action of $\mathrm{Sym}(n)$ on $V^{\otimes n}$ as follows. For each $\sigma \in \mathrm{Sym}(n)$ let

$$(v_1 \otimes \cdots \otimes v_n)\sigma = v_{1\sigma^{-1}} \otimes \cdots \otimes v_{n\sigma^{-1}},$$

for all $v_1, \ldots, v_n \in V$. Now for each $n \geq 2$ we define an element ω_n of $\mathbb{Z}\mathrm{Sym}(n)$, given by

$$\omega_n = (1 - (1\ 2))(1 - (1\ 2\ 3)) \cdots (1 - (1\ 2\ \cdots\ n)).$$

We can now define the embedding λ by using the elements ω_n. Let $n \geq 2$ and let $v_1, \ldots, v_n \in V$. Then the restriction of λ to the nth Lie power, $\lambda_n : L_n(V) \to V^{\otimes n}$, is the map given by

$$\lambda_n : [v_1, \ldots, v_n] \mapsto (v_1 \otimes \cdots \otimes v_n)\omega_n \tag{4.1.2}$$

and extended to the whole of $L_n(V)$ by linearity. It is easy to see that this holds for $n = 2$, since $[v_1, v_2]\lambda_2 = v_1 \otimes v_2 - v_2 \otimes v_1$, whilst

$$(v_1 \otimes v_2)\omega_2 = (v_1 \otimes v_2)(1 - (1\ 2)) = v_1 \otimes v_2 - v_2 \otimes v_1.$$

We proceed by induction on n. Suppose that

$$[v_1, \ldots, v_{n-1}]\lambda_{n-1} = (v_1 \otimes \cdots \otimes v_{n-1})\omega_{n-1},$$

for all $v_1, \ldots, v_{n-1} \in V$. Let $v_n \in V$, then, by definition,

$$\begin{aligned} [v_1, \ldots, v_{n-1}, v_n]\lambda_n &= [v_1, \ldots, v_{n-1}]\lambda_{n-1} \otimes v_n - v_n \otimes [v_1, \ldots, v_{n-1}]\lambda_{n-1} \\ &= (v_1 \otimes \cdots \otimes v_{n-1})\omega_{n-1} \otimes v_n - v_n \otimes (v_1 \otimes \cdots \otimes v_{n-1})\omega_{n-1}. \end{aligned}$$

On the other hand, since $\omega_n = \omega_{n-1}(1 - (1\ 2\ \cdots\ n))$, we have that

$$\begin{aligned}(v_1 \otimes \cdots \otimes v_{n-1} \otimes v_n)\omega_n &= (v_1 \otimes \cdots \otimes v_{n-1} \otimes v_n)\omega_{n-1}(1 - (1\ 2\ \cdots\ n)) \\ &= (v_1 \otimes \cdots \otimes v_{n-1})\omega_{n-1} \otimes v_n)(1 - (1\ 2\ \cdots\ n)) \\ &= (v_1 \otimes \cdots \otimes v_{n-1})\omega_{n-1} \otimes v_n - v_n \otimes (v_1 \otimes \cdots \otimes v_{n-1})\omega_{n-1}.\end{aligned}$$

Thus we have that $[v_1, \dots, v_n]\lambda_n = (v_1 \otimes \cdots \otimes v_n)\omega_n$ for all $n \geq 2$.

We state the following result without proof.

Theorem 4.1.1. The Dynkin-Specht-Wever Theorem [18], [53], [58].

Let K be a field of characteristic zero, or the ring of integers and let V be a free K-module. Then $u \in V^{\otimes n}$ is a Lie element if and only if $u\omega_n = nu$.

For example, we see that $(v_1 \otimes v_2)\omega_2 = v_1 \otimes v_2 - v_2 \otimes v_1 \neq 2v_1 \otimes v_2$, whilst

$$(v_1 \otimes v_2 - v_2 \otimes v_1)\omega_2 = v_1 \otimes v_2 - v_2 \otimes v_1 - v_2 \otimes v_1 + v_1 \otimes v_2 = 2(v_1 \otimes v_2 - v_2 \otimes v_1).$$

Thus, $v_1 \otimes v_2$ is not a Lie element, while $v_1 \otimes v_2 - v_2 \otimes v_1$ is a Lie element (indeed, $v_1 \otimes v_2 - v_2 \otimes v_1 = [v_1, v_2]$). Theorem 4.1.1 was proved independently by Dynkin [18] in 1947, Specht [53] in 1948 and Wever [58] in 1949. For further reference, see also Magnus [43] and Reutenauer [45].

It is also easy to see that for all $u \in V^{\otimes n}$ we have that $u\omega_n^2 = nu\omega_n$. Indeed, by (4.1.2), $u\omega_n$ is a Lie element, hence by Theorem 4.1.1 we have that $(u\omega_n)\omega_n = n(u\omega_n)$. Moreover, since the action of $\mathbb{Z}\mathrm{Sym}(n)$ on $V^{\otimes n}$ is faithful for $\dim V \geq n$, we have the following identity in the integral group ring $\mathbb{Z}\mathrm{Sym}(n)$,

$$\omega_n^2 = n\omega_n.$$

Hence, we see that the element $\frac{1}{n}\omega_n$ is an idempotent in $\mathbb{Q}\mathrm{Sym}(n)$.

The idempotent $\frac{1}{n}\omega_n \in \mathbb{Q}\mathrm{Sym}(n)$ is an example of a Lie idempotent. For a field K of characteristic zero, an element $e \in K\mathrm{Sym}(n)$ is called a Lie idempotent if e is idempotent in $K\mathrm{Sym}(n)$ and for V an n-dimensional vector space, the image of e acting on the right of the fine homogeneous component $T_\alpha(V)$, of multidegree $\alpha = (1^n)$, is given by $L_\alpha(V)\lambda_n$. We call $\frac{1}{n}\omega_n$ the Dynkin-Specht-Wever idempotent. Other examples of Lie idempotents are the Klyachko idempotent [33], the Reutenauer idempotent [45] and the Garsia idempotent [20].

In recent years Lie idempotents have attracted the attention of a number of authors due to the various combinatorial connections encountered. One such interesting connection is the fact that all of the idempotents mentioned above are idempotent elements of the Solomon descent algebra, a certain subring of $\mathbb{Z}\mathrm{Sym}(n)$ (see [48] for further reference).

4.2 A new factorisation

In this section we give an alternative description of the Dynkin-Specht-Wever idempotent. We begin by defining some elements of the symmetric group $\mathrm{Sym}(n)$. For $2 \leq i \leq n$ let ρ_i denote the cycle of length i given by $\rho_i = (1\ 2\ \cdots\ i)$. Then we have that $\omega_n = (1 - \rho_2)\cdots(1 - \rho_n)$ and, hence, we may write

$$\omega_n = \sum_{k=0}^{n-1}(-1)^k \sum \rho_{i_1}\cdots\rho_{i_k},$$

where the second summation ranges over all products $\rho_{i_1}\cdots\rho_{i_k}$ of length k, where $1 < i_1 < \cdots < i_k \leq n$ (we define the product of length zero to be 1). We next define $\alpha_k \in \mathrm{Sym}(n)$, for $2 \leq k \leq n$, to be the element given by

$$\alpha_k = \begin{pmatrix} 1 & 2 & 3 & \cdots & k-2 & k-1 & k & k+1 & k+2 & \cdots & n \\ k & k-1 & k-2 & \cdots & 3 & 2 & 1 & k+1 & k+2 & \cdots & n \end{pmatrix}.$$

We shall show that, for $n \geq 2$, each ω_n can be written in terms of the α_k, for $2 \leq k \leq n$. We begin with two propositions, relating the elements α_k to the elements ρ_i.

Proposition 4.2.1. *Let $n \geq 2$ and, for $2 \leq i, k \leq n$, let ρ_i and α_k be the elements of $\mathrm{Sym}(n)$ defined above. Then each α_k may be written as the product*

$$\alpha_k = \rho_2\rho_3\cdots\rho_k.$$

Proof. We show that the product $\rho_2 \cdots \rho_k$ maps each $j \in \{1, \ldots, n\}$ to the image of j under α_k. Since the product $\rho_2 \cdots \rho_k$ is just a composition of maps we have

$$\begin{array}{ccccccccccccc}
 & \rho_2 & & \rho_3 & & \rho_4 & \cdots & \rho_{k-2} & & \rho_{k-1} & & \rho_k & \\
1 & \to & 2 & \to & 3 & \to & \cdots & \to & k-2 & \to & k-1 & \to & k \\
2 & \to & 1 & \to & 2 & \to & \cdots & \to & k-3 & \to & k-2 & \to & k-1 \\
3 & \to & 3 & \to & 1 & \to & \cdots & \to & k-4 & \to & k-3 & \to & k-2 \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
k-2 & \to & k-2 & \to & k-2 & \to & \cdots & \to & 1 & \to & 2 & \to & 3 \\
k-1 & \to & k-1 & \to & k-1 & \to & \cdots & \to & k-1 & \to & 1 & \to & 2 \\
k & \to & k & \to & k & \to & \cdots & \to & k & \to & k & \to & 1 \\
k+1 & \to & k+1 & \to & k+1 & \to & \cdots & \to & k+1 & \to & k+1 & \to & k+1 \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
n & \to & n & \to & n & \to & \cdots & \to & n & \to & n & \to & n
\end{array}$$

and hence it is easy to see that $\alpha_k = \rho_2 \cdots \rho_k$. □

Proposition 4.2.2. *Let $n \geq 2$ and, for $2 \leq i \leq n$, let ρ_i and α_n be the elements of* Sym(n) *defined above. Let $1 \leq k < n$ and let $1 < i_1 < i_2 < \cdots < i_k \leq n$. Then*

$$\rho_{i_1} \cdots \rho_{i_k} \alpha_n = \rho_2 \cdots \rho_{i_1 - 1} \rho_{i_1 + 1} \cdots \rho_{i_k - 1} \rho_{i_k + 1} \cdots \rho_n.$$

Proof. We prove the proposition by induction on k. For $k = 1$ we must show that

$$\rho_i \alpha_n = \rho_2 \cdots \rho_{i-1} \rho_{i+1} \cdots \rho_n, \tag{4.2.1}$$

for all $1 < i \leq n$. We calculate the image of each element of $\{1, \ldots, n\}$ under the maps given by the left-hand and right-hand sides of (4.2.1). On the left-hand side we have:

$$\begin{array}{ccccc}
 & \rho_i & & \alpha_n & \\
1 & \to & 2 & \to & n-1 \\
2 & \to & 3 & \to & n-2 \\
\vdots & & \vdots & & \vdots \\
i-1 & \to & i & \to & n-i+1 \\
i & \to & 1 & \to & n \\
i+1 & \to & i+1 & \to & n-i \\
\vdots & & \vdots & & \vdots \\
n & \to & n & \to & 1.
\end{array}$$

Similarly, on the right-hand side we have:

$$\begin{array}{ccccccccccccc}
 & \rho_2 & & \rho_3 & \cdots & \rho_{i-1} & & \rho_{i+1} & & \rho_{i+2} & \cdots & \rho_n & \\
1 & \to & 2 & \to & \cdots & \to & i-1 & \to & i & \to & \cdots & \to & n-1 \\
2 & \to & 1 & \to & \cdots & \to & i-2 & \to & i-1 & \to & \cdots & \to & n-2 \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
i-1 & \to & i-1 & \to & \cdots & \to & 1 & \to & 2 & \to & \cdots & \to & n-i+1 \\
i & \to & i & \to & \cdots & \to & i & \to & i+1 & \to & \cdots & \to & n \\
i+1 & \to & i+1 & \to & \cdots & \to & i+1 & \to & 1 & \to & \cdots & \to & n-i \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
n & \to & n & \to & \cdots & \to & n & \to & n & \to & \cdots & \to & 1.
\end{array}$$

So the proposition holds for $k = 1$.

Next suppose that for all $n \geq 2$ and for all $l < k < n$, with $1 < i_1 < \cdots < i_l \leq n$, we have that $\rho_{i_1} \cdots \rho_{i_l} \alpha_n = \rho_2 \cdots \rho_{i_1 - 1} \rho_{i_1 + 1} \cdots \rho_{i_l - 1} \rho_{i_l + 1} \cdots \rho_n$. By induction we have that

$$\rho_{i_2} \cdots \rho_{i_k} \alpha_n = \rho_2 \cdots \rho_{i_2 - 1} \rho_{i_2 + 1} \cdots \rho_{i_k - 1} \rho_{i_k + 1} \cdots \rho_n$$

and hence

$$\begin{aligned}
\rho_{i_1} \rho_{i_2} \cdots \rho_{i_k} \alpha_n &= \rho_{i_1} (\rho_{i_2} \cdots \rho_{i_k} \alpha_n) \\
&= \rho_{i_1} (\rho_2 \cdots \rho_{i_2 - 1} \rho_{i_2 + 1} \cdots \rho_{i_k - 1} \rho_{i_k + 1} \cdots \rho_n).
\end{aligned}$$

Now, by associativity, we have

$$\rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}\alpha_n = (\rho_{i_1}\rho_2\cdots\rho_{i_2-1})(\rho_{i_2+1}\cdots\rho_{i_k-1}\rho_{i_k+1}\cdots\rho_n)$$

and hence, by Proposition 4.2.1, we may apply induction again to obtain

$$\rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}\alpha_n = \rho_2\cdots\rho_{i_1-1}\rho_{i_1+1}\cdots\rho_{i_2-1}\rho_{i_2+1}\cdots\rho_{i_k-1}\rho_{i_k+1}\cdots\rho_n,$$

as required. □

We now come to the main result of this chapter.

Theorem 4.2.3. *For all $n \geq 2$ we may write ω_n as a product*

$$\omega_n = (1-\alpha_2)(1+\alpha_3)\cdots(1+(-1)^{n-1}\alpha_n), \tag{4.2.2}$$

where, for $1 \leq k \leq n$, we define α_k to be the permutation

$$\alpha_k = \begin{pmatrix} 1 & 2 & 3 & \cdots & k-2 & k-1 & k & k+1 & k+2 & \cdots & n \\ k & k-1 & k-2 & \cdots & 3 & 2 & 1 & k+1 & k+2 & \cdots & n \end{pmatrix}.$$

Proof. It is easy to check that this is the case for small values of n. Indeed, we have $\omega_2 = 1-(1\ 2) = 1-\alpha_2$ and it is easy to see that

$$\omega_3 = (1-(1\ 2))(1-(1\ 2\ 3)) = (1-(1\ 2))(1+(1\ 3)) = (1-\alpha_2)(1+\alpha_3).$$

For $n \geq 2$ let σ_n denote the element of $\mathbb{Z}\mathrm{Sym}(n)$ given by the right-hand side of (4.2.2), that is,

$$\sigma_n = (1-\alpha_2)(1+\alpha_3)\cdots(1+(-1)^{n-1}\alpha_n).$$

We shall prove by induction on n that $\omega_n = \sigma_n$ for all $n \geq 2$. It is easy to see that $\sigma_n = \sigma_{n-1}(1+(-1)^{n-1}\alpha_n)$. By induction we have that $\sigma_n = \omega_{n-1}(1+(-1)^{n-1}\alpha_n)$.

Recall that

$$\omega_{n-1} = \sum_{k=0}^{n-2}(-1)^k \sum_{1<i_1<\cdots<i_k<n} \rho_{i_1}\cdots\rho_{i_k},$$

and hence,

$$\begin{aligned}\sigma_n = \omega_{n-1}(1+(-1)^{n-1}\alpha_n) &= \left(\sum_{k=0}^{n-2}(-1)^k \sum_{1<i_1<\cdots<i_k<n} \rho_{i_1}\cdots\rho_{i_k}\right)(1+(-1)^{n-1}\alpha_n) \\ &= \left(\sum_{k=0}^{n-2}(-1)^k \sum_{1<i_1<\cdots<i_k<n} \rho_{i_1}\cdots\rho_{i_k}\right) \\ &\quad + \left(\sum_{k=0}^{n-2}(-1)^{k+n-1} \sum_{1<i_1<\cdots<i_k<n} \rho_{i_1}\cdots\rho_{i_k}\alpha_n\right). \quad (4.2.3)\end{aligned}$$

Now, by Propositions 4.2.1 and 4.2.2 (applied to the terms containing α_n), equation (4.2.3) becomes

$$\begin{aligned}\sigma_n &= \left(\sum_{k=0}^{n-2}(-1)^k \sum_{1<i_1<\cdots<i_k<n} \rho_{i_1}\cdots\rho_{i_k}\right) + (-1)^{n-1}\rho_2\cdots\rho_n \\ &\quad + \sum_{k=1}^{n-2}(-1)^{k+n-1} \sum_{1<i_1<\cdots<i_k<n} \rho_2\cdots\rho_{i_1-1}\rho_{i_1+1}\cdots\rho_{i_k-1}\rho_{i_k+1}\cdots\rho_n. \quad (4.2.4)\end{aligned}$$

We may simplify equation (4.2.4) even further by making the following observations. First we notice that $(-1)^{k+n-1} = (-1)^{n-1-k}$ for all n and k. Secondly, we see that each product in the final summation of (4.2.4) is of length $n-1-k$. Thus, we may write

$$\begin{aligned}\sigma_n = &\quad \sum_{k=0}^{n-2}(-1)^k \sum_{1<i_1<\cdots<i_k<n} \rho_{i_1}\cdots\rho_{i_k} \\ &+ \sum_{m=1}^{n-1}(-1)^m \sum_{1<j_1<\cdots<j_{m-1}<n} \rho_{j_1}\cdots\rho_{j_{m-1}}\rho_n \\ = &\quad \sum_{k=0}^{n-1}(-1)^k \sum_{1<i_1<\cdots<i_k\leq n} \rho_{i_1}\cdots\rho_{i_k} = \omega_n,\end{aligned}$$

as required. □

We shall give an application of this result in the following chapter.

Chapter 5

Decompositions of Lie powers in prime characteristic

5.1 The Decomposition Theorem

Let G be a group, K a field and V a finite-dimensional KG-module. As usual, we wish to determine the KG-module structure of $L_n(V)$. From now on we look at the more difficult situation where K is a field of prime characteristic $p > 0$. In this case $L_n(V)$ is no longer semisimple and the problem takes on a different complexion. When $p \nmid n$ we have that $L_n(V)$ is isomorphic to a direct summand of $V^{\otimes n}$ and in this case we can exploit our knowledge of the tensor power in order to give information about the Lie power. In particular, Donkin and Erdmann [17] have described the indecomposable summands of $L_n(V)$ with $p \nmid n$ in the case where K is an infinite field and G is the general linear group $GL(V)$. However, their methods do not extend to arbitrary Lie powers.

Recently there have been some major developments in the study of Lie powers of degree divisible by p. As a first step Bryant and Stöhr [14] considered the case $n = p$ and showed that $L_p(V)$ has the following module direct sum decomposition: $L_p(V) = M_p(V) \oplus B_p(V)$ where $M_p(V)$ is the free metabelian Lie power and $B_p(V)$

is given by $L''(V) \cap L_p(V)$, which is identified with a direct summand of $V^{\otimes p}$ under the canonical embedding of $L_p(V)$ in $V^{\otimes p}$. Moreover, in the case where K is an infinite field and G is the general linear group, $GL(V)$, Bryant and Stöhr calculated the indecomposable summands of $B_p(V)$ together with their multiplicities, giving a complete description of the indecomposable summands of $L_p(V)$ in this case.

Erdmann and Schocker [19] next showed that the study of the pkth Lie power, where $p \nmid k$, can in some sense be reduced to the study of $L_p(L_k(V))$. Their analysis made significant use of the Solomon descent algebra (see [48] for further reference).

Taking this idea much further, Bryant and Schocker [10] were able to show that the study of Lie powers of arbitrary degree can be reduced to the study of Lie powers of p-power degree. The following result is the 'Decomposition Theorem', [10, Theorem 4.4].

Theorem 5.1.1. [10, Theorem 4.4]. *Let K be a field of prime characteristic p, G a group, and V a finite-dimensional KG-module. Let k be a positive integer not divisible by p. Then, for each non-negative integer m, there is a submodule $B_{p^m k}$ of $L_{p^m k}(V)$ such that $B_{p^m k}$ is isomorphic to a direct summand of $V^{\otimes p^m k}$ and*

$$L_{p^m k}(V) = L_{p^m}(B_k) \oplus L_{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L_1(B_{p^m k}).$$

(The Lie powers $L_{p^m}(B_k)$, $\ldots$, $L_1(B_{p^m k})$ may be regarded as subspaces of $L_{p^m k}(V)$ for the reasons explained in [10, Section 2].)

When $m = 0$, Theorem 5.1.1 gives that $L_k(V) = B_k$ and hence, when $m = 1$, we have

$$L_{pk}(V) = L_p(L_k(V)) \oplus B_{pk}. \tag{5.1.1}$$

Since $L_p(L_k(V))$ is a submodule of $L_{pk}(V)$, we have that $L_p(L_k(V))$ is a direct summand of $L_{pk}(V)$ if and only if there exists a projection $\phi : L_{pk}(V) \to L_p(L_k(V))$. Hence, if we can construct a projection map from $L_{pk}(V)$ onto $L_p(L_k(V))$ this will give an alternative proof of (5.1.1).

We shall use our new factorisation of ω_n in §5.2 to obtain a proof of the fact that (in characteristic 2) $L_2(L_k(V))$ is a direct summand of $L_{2k}(V)$ for all odd $k > 0$. We obtain this result by constructing a projection of $L_{2k}(V)$ onto $L_2(L_k(V))$. Our argument does not require V to be finite-dimensional.

In Section 5.3 we consider a small example, namely, the sixth Lie power in characteristic 2 and characteristic 3. We shall see in the following chapters that this simple example proves to be particularly illuminating. In Chapter 6 we shall show that the form of the module $B_6^{(3)}$ as described in Section 5.3.1 can be generalised, while in Chapter 7 we are able to apply our knowledge of the decomposition of the sixth Lie power to a group theoretic problem.

5.2 A projection of $L_{2k}(V)$ onto $L_2(L_k(V))$

Throughout this chapter we shall write $a_1a_2\cdots a_n$ to mean $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in order to simplify the notation.

Theorem 5.2.1. *Let K be a field of characteristic $p \geq 0$, G a group and let V be a KG-module. Let $k > 0$ be invertible in K and let $\phi_{(2,k)} : L_{2k}(V) \to L_2(L_k(V))$ be the KG-module homomorphism given by $\phi_{(2,k)} = \frac{(-1)^{k-1}}{2k^2}\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)}$, where λ_{2k} is the canonical embedding of $L_{2k}(V)$ into $V^{\otimes 2k}$,*

$$\begin{aligned} \theta_{(2,k)} : a_1\cdots a_k a_{k+1}\cdots a_{2k} &\mapsto a_1\cdots a_k a_{2k}\cdots a_{k+1} \\ \text{and } \pi_{(2,k)} : a_1\cdots a_k a_{k+1}\cdots a_{2k} &\mapsto [[a_1,\ldots,a_k],[a_{k+1},\ldots,a_{2k}]]. \end{aligned}$$

Then $\phi_{(2,k)}$ is a projection of $L_{2k}(V)$ onto $L_2(L_k(V))$. Hence, $L_2(L_k(V))$ is a KG-module direct summand of $L_{2k}(V)$.

Proof. We first remark that $\phi_{(2,k)}$ is well-defined over K, since, as we shall see, if working over $\mathbb{Z}$ we have that $v\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)} \in 2L_2(L_k(V))$ for all $v \in L_{2k}(V)$. Thus we may divide by 2. Also, we have that k is invertible in K, so the definition of $\phi_{(2,k)}$ makes sense over K. We shall prove:

(i) $v\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)} \in 2L_2(L_k(V)),\ \forall v \in L_{2k}(V)$,

(ii) $u\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)} = 2k^2u,\ \forall u \in L_2(L_k(V))$.

Hence,

$$\begin{aligned}
v\phi_{(2,k)}^2 &= \left(\frac{1}{2k^2}v\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)}\right)\phi_{(2,k)} \\
&= \frac{1}{2k^2}\left(2\sum_i u_i\right)\phi_{(2,k)}, \quad \text{where } u_i \in L_2(L_k(V)) \\
&= \frac{1}{2k^2}\left(2\sum_i u_i\phi_{(2,k)}\right) = \frac{1}{2k^2}\left(\frac{2}{2k^2}\sum_i u_i\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)}\right) \\
&= \frac{1}{2k^2}\left(\frac{2}{2k^2}\sum_i 2k^2u_i\right) = \frac{1}{2k^2}\left(2\sum_i u_i\right) \\
&= \frac{1}{2k^2}v\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)} = v\phi_{(2,k)}.
\end{aligned}$$

Thus $\phi_{(2,k)}$ is a projection. Since $\phi_{(2,k)}$ is a projection of $L_{2k}(V)$ onto one of its submodules, $L_2(L_k(V))$, we must have that $L_2(L_k(V))$ is a direct summand of $L_{2k}(V)$ over any field of characteristic $p \geq 0$, $p \nmid k$. A complement of $L_2(L_k(V))$ inside $L_{2k}(V)$ is given by $L_{2k}(V)(1 - \phi_{(2,k)})$. It remains to prove (i) and (ii).

As promised, our proof employs our new found factorisation of ω_{2k}. Let

$$\begin{aligned}
\alpha : a_1 \cdots a_k a_{k+1} \cdots a_{2k} &\mapsto a_{2k} \cdots a_{k+1} a_k \cdots a_1, \\
\beta : a_1 \cdots a_k a_{k+1} \cdots a_{2k} &\mapsto a_{k+1} \cdots a_{2k} a_1 \cdots a_k.
\end{aligned}$$

Then $\alpha\theta_{(2,k)} = \theta_{(2,k)}\beta$. Indeed,

$$(a_1 \cdots a_k a_{k+1} \cdots a_{2k})\alpha\theta_{(2,k)} = a_{2k} \cdots a_{k+1} a_1 \cdots a_k,$$

$$(a_1 \cdots a_k a_{k+1} \cdots a_{2k})\theta_{(2,k)}\beta = a_{2k} \cdots a_{k+1} a_1 \cdots a_k.$$

Recall from Chapter 4 that $\omega_{2k} = \omega_{2k-1}(1 - \alpha_{2k})$, where α_{2k} is given by

$$\alpha_{2k} = \begin{pmatrix} 1 & 2 & 3 & \cdots & 2k-2 & 2k-1 & 2k \\ 2k & 2k-1 & 2k-2 & \cdots & 3 & 2 & 1 \end{pmatrix}.$$

So $\omega_{2k} = \omega_{2k-1}(1-\alpha)$. Finally, we have that $(1-\beta)\pi_{(2,k)} = 2\pi_{(2,k)}$. Indeed,

$$\begin{aligned}(a_1 \cdots a_k a_{k+1} \cdots a_{2k})(1-\beta)\pi_{(2,k)} &= (a_1 \cdots a_k a_{k+1} \cdots a_{2k})\pi_{(2,k)} \\ &\quad -(a_{k+1} \cdots a_{2k} a_1 \cdots a_k)\pi_{(2,k)} \\ &= [[a_1, \ldots, a_k], [a_{k+1}, \ldots, a_{2k}]] \\ &\quad -[[a_{k+1}, \ldots, a_{2k}], [a_1, \ldots, a_k]] \\ &= 2[[a_1, \ldots, a_k], [a_{k+1}, \ldots, a_{2k}]] \\ &= 2(a_1 \cdots a_k a_{k+1} \cdots a_{2k})\pi_{(2,k)}.\end{aligned}$$

Since $L_{2k}(V)$ is spanned by left-normed monomials, it is enough to prove (i) in the case $v = [a_1, \ldots, a_k, a_{k+1}, \ldots, a_{2k}]$. Then

$$\begin{aligned}v\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)} &= (a_1 \cdots a_k a_{k+1} \cdots a_{2k})\omega_{2k}\theta_{(2,k)}\pi_{(2,k)} \\ &= (a_1 \cdots a_k a_{k+1} \cdots a_{2k})\omega_{2k-1}(1-\alpha)\theta_{(2,k)}\pi_{(2,k)} \\ &= (a_1 \cdots a_k a_{k+1} \cdots a_{2k})\omega_{2k-1}(\theta_{(2,k)} - \alpha\theta_{(2,k)})\pi_{(2,k)} \\ &= (a_1 \cdots a_k a_{k+1} \cdots a_{2k})\omega_{2k-1}(\theta_{(2,k)} - \theta_{(2,k)}\beta)\pi_{(2,k)} \\ &= (a_1 \cdots a_k a_{k+1} \cdots a_{2k})\omega_{2k-1}\theta_{(2,k)}(1-\beta)\pi_{(2,k)} \\ &= 2(a_1 \cdots a_k a_{k+1} \cdots a_{2k})\omega_{2k-1}\theta_{(2,k)}\pi_{(2,k)} \in 2L_2(L_k(V)).\end{aligned}$$

Since $L_2(L_k(V))$ is spanned by monomials of the form $[[a_1, \ldots, a_k], [a_{k+1}, \ldots, a_{2k}]]$, it is enough to prove (ii) in the case $u = [[a_1, \ldots, a_k], [a_{k+1}, \ldots, a_{2k}]]$. Then,

$$\begin{aligned}u\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)} &= ((a_1 \cdots a_k)\omega_k(a_{k+1} \cdots a_{2k})\omega_k)\theta_{(2,k)}\pi_{(2,k)} \\ &\quad -((a_{k+1} \cdots a_{2k})\omega_k(a_1 \cdots a_k)\omega_k)\theta_{(2,k)}\pi_{(2,k)} \\ &= ((a_1 \cdots a_k)\omega_k(a_{k+1} \cdots a_{2k})\omega_{k-1}(1 \pm \alpha_k))\theta_{(2,k)}\pi_{(2,k)} \\ &\quad -((a_{k+1} \cdots a_{2k})\omega_k(a_1 \cdots a_k)\omega_{k-1}(1 \pm \alpha_k))\theta_{(2,k)}\pi_{(2,k)}.\end{aligned}$$

Since

$$\begin{aligned}((a_1\cdots a_k)(a_{k+1}\cdots a_{2k})(1\pm\alpha_k))\theta_{(2,k)} &= (a_1\cdots a_k a_{k+1}\cdots a_{2k})\theta_{(2,k)}\\ &\quad \pm(a_1\cdots a_k a_{2k}\cdots a_{k+1})\theta_{(2,k)}\\ &= (a_1\cdots a_k a_{2k}\cdots a_{k+1})\\ &\quad \pm(a_1\cdots a_k a_{k+1}\cdots a_{2k})\\ &= \pm(a_1\cdots a_k)(a_{k+1}\cdots a_{2k})(1\pm\alpha_k),\end{aligned}$$

we obtain

$$\begin{aligned}u\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)} &= (-1)^{k-1}((a_1\cdots a_k)\omega_k(a_{k+1}\cdots a_{2k})\omega_{k-1}(1\pm\alpha_k))\pi_{(2,k)}\\ &\quad -(-1)^{k-1}((a_{k+1}\cdots a_{2k})\omega_k(a_1\cdots a_k)\omega_{k-1}(1\pm\alpha_k))\pi_{(2,k)}\\ &= (-1)^{k-1}((a_1\cdots a_k)\omega_k(a_{k+1}\cdots a_{2k})\omega_k)\pi_{(2,k)}\\ &\quad -(-1)^{k-1}((a_{k+1}\cdots a_{2k})\omega_k(a_1\cdots a_k)\omega_k)\pi_{(2,k)}\\ &= (-1)^{k-1}u\lambda_{2k}\pi_{(2,k)}.\end{aligned}$$

It is now straightforward to show that $u\lambda_{2k}\pi_{(2,k)} = 2k^2u$,

$$\begin{aligned}u\lambda_{2k}\pi_{(2,k)} &= ([a_1,\ldots,a_k]\lambda_k\otimes[a_{k+1},\ldots,a_{2k}]\lambda_k)\,\pi_{(2,k)}\\ &\quad -([a_{k+1},\ldots,a_{2k}]\lambda_k\otimes[a_1,\ldots,a_k]\lambda_k)\,\pi_{(2,k)}\\ &= ((a_1\cdots a_k)\omega_k(a_{k+1}\cdots a_{2k})\omega_k-(a_{k+1}\cdots a_{2k})\omega_k(a_1\cdots a_k)\omega_k)\,\pi_{(2,k)}\\ &= [(a_1\cdots a_k)\omega_k\pi_k,(a_{k+1}\cdots a_{2k})\omega_k\pi_k]\\ &\quad -[(a_{k+1}\cdots a_{2k})\omega_k\pi_k,(a_1\cdots a_k)\omega_k\pi_k]\\ &= [[a_1,\ldots,a_k]\lambda_k\pi_k,[a_{k+1},\ldots,a_{2k}]\lambda_k\pi_k]\\ &\quad -[[a_{k+1},\ldots,a_{2k}]\lambda_k\pi_k,[a_1,\ldots,a_k]\lambda_k\pi_k]\\ &= 2[[a_1,\ldots,a_k]\lambda_k\pi_k,[a_{k+1},\ldots,a_{2k}]\lambda_k\pi_k]\\ &= 2k^2[[a_1,\ldots,a_k],[a_{k+1},\ldots,a_{2k}]]\\ &= 2k^2u.\end{aligned}$$

This completes the proof. □

In particular, we have shown that in characteristic $p = 2$ we have that

$$L_{2k}(V) = L_2(L_k(V)) \oplus B_{2k},$$

for all odd $k > 0$. Unfortunately, our proof does not yield any new information about the modules B_{2k}.

We conclude this section with a few remarks. Notice that the map $\phi'_{(2,k)}$ given as the composite $\frac{1}{2k^2}\lambda_{2k}\pi_{(2,k)}$ is not a projection. Indeed, although we have that $u\phi'_{(2,k)} = u$ for all $u \in L_2(L_k(V))$, the image of $\lambda_{2k}\pi_{(2,k)}$ is not divisible by 2 over $\mathbb{Z}$. Hence, in characteristic 2, $\frac{1}{2k^2}\lambda_{2k}\pi_{(2,k)}$ is not a well defined map.

The key idea is to choose $\theta_{(2,k)} : V^{\otimes 2k} \to V^{\otimes 2k}$ in such a way that the composite $\phi_{2k} = \frac{1}{2k^2}\lambda_{2k}\theta_{(2,k)}\pi_{(2,k)}$ is a well defined projection. It would be nice to generalise this approach to the case $p > 2$. That is, let

$$\phi_{(p,k)} : L_{pk}(V) \to L_p(L_k(V))$$

be given as the composite $\phi_{(p,k)} = \frac{1}{pk^p}\lambda_{pk}\theta_{(p,k)}\pi_{(p,k)}$ where

$$L_{pk}(V) \xrightarrow{\lambda_{pk}} V^{\otimes pk} \xrightarrow{\theta_{(p,k)}} V^{\otimes pk} \xrightarrow{\pi_{(p,k)}} L_p(L_k(V)),$$

with

$$\begin{aligned} \lambda_{pk} : [a_1, \ldots, a_{pk}] &\mapsto (a_1 \cdots a_{pk})\omega_{pk}, \\ \pi_{(p,k)} : a_1 \cdots a_k \cdots a_{(p-1)k+1} \cdots a_{pk} &\mapsto [[a_1, \ldots, a_k], \ldots, [a_{(p-1)k+1}, \ldots, a_{pk}]] \end{aligned}$$

and $\theta_{(p,k)} \in \mathbb{Z}\mathrm{Sym}(pk)$. We might wonder if we can always choose $\theta_{(p,k)}$ such that $\phi_{(p,k)}$ is a projection. In fact, making such a choice is not at all easy, as we shall see from the following example.

Finally, we note that $u\lambda_{pk}\pi_{(p,k)} = pk^p u$ for all $u \in L_p(L_k(V))$.

5.3 An illustrative example

In this section we consider decompositions of the sixth Lie power $L_6(V)$ in characteristic 2 and characteristic 3. We shall begin with the somewhat easier case of characteristic 3.

5.3.1 The sixth Lie power in characteristic three

By Theorem 5.1.1 (the Decomposition Theorem) we have that for finite-dimensional modules V,

$$L_6(V) = L_3(L_2(V)) \oplus B_6^{(3)},$$

where we write $B_6^{(3)}$, and later $B_6^{(2)}$, to denote the modules B_6 of Theorem 5.1.1 in characteristic 3 and 2 respectively. We shall verify that $L_3(L_2(V))$ is a direct summand of $L_6(V)$ by constructing a projection $\phi_{(3,2)} : L_6(V) \to L_3(L_2(V))$. Since $L_3(L_2(V))$ is a submodule of $L_6(V)$, we must then have that $L_3(L_2(V))$ is a direct summand of $L_6(V)$. We construct our projection as a composition of maps $\phi_{(3,2)} = \frac{1}{24}\lambda_6\theta_{(3,2)}\pi_{(3,2)}$, where

$$L_6(V) \xrightarrow{\lambda_6} V^{\otimes 6} \xrightarrow{\theta_{(3,2)}} V^{\otimes 6} \xrightarrow{\pi_{(3,2)}} L_3(L_2(V)).$$

Here, λ_6 denotes the canonical embedding of $L_6(V)$ into $V^{\otimes 6}$, $\theta_{(3,2)} \in \mathbb{Z}\mathrm{Sym}(6)$ and $\pi_{(3,2)}$ is the natural projection of $V^{\otimes 6}$ onto $L_3(L_2(V))$:

$$\begin{aligned} \lambda_6 : [a_1, a_2, a_3, a_4, a_5, a_6] &\mapsto (a_1a_2a_3a_4a_5a_6)\omega_6, \\ \pi_{(3,2)} : a_1a_2a_3a_4a_5a_6 &\mapsto [[a_1, a_2], [a_3, a_4], [a_5, a_6]]. \end{aligned}$$

We shall show that this map is a well-defined projection for arbitrary modules V and hence that $L_3(L_2(V))$ is a direct summand of $L_6(V)$ for all modules V (not just finite-dimensional modules).

We then give a description of $B_6^{(3)}$ in terms of our map $\phi_{(3,2)}$. Indeed,

$$B_6^{(3)} = L_6(V)(1 - \phi_{(3,2)}) = < (abcdef)e_{(3,2)}\pi_6 : a, b, c, d, e, f \in V >,$$

where $e_{(3,2)}$ is an idempotent element of $\mathbb{Z}_3\mathrm{Sym}(6)$.

Finally, we show that there exists an isomorphism,

$$B_6^{(3)} \cong L_2(V) \otimes S_2(V) \otimes S_2(V), \tag{5.3.1}$$

of KG-modules. It follows immediately that $B_6^{(3)}$ is isomorphic to a direct summand of $V^{\otimes 6}$. We prove the existence of the isomorphism (5.3.1) by constructing a surjection ψ_6 of $L_6(V)$ onto $L_2(V) \otimes S_2(V) \otimes S_2(V)$ with kernel $L_3(L_2(V))$.

We begin by defining a projection of $L_6(V)$ onto $L_3(L_2(V))$.

Theorem 5.3.1. *Let K be a field of characteristic $p \neq 2$, G a group and let V be a KG-module. Let $\phi_{(3,2)} : L_6(V) \to L_3(L_2(V))$ be the KG-module homomorphism given by $\phi_{(3,2)} = \frac{1}{24}\lambda_6\theta_{(3,2)}\pi_{(3,2)}$, where λ_6 is the canonical embedding of $L_6(V)$ into $V^{\otimes 6}$,*

$$\begin{aligned} \theta_{(3,2)} : a_1a_2a_3a_4a_5a_6 &\mapsto (a_1a_2a_3a_4a_5a_6)(1-(1\ 3)-(1\ 4)) \\ \textit{and } \pi_{(3,2)} : a_1a_2a_3a_4a_5a_6 &\mapsto [[a_1,a_2],[a_3,a_4],[a_5,a_6]]. \end{aligned}$$

Then $\phi_{(3,2)}$ is a projection of $L_6(V)$ onto $L_3(L_2(V))$. Hence, $L_3(L_2(V))$ is a KG-module direct summand of $L_6(V)$.

Proof. We first remark that $\phi_{(3,2)}$ is well-defined over K, since, as we shall see, if working over $\mathbb{Z}$ we have that $v\lambda_6\theta_{(3,2)}\pi_{(3,2)} \in 3L_3(L_2(V))$ for all $v \in L_6(V)$. Thus we may divide by 3. Also, we have that 2 is invertible in K, so the definition of $\phi_{(3,2)}$ makes sense over K. We shall prove:

(i) $v\lambda_6\theta_{(3,2)}\pi_{(3,2)} \in 3L_3(L_2(V))$ for all $v \in L_6(V)$ and

(ii) $u\lambda_6\theta_{(3,2)}\pi_{(3,2)} = 24u$ for all $u \in L_3(L_2(V))$.

Hence,

$$\begin{aligned} v\phi_{(3,2)}^2 &= \left(\frac{1}{24}v\lambda_6\theta_{(3,2)}\pi_{(3,2)}\right)\phi_{(3,2)} \\ &= \frac{1}{24}\left(3\sum_i u_i\right)\phi_{(3,2)}, \quad \text{where } u_i \in L_3(L_2(V)) \\ &= \frac{1}{24}\left(3\sum_i u_i\phi_{(3,2)}\right) = \frac{1}{24}\left(\frac{3}{24}\sum_i u_i\lambda_6\theta_{(3,2)}\pi_{(3,2)}\right) \\ &= \frac{1}{24}\left(\frac{3}{24}\sum_i 24u_i\right) = \frac{1}{24}\left(3\sum_i u_i\right) = \frac{1}{24}v\lambda_6\theta_{(3,2)}\pi_{(3,2)} = v\phi_{(3,2)}. \end{aligned}$$

Thus $\phi_{(3,2)}$ is a projection. Since $\phi_{(3,2)}$ is a projection of $L_6(V)$ onto one of its submodules, $L_3(L_2(V))$, we must have that $L_3(L_2(V))$ is a direct summand of $L_6(V)$ over any field of characteristic $p \neq 2$. A complement of $L_3(L_2(V))$ inside $L_6(V)$ is given by $L_6(V)(1 - \phi_{(3,2)})$. It remains to prove (i) and (ii).

Since $L_6(V)$ is spanned by left-normed monomials, it is enough to prove (i) in the case $v = [a, b, c, d, e, f] \in L_6(V)$. Then $v\lambda_6\theta_{(3,2)} = v\lambda_6 - v\lambda_6(1\ 3) - v\lambda_6(1\ 4)$, where

$$\begin{array}{rcrrrr}
v\lambda_6 & = & ab\,cd\,ef & -ba\,cd\,ef & -ca\,bd\,ef & +cb\,ad\,ef \\
& & -da\,bc\,ef & +db\,ac\,ef & +dc\,ab\,ef & -dc\,ba\,ef \\
& & -ea\,bc\,df & +eb\,ac\,df & +ec\,ab\,df & -ec\,ba\,df \\
& & +ed\,ab\,cf & -ed\,ba\,cf & -ed\,ca\,bf & +ed\,cb\,af \\
& & -fa\,bc\,de & +fb\,ac\,de & +fc\,ab\,de & -fc\,ba\,de \\
& & +fd\,ab\,ce & -fd\,ba\,ce & -fd\,ca\,be & +fd\,cb\,ae \\
& & +fe\,ab\,cd & -fe\,ba\,cd & -fe\,ca\,bd & +fe\,cb\,ad \\
& & -fe\,da\,bc & +fe\,db\,ac & +fe\,dc\,ab & -fe\,dc\,ba,
\end{array}$$

$$\begin{array}{rcrrrr}
-v\lambda_6(1\ 3) & = & -cb\,ad\,ef & +ca\,bd\,ef & +ba\,cd\,ef & -ab\,cd\,ef \\
& & +ba\,dc\,ef & -ab\,dc\,ef & -ac\,db\,ef & +bc\,da\,ef \\
& & +ba\,ec\,df & -ab\,ec\,df & -ac\,eb\,df & +bc\,ea\,df \\
& & -ad\,eb\,cf & +bd\,ea\,cf & +cd\,ea\,bf & -cd\,eb\,af \\
& & +ba\,fc\,de & -ab\,fc\,de & -ac\,fb\,de & +bc\,fa\,de \\
& & -ad\,fb\,ce & +bd\,fa\,ce & +cd\,fa\,be & -cd\,fb\,ae \\
& & -ae\,fb\,cd & +be\,fa\,cd & +ce\,fa\,bd & -ce\,fb\,ad \\
& & +de\,fa\,bc & -de\,fb\,ac & -de\,fc\,ab & +de\,fc\,ba
\end{array}$$

and

$$
\begin{aligned}
-v\lambda_6(1\,4) \quad = \quad & -db\ ca\ ef \quad +da\ cb\ ef \quad +da\ bc\ ef \quad -db\ ac\ ef \\
& +ca\ bd\ ef \quad -cb\ ad\ ef \quad -bc\ ad\ ef \quad +ac\ bd\ ef \\
& +ca\ be\ df \quad -cb\ ae\ df \quad -bc\ ae\ df \quad +ac\ be\ df \\
& -bd\ ae\ cf \quad +ad\ be\ cf \quad +ad\ ce\ bf \quad -bd\ ce\ af \\
& +ca\ bf\ de \quad -cb\ af\ de \quad -bc\ af\ de \quad +ac\ bf\ de \\
& -bd\ af\ ce \quad +ad\ bf\ ce \quad +ad\ cf\ be \quad -bd\ cf\ ae \\
& -be\ af\ cd \quad +ae\ bf\ cd \quad +ae\ cf\ bd \quad -be\ cf\ ad \\
& +ae\ df\ bc \quad -be\ df\ ac \quad -ce\ df\ ab \quad +ce\ df\ ba.
\end{aligned}
$$

By anticommutativity, we see that all the elements in red cancel when we apply the map $\pi_{(3,2)}$ to $v\lambda_6\theta_{(3,2)}$. Hence $v\lambda_6\theta_{(3,2)}\pi_{(3,2)}$ is equal to the following expression:

$$
\begin{aligned}
& [[a,b],[c,d],[e,f]] && -[[b,a],[c,d],[e,f]] && -[[c,a],[b,d],[e,f]] && +[[c,b],[a,d],[e,f]] \\
& -[[d,a],[b,c],[e,f]] && +[[d,b],[a,c],[e,f]] && +[[d,c],[a,b],[e,f]] && -[[d,c],[b,a],[e,f]] \\
& -[[e,a],[b,c],[d,f]] && +[[e,b],[a,c],[d,f]] && +[[e,c],[a,b],[d,f]] && -[[e,c],[b,a],[d,f]] \\
& +[[e,d],[a,b],[c,f]] && -[[e,d],[b,a],[c,f]] && -[[e,d],[c,a],[b,f]] && +[[e,d],[c,b],[a,f]] \\
& -[[f,a],[b,c],[d,e]] && +[[f,b],[a,c],[d,e]] && +[[f,c],[a,b],[d,e]] && -[[f,c],[b,a],[d,e]] \\
& +[[f,d],[a,b],[c,e]] && -[[f,d],[b,a],[c,e]] && -[[f,d],[c,a],[b,e]] && +[[f,d],[c,b],[a,e]] \\
& +[[f,e],[a,b],[c,d]] && -[[f,e],[b,a],[c,d]] && -[[f,e],[c,a],[b,d]] && +[[f,e],[c,b],[a,d]] \\
& -[[f,e],[d,a],[b,c]] && +[[f,e],[d,b],[a,c]] && +[[f,e],[d,c],[a,b]] && -[[f,e],[d,c],[b,a]] \\
& +[[b,a],[e,c],[d,f]] && -[[a,b],[e,c],[d,f]] && -[[a,c],[e,b],[d,f]] && +[[b,c],[e,a],[d,f]] \\
& -[[a,d],[e,b],[c,f]] && +[[b,d],[e,a],[c,f]] && +[[c,d],[e,a],[b,f]] && -[[c,d],[e,b],[a,f]] \\
& +[[b,a],[f,c],[d,e]] && -[[a,b],[f,c],[d,e]] && -[[a,c],[f,b],[d,e]] && +[[b,c],[f,a],[d,e]] \\
& -[[a,d],[f,b],[c,e]] && +[[b,d],[f,a],[c,e]] && +[[c,d],[f,a],[b,e]] && -[[c,d],[f,b],[a,e]] \\
& -[[a,e],[f,b],[c,d]] && +[[b,e],[f,a],[c,d]] && +[[c,e],[f,a],[b,d]] && -[[c,e],[f,b],[a,d]] \\
& +[[d,e],[f,a],[b,c]] && -[[d,e],[f,b],[a,c]] && -[[d,e],[f,c],[a,b]] && +[[d,e],[f,c],[b,a]] \\
& -[[b,d],[a,e],[c,f]] && +[[a,d],[b,e],[c,f]] && +[[a,d],[c,e],[b,f]] && -[[b,d],[c,e],[a,f]] \\
& -[[b,d],[a,f],[c,e]] && +[[a,d],[b,f],[c,e]] && +[[a,d],[c,f],[b,e]] && -[[b,d],[c,f],[a,e]] \\
& -[[b,e],[a,f],[c,d]] && +[[a,e],[b,f],[c,d]] && +[[a,e],[c,f],[b,d]] && -[[b,e],[c,f],[a,d]] \\
& +[[a,e],[d,f],[b,c]] && -[[b,e],[d,f],[a,c]] && -[[c,e],[d,f],[a,b]] && +[[c,e],[d,f],[b,a]].
\end{aligned}
$$

The terms above have been coloured to make it clear how the above expression can be simplified. Gathering together like terms and using the Jacobi identity (repeatedly) yields:

$$\begin{aligned} v\lambda_6\theta_{(3,2)}\pi_{(3,2)} = \quad & 6[[a,b],[c,d],[e,f]] \quad +3[[a,c],[b,d],[e,f]] \quad +3[[a,d],[b,c],[e,f]] \\ & +6[[a,b],[c,e],[d,f]] \quad +3[[a,c],[b,e],[d,f]] \quad +3[[a,e],[b,c],[d,f]] \\ & +6[[a,b],[c,f],[d,e]] \quad +3[[a,d],[b,e],[c,f]] \quad +3[[a,e],[b,d],[c,f]] \\ & +3[[a,c],[b,f],[d,e]] \quad +3[[a,e],[b,f],[c,d]] \quad +3[[a,d],[b,f],[c,e]] \\ & +3[[a,f],[b,c],[d,e]] \quad +3[[a,f],[b,d],[c,e]] \quad +3[[a,f],[b,e],[c,d]]. \end{aligned}$$

Hence we have $v\lambda_6\theta_{(3,2)}\pi_{(3,2)} \in 3L_3(L_2)$ as required.

Since $L_3(L_2(V))$ is spanned by monomials it is enough to prove (ii) in the case $u = [[a,b],[c,d],[e,f]] \in L_3(L_2(V))$. Then $u\lambda_6\theta = u\lambda_6 - u\lambda_6(1\ 3) - u\lambda_6(1\ 4)$, where

$$\begin{aligned} u\lambda_6 \quad = \quad & ab\ cd\ ef \quad -ba\ cd\ ef \quad -ab\ dc\ ef \quad +ba\ dc\ ef \\ & -ab\ cd\ fe \quad +ba\ cd\ fe \quad +ab\ dc\ fe \quad -ba\ dc\ fe \\ & -cd\ ab\ ef \quad +cd\ ba\ ef \quad +dc\ ab\ ef \quad -dc\ ba\ ef \\ & +cd\ ab\ fe \quad -cd\ ba\ fe \quad -dc\ ab\ fe \quad +dc\ ba\ fe \\ & -ef\ ab\ cd \quad +ef\ ba\ cd \quad +ef\ ab\ dc \quad -ef\ ba\ dc \\ & +fe\ ab\ cd \quad -fe\ ba\ cd \quad -fe\ ab\ dc \quad +fe\ ba\ dc \\ & +ef\ cd\ ab \quad -ef\ cd\ ba \quad -ef\ dc\ ab \quad +ef\ dc\ ba \\ & -fe\ cd\ ab \quad +fe\ cd\ ba \quad +fe\ dc\ ab \quad -fe\ dc\ ba, \\ -u\lambda_6(1\ 3) \quad = \quad & -cb\ ad\ ef \quad +ca\ bd\ ef \quad +db\ ac\ ef \quad -da\ bc\ ef \\ & +cb\ ad\ fe \quad -ca\ bd\ fe \quad -db\ ac\ fe \quad +da\ bc\ fe \\ & +ad\ cb\ ef \quad -bd\ ca\ ef \quad -ac\ db\ ef \quad +bc\ da\ ef \\ & -ad\ cb\ fe \quad +bd\ ca\ fe \quad +ac\ db\ fe \quad -bc\ da\ fe \\ & +af\ eb\ cd \quad -bf\ ea\ cd \quad -af\ eb\ dc \quad +bf\ ea\ dc \\ & -ae\ fb\ cd \quad +be\ fa\ cd \quad +ae\ fb\ dc \quad -be\ fa\ dc \\ & -cf\ ed\ ab \quad +cf\ ed\ ba \quad +df\ ec\ ab \quad -df\ ec\ ba \\ & +ce\ fd\ ab \quad -ce\ fd\ ba \quad -de\ fc\ ab \quad +de\ fc\ ba \end{aligned}$$

and

$$\begin{array}{rcllll} u\lambda_6(1\ 4) & = & -db\ ca\ ef & +da\ cb\ ef & +cb\ da\ ef & -ca\ db\ ef \\ & & +db\ ca\ fe & -da\ cb\ fe & -cb\ da\ fe & +ca\ db\ fe \\ & & +bd\ ac\ ef & -ad\ bc\ ef & -bc\ ad\ ef & +ac\ bd\ ef \\ & & -bd\ ac\ fe & +ad\ bc\ fe & +bc\ ad\ fe & -ac\ bd\ fe \\ & & +bf\ ae\ cd & -af\ be\ cd & -bf\ ae\ dc & +af\ be\ dc \\ & & -be\ af\ cd & +ae\ bf\ cd & +be\ af\ dc & -ae\ bf\ dc \\ & & -df\ ce\ ab & +df\ ce\ ba & +cf\ de\ ab & -cf\ de\ ba \\ & & +de\ cf\ ab & -de\ cf\ ba & -ce\ df\ ab & +ce\ df\ ba. \end{array}$$

It can be seen, using anticommutativity, that both $u\lambda_6(1\ 3)$ and $u\lambda_6(1\ 4)$ are mapped to zero by $\pi_{(3,2)}$ and hence,

$$u\lambda_6\theta_{(3,2)}\pi_{(3,2)} = u\lambda_6\pi_{(3,2)} = 24u,$$

as required. □

In fact, we have the following more general result:

Proposition 5.3.2. *Let K be a field, G a group, and let V be a KG-module. Let u be the element of $L_p(L_2(V))$ given by $u = [[a_1, a_2], [a_3, a_4], [a_5, a_6], \ldots, [a_{2p-1}, a_{2p}]]$. Let λ_{2p} denote the canonical embedding of $L_{2p}(V)$ into $V^{\otimes 2p}$, let $\theta_{(p,2)}$ be the element of $\mathbb{Z}\mathrm{Sym}(2p)$ given by $\theta_{(p,2)} = 1 - (1\ 3) - (1\ 4)$ and let $\pi_{(p,2)}$ be the natural projection of $V^{\otimes 2p}$ onto $L_p(L_2(V))$. Then*

$$u\lambda_{2p}\theta_{(p,2)}\pi_{(p,2)} = u\lambda_{2p}\pi_{(p,2)} = p2^p u.$$

Proof. We have the following commutative diagram of KG-modules:

$$\begin{array}{ccc} L_p(L_2(V)) & \xrightarrow{\gamma} & L_p(V^{\otimes 2}) \\ \downarrow{\gamma'} & & \downarrow{\delta} \\ (L_2(V))^{\otimes p} & \xrightarrow{\delta'} & V^{\otimes 2p} \end{array}$$

where the maps γ, γ', δ and δ' are given as follows:

$$
\begin{aligned}
\gamma : [[a_1, a_2], \ldots, [a_{2p-1}, a_{2p}]] &\mapsto (a_1 a_2 \cdots a_{2p-1} a_{2p})(1 - t_1)(1 - t_2) \cdots (1 - t_p)\gamma_{2p}, \\
\delta : [a_1 \otimes a_2, \ldots, a_{2p-1} \otimes a_{2p}] &\mapsto (a_1 a_2 \cdots a_{2p-1} a_{2p})(1 - \tau_2) \cdots (1 - \tau_p), \\
\gamma' : [[a_1, a_2], \ldots, [a_{2p-1}, a_{2p}]] &\mapsto (a_1 a_2 \cdots a_{2p-1} a_{2p})(1 - \tau_2) \cdots (1 - \tau_p)\gamma'_{2p}, \\
\delta' : [a_1, a_2] \otimes \cdots \otimes [a_{2p-1}, a_{2p}] &\mapsto (a_1 a_2 \cdots a_{2p-1} a_{2p})(1 - t_1)(1 - t_2) \cdots (1 - t_p),
\end{aligned}
$$

with

$$
\begin{aligned}
t_i = (2i - 1 \;\; 2i), &\qquad \tau_i = (1\; 3\; \cdots\; 2i - 1)(2\; 4\; \cdots\; 2i) \in \mathrm{Sym}(2p), \\
\gamma_{2p} : a_1 a_2 \cdots a_{2p-1} a_{2p} &\mapsto [a_1 \otimes a_2, \ldots, a_{2p-1} \otimes a_{2p}], \\
\text{and } \gamma'_{2p} : a_1 a_2 \cdots a_{2p-1} a_{2p} &\mapsto [a_1, a_2] \otimes \cdots \otimes [a_{2p-1}, a_{2p}].
\end{aligned}
$$

It is easy to see that $\gamma\delta = \gamma'\delta' = \lambda_{2p}$. Thus, since

$$
\lambda_{2p} : [[a_1 a_2], \ldots, [a_{2p-1} a_{2p}]] \mapsto (a_1 a_2 \cdots a_{2p-1} a_{2p})\delta_{2p},
$$

where $\delta_{2p} \in \mathbb{Z}\mathrm{Sym}(2p)$, we must have that

$$
\begin{aligned}
\delta_{2p} &= (1 - t_1)(1 - t_2) \cdots (1 - t_p)(1 - \tau_2) \cdots (1 - \tau_p) \\
&= (1 - \tau_2) \cdots (1 - \tau_p)(1 - t_1)(1 - t_2) \cdots (1 - t_p).
\end{aligned}
$$

Since the t_i are disjoint we may permute the factors containing the t_i as follows:

$$
(1 - t_1)(1 - t_2) \cdots (1 - t_p) = (1 - t_{1\sigma})(1 - t_{2\sigma}) \cdots (1 - t_{p\sigma})
$$

for all $\sigma \in \mathrm{Sym}(p)$. Hence, we may write

$$
\begin{aligned}
\delta_{2p} &= (1 - t_{1\sigma})(1 - t_{2\sigma}) \cdots (1 - t_{p\sigma})(1 - \tau_2) \cdots (1 - \tau_p) \\
&= (1 - \tau_2) \cdots (1 - \tau_p)(1 - t_{1\sigma})(1 - t_{2\sigma}) \cdots (1 - t_{p\sigma}).
\end{aligned}
$$

In particular, we can choose $t_{p\sigma} = t_1 = (1\ 2)$. Now,

$$\begin{aligned}
(b_1b_2b_3b_4\cdots b_{2p-1}b_{2p})(1-(1\ 2))\theta_{(p,2)}\pi_{(p,2)} &= (b_1b_2b_3b_4\cdots b_{2p-1}b_{2p})\theta_{(p,2)}\pi_{(p,2)} \\
&\quad -(b_2b_1b_3b_4\cdots b_{2p-1}b_{2p})\theta_{(p,2)}\pi_{(p,2)} \\
&= b_1b_2b_3b_4\cdots b_{2p-1}b_{2p}\pi_{(p,2)} \\
&\quad -b_2b_1b_3b_4\cdots b_{2p-1}b_{2p}\pi_{(p,2)} \\
&\quad -b_3b_2b_1b_4\cdots b_{2p-1}b_{2p}\pi_{(p,2)} \\
&\quad +b_3b_1b_2b_4\cdots b_{2p-1}b_{2p}\pi_{(p,2)} \\
&\quad -b_4b_2b_3b_1\cdots b_{2p-1}b_{2p}\pi_{(p,2)} \\
&\quad +b_4b_1b_3b_2\cdots b_{2p-1}b_{2p}\pi_{(p,2)} \\
&= 2[[b_1,b_2],[b_3,b_4],\ldots[b_{2p-1},b_{2p}]] \\
&= 2(b_1b_2b_3b_4\cdots b_{2p-1}b_{2p})\pi_{(p,2)}
\end{aligned}$$

and

$$\begin{aligned}
(b_1b_2b_3b_4\cdots b_{2p-1}b_{2p})(1-(1\ 2))\pi_{(p,2)} &= (b_1b_2b_3b_4\cdots b_{2p-1}b_{2p})\pi_{(p,2)} \\
&\quad -(b_2b_1b_3b_4\cdots b_{2p-1}b_{2p})\pi_{(p,2)} \\
&= 2[[b_1,b_2],[b_3,b_4],\ldots[b_{2p-1},b_{2p}]] \\
&= 2(b_1b_2b_3b_4\cdots b_{2p-1}b_{2p})\pi_{(p,2)}.
\end{aligned}$$

Hence $\lambda_{2p}\theta_{(p,2)}\pi_{(p,2)} = \lambda_{2p}\pi_{(p,2)}$, which gives

$$u\lambda_{2p}\theta_{(p,2)}\pi_{(p,2)} = u\lambda_{2p}\pi_{(p,2)} = p2^pu$$

as required. □

We now give a description of $B_6^{(3)}$ of $L_3(L_2(V))$ in $L_6(V)$ using our projection $\phi_{(3,2)}$.

Corollary 5.3.3. *Let K be a field of characteristic 3, G a group, and V a KG-module. A KG-module complement $B_6^{(3)}$ of $L_3(L_2(V))$ in $L_6(V)$ is given by*

$$B_6^{(3)} =< (abcdef)e_6^{(3)}\pi_6 : a,b,c,d,e,f \in V >$$

where $e_6^{(3)} = 1 + \beta_1\beta_2\beta_3\beta_4 \in \mathbb{Z}_3\mathrm{Sym}(6)$, *with*

$$\begin{array}{llll} \beta_1 = & 2 + (2\,3) + (2\,3\,4) + (2\,3\,4\,5) + (2\,3\,4\,5\,6), & \beta_2 = & 1 + (4\,5) + (4\,5\,6), \\ \beta_3 = & 1 + 2(5\,6), & \beta_4 = & 1 + 2(3\,4). \end{array}$$

Proof. We have that

$$\begin{aligned} B_6^{(3)} &= L_6(V)(1-\phi_6^{(3)}) \\ &= < [a,b,c,d,e,f](1-\phi_6^{(3)}) : a,b,c,d,e,f \in V > . \end{aligned}$$

Applying this map to a left-normed commutator and multiplying both sides by 8, we see that

$$8[a,b,c,d,e,f](1-\phi_{(3,2)}) =$$

$$\begin{array}{rrrr}
6[a,b,c,d,e,f] & +2[a,b,d,c,e,f] & +2[a,b,c,d,f,e] & -2[a,b,d,c,f,e] \\
-2[a,b,c,e,d,f] & +2[a,b,e,c,d,f] & +2[a,b,c,e,f,d] & -2[a,b,e,c,f,d] \\
-2[a,b,c,f,d,e] & +2[a,b,f,c,d,e] & +2[a,b,c,f,e,d] & -2[a,b,f,c,e,d] \\
-\ [a,c,b,d,e,f] & +\ [a,c,d,b,e,f] & +\ [a,c,b,d,f,e] & -\ [a,c,d,b,f,e] \\
-\ [a,c,b,e,d,f] & +\ [a,c,e,b,d,f] & +\ [a,c,b,e,f,d] & -\ [a,c,e,b,f,d] \\
-\ [a,c,b,f,d,e] & +\ [a,c,f,b,d,e] & +\ [a,c,b,f,e,d] & -\ [a,c,f,b,e,d] \\
-\ [a,d,b,c,e,f] & +\ [a,d,c,b,e,f] & +\ [a,d,b,c,f,e] & -\ [a,d,c,b,f,e] \\
-\ [a,d,b,e,c,f] & +\ [a,d,e,b,c,f] & +\ [a,d,b,e,f,c] & -\ [a,d,e,b,f,c] \\
-\ [a,d,b,f,c,e] & +\ [a,d,f,b,c,e] & +\ [a,d,b,f,e,c] & -\ [a,d,f,b,e,c] \\
-\ [a,e,b,c,d,f] & +\ [a,e,c,b,d,f] & +\ [a,e,b,c,f,d] & -\ [a,e,c,b,f,d] \\
-\ [a,e,b,d,c,f] & +\ [a,e,d,b,c,f] & +\ [a,e,b,d,f,c] & -\ [a,e,d,b,f,c] \\
-\ [a,e,b,f,c,d] & +\ [a,e,f,b,c,d] & +\ [a,e,b,f,d,c] & -\ [a,e,f,b,d,c] \\
-\ [a,f,b,c,d,e] & +\ [a,f,c,b,d,e] & +\ [a,f,b,c,e,d] & -\ [a,f,c,b,e,d] \\
-\ [a,f,b,d,c,e] & +\ [a,f,d,b,c,e] & +\ [a,f,b,d,e,c] & -\ [a,f,d,b,e,c] \\
-\ [a,f,b,e,c,d] & +\ [a,f,e,b,c,d] & +\ [a,f,b,e,d,c] & -\ [a,f,e,b,d,c].
\end{array}$$

In characteristic 3 this yields

$[a,b,c,d,e,f](1-\phi_{(3,2)}) =$

$$\begin{array}{llll}
[a,b,d,c,e,f] & +\ [a,b,c,d,f,e] & +2[a,b,d,c,f,e] & \\
+2[a,b,c,e,d,f] & +\ [a,b,e,c,d,f] & +\ [a,b,c,e,f,d] & +2[a,b,e,c,f,d] \\
+2[a,b,c,f,d,e] & +\ [a,b,f,c,d,e] & +\ [a,b,c,f,e,d] & +2[a,b,f,c,e,d] \\
+\ [a,c,b,d,e,f] & +2[a,c,d,b,e,f] & +2[a,c,b,d,f,e] & +\ [a,c,d,b,f,e] \\
+\ [a,c,b,e,d,f] & +2[a,c,e,b,d,f] & +2[a,c,b,e,f,d] & +\ [a,c,e,b,f,d] \\
+\ [a,c,b,f,d,e] & +2[a,c,f,b,d,e] & +2[a,c,b,f,e,d] & +\ [a,c,f,b,e,d] \\
+\ [a,d,b,c,e,f] & +2[a,d,c,b,e,f] & +2[a,d,b,c,f,e] & +\ [a,d,c,b,f,e] \\
+\ [a,d,b,e,c,f] & +2[a,d,e,b,c,f] & +2[a,d,b,e,f,c] & +\ [a,d,e,b,f,c] \\
+\ [a,d,b,f,c,e] & +2[a,d,f,b,c,e] & +2[a,d,b,f,e,c] & +\ [a,d,f,b,e,c] \\
+\ [a,e,b,c,d,f] & +2[a,e,c,b,d,f] & +2[a,e,b,c,f,d] & +\ [a,e,c,b,f,d] \\
+\ [a,e,b,d,c,f] & +2[a,e,d,b,c,f] & +2[a,e,b,d,f,c] & +\ [a,e,d,b,f,c] \\
+\ [a,e,b,f,c,d] & +2[a,e,f,b,c,d] & +2[a,e,b,f,d,c] & +\ [a,e,f,b,d,c] \\
+\ [a,f,b,c,d,e] & +2[a,f,c,b,d,e] & +2[a,f,b,c,e,d] & +\ [a,f,c,b,e,d] \\
+\ [a,f,b,d,c,e] & +2[a,f,d,b,c,e] & +2[a,f,b,d,e,c] & +\ [a,f,d,b,e,c] \\
+\ [a,f,b,e,c,d] & +2[a,f,e,b,c,d] & +2[a,f,b,e,d,c] & +\ [a,f,e,b,d,c],
\end{array}$$

and it is straightforward to check that this is equal to $(abcdef)e_6^{(3)}\pi_6$, as required.

$\square$

We now show that $B_6^{(3)} \cong L_2(V) \otimes S_2(V) \otimes S_2(V)$ by defining a surjection

$$\psi_6 : L_6(V) \to L_2(V) \otimes S_2(V) \otimes S_2(V).$$

Theorem 5.3.4. *Let K be a field of characteristic 3, G a group, and V a KG-module. Then*

$$L_6(V) \cong L_3(L_2(V)) \oplus [L_2(V) \otimes S_2(V) \otimes S_2(V)]. \tag{5.3.2}$$

We shall show that in characteristic 3 there exists a KG-module epimorphism ψ_6 of $L_6(V)$ onto $L_2(V) \otimes S_2(V) \otimes S_2(V)$ with $L_3(L_2(V)) \subseteq \ker(\psi_6)$, thus giving

$$\operatorname{Im}\psi_6 = L_2(V) \otimes S_2(V) \otimes S_2(V) \cong L_6(V)/\ker(\psi_6).$$

In fact, for such an epimorphism ψ_6 we have that the kernel of ψ_6 coincides with $L_3(L_2(V))$. Indeed, suppose first that V is finite-dimensional, $\dim V = r$, say. Then

$$\begin{aligned}
\dim\ker(\psi_6) &= \dim L_6(V) - \dim\left(L_2(V) \otimes S_2(V) \otimes S_2(V)\right) \\
&= \frac{1}{6}(r^6 - r^3 - r^2 + r) - \frac{1}{2}(r^2 - r)\frac{1}{2}(r^2 + r)\frac{1}{2}(r^2 + r) \\
&= \frac{1}{3}((\frac{1}{2}(r^2 - r))^3 - \frac{1}{2}(r^2 - r)) \\
&= \dim L_3(L_2(V)).
\end{aligned}$$

Since $L_3(L_2(V)) \subseteq \ker(\psi_6)$, we deduce that, for finite-dimensional modules V, we must have $L_3(L_2(V)) = \ker(\psi_6)$. Hence (5.3.2) holds for all finite-dimensional KG-modules V.

Next suppose that V is infinite-dimensional and let $u \in \ker(\psi_6)$. We shall show that $u \in L_3(L_2(V))$ and, hence, $\ker(\psi_6) = L_3(L_2(V))$, giving that (5.3.2) holds for arbitrary modules V.

Let $u \in L_6(V)$. Then we can write u as a linear combination of standard monomials of degree 6,

$$u = \sum_{i=1}^{k} u_i.$$

Let X be the set of free generators of $L(V)$ occurring in the u_i. Since u is a linear combination of finitely many standard monomials, we must have that X is a finite set, $X = \{x_1, \ldots, x_r\}$, say. Let W be the K-span of X. Thus W is a finite-dimensional K-submodule of V and $u \in L(W)$. Now suppose that $u \in \ker(\psi_6)$. Since ψ_6 restricted to $L_6(W)$ has kernel $L_3(L_2(W))$ we have that $u \in L_3(L_2(W)) \subseteq L_3(L_2(V))$. Hence, we have that $\ker(\psi_6) = L_3(L_2(V))$ and

$$\begin{aligned} \operatorname{Im}\psi_6^{(3)} = L_2(V) \otimes S_2(V) \otimes S_2(V) &\cong L_6(V)/\ker(\psi_6) \\ &\cong L_6(V)/L_3(L_2(V)) \\ &\cong B_6^{(3)}, \end{aligned}$$

for arbitrary KG-modules V. It remains to prove the existence of ψ_6.

Lemma 5.3.5. *Let K be a field, G a group, and V a KG-module. Let $\sigma \in \mathrm{Sym}(6)$ and let $\psi_\sigma : L_6(V) \to L_2(V) \otimes S_2(V) \otimes S_2(V)$ be the KG-module homomorphism given as the composite $\psi_\sigma = \lambda_6 \sigma \nu$ where*

$$\begin{aligned} \lambda_6 : [a_1, a_2, a_3, a_4, a_5, a_6] &\mapsto (a_1a_2a_3a_4a_5a_6)\omega_6, \\ \sigma : a_1a_2a_3a_4a_5a_6 &\mapsto (a_1a_2a_3a_4a_5a_6)\sigma, \\ \nu : a_1a_2a_3a_4a_5a_6 &\mapsto [a_1, a_2] \otimes a_3 \circ a_4 \otimes a_5 \circ a_6. \end{aligned}$$

Let H be the subgroup of $\mathrm{Sym}(6)$ given by $H =< (1\ 2), (3\ 4), (5\ 6), (1\ 6)(2\ 5)(3\ 4) >$. Then $\psi_\sigma = \pm\psi_{h\sigma}$ for all $h \in H$.

Proof. It is easy to see that

$$\begin{aligned} a_1a_2a_3a_4a_5a_6(1\ 2)\nu &= [a_2, a_1] \otimes a_3 \circ a_4 \otimes a_5 \circ a_6 \\ &= -[a_1, a_2] \otimes a_3 \circ a_4 \otimes a_5 \circ a_6, \\ a_1a_2a_3a_4a_5a_6(3\ 4)\nu &= [a_1, a_2] \otimes a_4 \circ a_3 \otimes a_5 \circ a_6 \\ &= [a_1, a_2] \otimes a_3 \circ a_4 \otimes a_5 \circ a_6, \\ \text{and} \qquad a_1a_2a_3a_4a_5a_6(5\ 6)\nu &= [a_1, a_2] \otimes a_3 \circ a_4 \otimes a_6 \circ a_5 \\ &= [a_1, a_2] \otimes a_3 \circ a_4 \otimes a_5 \circ a_6. \end{aligned}$$

We also have that $\omega_6 = -\omega_6(1\ 6)(2\ 5)(3\ 4)$.

For all $\sigma \in \mathrm{Sym}(6)$ let $([a_1, a_2] \otimes a_3 \circ a_4 \otimes a_5 \circ a_6)\sigma$ denote

$$[a_{1\sigma^{-1}}, a_{2\sigma^{-1}}] \otimes a_{3\sigma^{-1}} \circ a_{4\sigma^{-1}} \otimes a_{5\sigma^{-1}} \circ a_{6\sigma^{-1}}.$$

Then it is easy to see that $(a_1a_2a_3a_4a_5a_6)\sigma\nu = (a_1a_2a_3a_4a_5a_6)\nu\sigma$. This gives,

$$\begin{aligned}(a_1a_2a_3a_4a_5a_6)\psi_{(1\ 2)\sigma} &= (a_1a_2a_3a_4a_5a_6)\omega_6(1\ 2)\sigma\nu \\ &= (a_1a_2a_3a_4a_5a_6)\omega_6(1\ 2)\nu\sigma \\ &= -(a_1a_2a_3a_4a_5a_6)\omega_6\nu\sigma \\ &= -(a_1a_2a_3a_4a_5a_6)\omega_6\sigma\nu \\ &= -(a_1a_2a_3a_4a_5a_6)\psi_\sigma.\end{aligned}$$

Similarly, we see that

$$\begin{aligned}(a_1a_2a_3a_4a_5a_6)\psi_{(3\ 4)\sigma} &= (a_1a_2a_3a_4a_5a_6)\psi_\sigma, \\ \text{and} \quad (a_1a_2a_3a_4a_5a_6)\psi_{(5\ 6)\sigma} &= (a_1a_2a_3a_4a_5a_6)\psi_\sigma.\end{aligned}$$

Now,

$$\begin{aligned}(a_1a_2a_3a_4a_5a_6)\psi_{(1\ 6)(2\ 5)(3\ 4)\sigma} &= (a_1a_2a_3a_4a_5a_6)\omega_6(1\ 6)(2\ 5)(3\ 4)\sigma\nu \\ &= -(a_1a_2a_3a_4a_5a_6)\omega_6\sigma\nu \\ &= -(a_1a_2a_3a_4a_5a_6)\psi_\sigma.\end{aligned}$$

Finally, let $h_1, h_2 \in \{(1\ 2), (3\ 4), (5\ 6), (1\ 6)(2\ 5)(3\ 4)\}$ be two distinct generators of H. Then we have that $\psi_{h_1h_2\sigma} = \pm\psi_\sigma$. Indeed,

$$\begin{aligned}(a_1a_2a_3a_4a_5a_6)\psi_{h_1h_2\sigma} &= (a_1a_2a_3a_4a_5a_6)\omega_6h_1h_2\sigma\nu \\ &= (a_1a_2a_3a_4a_5a_6)\omega_6h_1h_2\nu\sigma.\end{aligned}$$

Now, if h_2 is a transposition, we may write

$$(a_1a_2a_3a_4a_5a_6)\psi_{h_1h_2\sigma} = \pm(a_1a_2a_3a_4a_5a_6)\omega_6h_1\nu\sigma$$

and hence it follows that

$$(a_1a_2a_3a_4a_5a_6)\psi_{h_1h_2\sigma} = \pm(a_1a_2a_3a_4a_5a_6)\omega_6\nu\sigma = \pm\psi_\sigma.$$

If $h_2 = (1\ 6)(2\ 5)(3\ 4)$ we see that

$$\begin{aligned}(1\ 2)(1\ 6)(2\ 5)(3\ 4) &= (1\ 6)(2\ 5)(3\ 4)(5\ 6),\\ (3\ 4)(1\ 6)(2\ 5)(3\ 4) &= (1\ 6)(2\ 5)(3\ 4)(3\ 4),\\ (5\ 6)(1\ 6)(2\ 5)(3\ 4) &= (1\ 6)(2\ 5)(3\ 4)(1\ 2).\end{aligned}$$

Hence, in this case, we can write $h_1h_2 = h_2h_1'$ for some transposition h_1' of H. The result follows. □

So we have that $\psi_\tau = \pm\psi_\sigma$ for all $\tau \in H\sigma$. For each coset $H\sigma$, we set $\psi_{H\sigma} = \psi_\sigma$, where σ is a distinguished representative of $H\sigma$.

Lemma 5.3.6. *Let K be a field of characteristic 3, let G be a group and let V be a KG-module. Let $\psi_{H\sigma}$ be defined as above. Then $\psi_{H\sigma}$ is a KG-module epimorphism of $L_6(V)$ onto $L_2(V) \otimes S_2(V) \otimes S_2(V)$ with $L_3(L_2(V))$ contained in its kernel if and only if $H\sigma$ is one of the following cosets:*

$$H(2\ 3\ 4), H(1\ 3\ 4\ 2), H(1\ 3\ 5\ 2)(4\ 6), H(2\ 5\ 3\ 6\ 4), H(1\ 5\ 3\ 6\ 4\ 2),\ \textit{or}\ H(1\ 5\ 2).$$

Proof. We shall find it helpful to use the image line notation for permutations. For example, the image line of $(2\ 3\ 4) \in \mathrm{Sym}(6)$ is 134256. Now, since the action of $\mathrm{Sym}(6)$ on $V^{\otimes 6}$ is given by

$$(a_1a_2a_3a_4a_5a_6)\sigma = a_{1\sigma^{-1}}a_{2\sigma^{-1}}a_{3\sigma^{-1}}a_{4\sigma^{-1}}a_{5\sigma^{-1}}a_{6\sigma^{-1}},$$

it is informative to record the image line of the inverse of each of our permutations. The image lines of the inverse of each of the above representative permutations are:

142356, 241356, 251634, 145623, 245613 and 253416.

There are $\frac{6!}{|H|} = \frac{720}{16} = 45$ cosets $H\sigma$. We shall first check how many of the maps $\psi_{H\sigma}$ have the property that $L_3(L_2(V)) \subseteq \ker\psi_{H\sigma}$. A straight-forward calculation shows that the only choices of $H\sigma$ which result in a map $\psi_{H\sigma}$ containing $L_3(L_2(V))$ in its kernel are given by the following representative inverse image lines:

$$\begin{array}{cccccc} 123456, & 152634, & 241356, & 125634, & 153426, & 245613, \\ 132456, & 162534, & 251634, & 135624, & 163425, & 253416, \\ 142356, & 231456, & 341256, & 145623, & 235614, & 345612. \end{array}$$

Indeed, we simply apply each map $\psi_{H\sigma}$ to a monomial element of $L_3(L_2(V))$. Next, let $x, y \in V$. Then it is easy to check that

$[x, y, y, y, x, x]\psi_{H\sigma} = 0, \quad \forall\sigma^{-1} \in \{123456, 125634, 132456, 135624, 231456, 235614\},$

$[[x, y, y, y], [x, y]]\psi_{H\sigma} = 0, \quad \forall\sigma^{-1} \in \{152634, 153426, 341256, 345612\}.$

Thus, we see that $L_3(L_2(V)) \subset \ker\psi_{H\sigma}$ and hence $\psi_{H\sigma}$ is not surjective in each of these cases. We prove that, of the remaining maps, the six listed in Lemma 5.3.6 are all surjective. Notice that if $\psi_{H\sigma}$ is a surjection then so is $\psi_{H\sigma(3\ 5)(4\ 6)}$, since

$$(a_1a_2a_3a_4a_5a_6)\sigma = a_{1\sigma^{-1}}a_{2\sigma^{-1}}a_{3\sigma^{-1}}a_{4\sigma^{-1}}a_{5\sigma^{-1}}a_{6\sigma^{-1}},$$

$$(a_1a_2a_3a_4a_5a_6)\sigma(3\ 5)(4\ 6) = a_{1\sigma^{-1}}a_{2\sigma^{-1}}a_{5\sigma^{-1}}a_{6\sigma^{-1}}a_{3\sigma^{-1}}a_{4\sigma^{-1}},$$

and the map given by $[a_1, a_2] \otimes a_3 \circ a_4 \otimes a_5 \circ a_6 \mapsto [a_1, a_2] \otimes a_5 \circ a_6 \otimes a_3 \circ a_4$ is an isomorphism. Thus we only consider the representatives $\sigma^{-1} \in \{142356, 241356, 251634\}$.

Let y be an arbitrary element of $L_2(V) \otimes S_2(V) \otimes S_2(V)$. For each map $\psi_{H\sigma}$ above, we need to find a preimage for y in $L_6(V)$. Since $L_2(V) \otimes S_2(V) \otimes S_2(V)$ is spanned by elements of the form $[a, b] \otimes c \circ d \otimes e \circ f$, it is enough to find a preimage for $y = [a, b] \otimes c \circ d \otimes e \circ f$. We see that such a preimage (if it exists) must be contained in the fine homogeneous component of multidegree (1^6) in $L(W)$, where $W = \langle a, b, c, d, e, f\rangle \subseteq V$. We set $a > b > c > d > e > f$ and let $\{u_1, \ldots, u_{90}\}$ be a fixed Hall basis of the fine homogeneous component $L_{(1^6)}(W)$, modulo $L_{(1^3)}(L_{(1^2)}(W))$. Let $\{v_1, \ldots, v_{90}\}$ be a fixed basis of $L_{(1^2)}(W) \otimes S_{(1^2)}(W) \otimes S_{(1^2)}(W)$ with $v_1 = y$. Let $\underline{u} = (u_1, \ldots, u_{90})$ and $\underline{v} = (v_1, \ldots, v_{90})$. Then for each $\psi_{H\sigma}$ we can write

$$\underline{u}\psi_{H\sigma} = A_\sigma\underline{v},$$

where A_σ is a 90×90 matrix with entries in $\mathbb{Z}_3$. If A_σ is invertible, $A_\sigma^{-1} = B_\sigma = (\beta_{i,j})$ say, then we can write

$$\underline{v} = B_\sigma\underline{u}\psi_{H\sigma}.$$

This gives

$$y = [a,b] \otimes c \circ d \otimes e \circ f = v_1 = \left(\sum_{j=1}^{90} \beta_{1,j} u_j \right) \psi_{H\sigma},$$

and hence we have a preimage. Therefore, in order to complete the proof of the lemma, we must check that A_σ is invertible for each $\sigma^{-1} \in \{142356, 241356, 251634\}$. This can be easily verified by means of a computer program (we also find that A_σ is not invertible for $\sigma^{-1} \in \{62534, 163425\}$). We give the preimage of $y = [a,b] \otimes c \circ d \otimes e \circ f$ in the case where $\sigma^{-1} = 142356$, that is, $\sigma = (2\ 3\ 4)$, below to illustrate.

$$\begin{array}{llll}
-[a,f,e,d,c,b] & +[b,f,e,d,c,a] & +[[a,d,c,b],[e,f]] & -[[b,d,c,a],[e,f]] \\
+[[a,e,d,c],[b,f]] & +[[c,e,d,a],[b,f]] & +[[d,e,c,a],[b,f]] & -[[b,e,d,c],[a,f]] \\
-[[c,e,d,b],[a,f]] & -[[d,e,c,b],[a,f]] & +[[a,f,d,c],[b,e]] & +[[c,f,d,a],[b,e]] \\
+[[d,f,c,a],[b,e]] & -[[b,f,d,c],[a,e]] & -[[c,f,d,b],[a,e]] & -[[d,f,c,b],[a,e]] \\
+[[a,f,e,b],[c,d]] & -[[b,f,e,a],[c,d]] & +[[e,f,c,a],[b,d]] & -[[e,f,c,b],[a,d]] \\
+[[e,f,d,a],[b,c]] & -[[e,f,d,b],[a,c]] & +[[c,f,e,d],[a,b]] & +[[d,f,e,c],[a,b]] \\
+[[e,f,d,c],[a,b]] & +[[a,f,b],[c,e,d]] & +[[a,f,b],[d,e,c]] & -[[a,f,c],[b,e,d]] \\
-[[a,f,c],[d,e,b]] & -[[a,f,d],[b,e,c]] & -[[a,f,d],[c,e,b]] & +[[a,f,e],[c,d,b]] \\
+[[a,e,b],[c,f,d]] & +[[a,e,b],[d,f,c]] & -[[a,e,c],[b,f,d]] & -[[a,e,c],[d,f,b]] \\
-[[a,e,d],[b,f,c]] & -[[a,e,d],[c,f,b]] & -[[a,d,b],[c,f,e]] & +[[a,d,c],[e,f,b]] \\
-[[a,c,b],[d,f,e]] & -[[b,f,a],[c,e,d]] & -[[b,f,a],[d,e,c]] & +[[b,f,c],[d,e,a]] \\
+[[b,f,d],[c,e,a]] & -[[b,f,e],[c,d,a]] & -[[b,e,a],[c,f,d]] & -[[b,e,a],[d,f,c]] \\
+[[b,e,c],[d,f,a]] & +[[b,e,d],[c,f,a]] & +[[b,d,a],[c,f,e]] & -[[b,d,c],[e,f,a]] \\
+[[c,d,a],[e,f,b]] & -[[c,d,b],[e,f,a]] & +[[b,c,a],[d,f,e]] &
\end{array}$$

The preimages for the other five maps can be found in Appendix A. This completes the proof of Lemma 5.3.6 and hence of Theorem 5.3.4. □

Finally, we verify that $B_6^{(3)}$ is isomorphic to a direct summand of $V^{\otimes 6}$. Indeed, in characteristic 3 we have that $V^{\otimes 2} \cong L_2(V) \oplus S_2(V)$. Hence, by identifying $V^{\otimes 6}$ with

$(V^{\otimes 2})^{\otimes 3}$ in the obvious way, we have that

$$\begin{aligned} V^{\otimes 6} &\cong (L_2(V) \oplus S_2(V))^{\otimes 3} \\ &\cong \quad [L_2(V) \otimes L_2(V) \otimes L_2(V)] \oplus [L_2(V) \otimes L_2(V) \otimes S_2(V)] \\ &\quad \oplus [L_2(V) \otimes S_2(V) \otimes L_2(V)] \oplus [L_2(V) \otimes S_2(V) \otimes S_2(V)] \\ &\quad \oplus [S_2(V) \otimes L_2(V) \otimes L_2(V)] \oplus [S_2(V) \otimes L_2(V) \otimes S_2(V)] \\ &\quad \oplus [S_2(V) \otimes S_2(V) \otimes L_2(V)] \oplus [S_2(V) \otimes S_2(V) \otimes S_2(V)] . \end{aligned}$$

Since $B_6^{(3)} \cong L_2(V) \otimes S_2(V) \otimes S_2(V)$, we see that $B_6^{(3)}$ is isomorphic to a direct summand of $V^{\otimes 6}$.

5.3.2 The sixth Lie power in characteristic two

In this subsection K denotes a field of characteristic 2. By Theorem 5.1.1 (the Decomposition Theorem), we have that for finite-dimensional modules V,

$$L_6(V) = L_2(L_3(V)) \oplus B_6^{(2)},$$

where $B_6^{(2)}$ denotes the module B_6 of Theorem 5.1.1 in characteristic 2. We have already seen, by Theorem 5.2.1, that $L_2(L_3(V))$ is a direct summand of $L_6(V)$ for arbitrary KG-modules V.

We now give a description of $B_6^{(2)}$ using the projection $\phi_{(2,3)}$ of Theorem 5.2.1.

Corollary 5.3.7. *Let K be a field of characteristic 2, G a group, and V a KG-module. A KG-module complement $B_6^{(2)}$ of $L_2(L_3(V))$ in $L_6(V)$ is given by*

$$B_6^{(2)} =< (abcdef)e_6^{(2)}\pi_6 : a, b, c, d, e, f \in V >$$

where $e_6^{(2)} = (4\ 6) + (5\ 6) + (4\ 6\ 5) + \kappa_1\kappa_2\kappa_3 \in \mathbb{Z}_2\mathrm{Sym}(6)$, *with*

$$\begin{aligned} \kappa_1 &= (3\ 6) + (3\ 6\ 4) + (3\ 6\ 4\ 5) \\ \kappa_2 &= 1 + (1\ 3\ 2\ 6) + (1\ 6\ 2\ 3) \\ \kappa_3 &= 1 + (4\ 5\ 6) + (4\ 6) + (4\ 5). \end{aligned}$$

Proof. We have that

$$\begin{aligned} B_6^{(2)} &= L_6(V)(1-\phi_{(2,3)}) \\ &= < [a,b,c,d,e,f](1-\phi_{(2,3)}) : a,b,c,d,e,f \in V > . \end{aligned}$$

Applying this map to a left-normed commutator gives

$$[a,b,c,d,e,f](1-\phi_{(2,3)}) = [a,b,c,d,e,f](1-\frac{1}{18}\lambda_6\theta_{(2,3)}\pi_{(2,3)}),$$

which can be expanded to give the following expression:

$$\frac{1}{9}\begin{pmatrix} 9[a,b,c,d,e,f] & -3[[a,b,c],[f,e,d]] & \\ -\ [[a,b,d],[f,e,c]] & +\ [[c,d,a],[f,e,b]] & -\ [[c,d,b],[f,e,a]] \\ -\ [[a,b,e],[f,d,c]] & +\ [[c,e,a],[f,d,b]] & -\ [[c,e,b],[f,d,a]] \\ +\ [[a,b,f],[d,e,c]] & +\ [[d,e,a],[f,c,b]] & -\ [[d,e,b],[f,c,a]] \end{pmatrix}.$$

So, in characteristic 2 we see that $[a,b,c,d,e,f](1-\phi_{(2,3)})$ is equal to

$$\begin{aligned} & [a,b,c,d,e,f] && +[[a,b,c],[f,e,d]] \\ + & [[a,b,d],[f,e,c]] && +[[c,d,a],[f,e,b]] && +[[c,d,b],[f,e,a]] \\ + & [[a,b,e],[f,d,c]] && +[[c,e,a],[f,d,b]] && +[[c,e,b],[f,d,a]] \\ + & [[a,b,f],[d,e,c]] && +[[d,e,a],[f,c,b]] && +[[d,e,b],[f,c,a]]. \end{aligned}$$

Next, using the Jacobi identity yields the fact that, in characteristic 2,

$$[[a,b,c],[d,e,f]] = [a,b,c,d,e,f]+[a,b,c,f,d,e]+[a,b,c,f,e,d]+[a,b,c,e,d,f].$$

The result follows. □

We conclude this chapter with a few remarks. First, notice that over a field of characteristic 2 the third tensor power has the following direct sum decomposition.

$$V^{\otimes 3} \cong L_3(V) \oplus L_3'(V) \oplus A_3(V)$$

where

$$\begin{aligned} L_3(V) &\cong V^{\otimes 3}\omega_3, \quad \text{with } \omega_3 = (1-(1\,2))(1-(1\,2\,3)), \\ L'_3(V) &\cong V^{\otimes 3}\omega'_3 \cong L_3(V), \quad \text{with } \omega'_3 = (2\,3)\omega'_3(2\,3), \\ \text{and } A_3(V) &\cong V^{\otimes 3}a_3, \quad \text{with } a_3 = 1+(1\,2\,3)+(1\,3\,2). \end{aligned}$$

Indeed, it is easily verified that ω_3, ω'_3 and a_3 are mutually orthogonal idempotents which sum to 1 in $\mathbb{Z}_2\mathrm{Sym}(6)$.

Since $B_6^{(3)}$ can be described up to isomorphism as a tensor product of $L_2(V)$ and its complement inside $V^{\otimes 2}$, namely $S_2(V)$, we might hope that a similar description can be found for $B_6^{(2)}$.

Conjecture 5.3.8. *Let K be a field of characteristic 2, let G be a group, V a KG-module. Then the module $B_6^{(2)}$ of Theorem 5.1.1 satisfies*

$$B_6^{(2)} \cong L_3(V) \otimes A_3(V).$$

Indeed, for finite-dimensional modules V, we can check that the dimensions of $B_6^{(2)}$ and $L_3(V) \otimes A_3(V)$ are equal. However, we have been unable to construct an explicit isomorphism (or indeed a surjection of $L_6(V)$ onto $L_3(V) \otimes A_3(V)$, as in §5.3.1), to support this claim. In fact, for finite-dimensional modules V, we shall see in the following chapter that Conjecture 5.3.8 does indeed hold. Moreover, we shall see that in general every module $B_{p^m k}$ can be expressed in terms of direct summands of the kth tensor power.

Chapter 6

Lie powers and Witt vectors

The majority of this chapter represents joint work with R. M. Bryant. Sections 1–4 comprise the results of a joint paper [7]. In Section 5 we outline a few specific consequences of these results, relating back to Chapter 5. All of our results are stated for finite-dimensional modules. In Section 6, we discuss how far these results can be generalised to arbitrary modules.

6.1 Introduction

Let G be a group and K a field. For any finite-dimensional KG-module V, let $L(V)$ be the free Lie algebra on V (the free Lie algebra generated by any basis of V), and regard $L(V)$ as a KG-module on which each element of G acts as a Lie algebra automorphism. Each homogeneous component $L_n(V)$ is a finite-dimensional submodule of $L(V)$, called the nth Lie power of V.

The central problem on Lie powers is to describe the modules $L_n(V)$ up to isomorphism. In this chapter we continue to concentrate upon the harder case, when K has prime characteristic p.

As we have already seen (Chapter 5), one of the fundamental results in characteristic p is the Decomposition Theorem of Bryant and Schocker [10], stated here as

Theorem 5.1.1. This reduces the study of arbitrary Lie powers of V to the study of certain Lie powers of p-power degree, namely, Lie powers $L_{p^i}(B_n)$, where, for each n, B_n is isomorphic to a certain direct summand of the nth tensor power $V^{\otimes n}$.

Let us write $n = p^m k$, where k is not divisible by p. In [11], a recursive formula was given for the modules $B_k, B_{pk}, B_{p^2k}, \ldots$, up to isomorphism. We state this as Theorem 6.2.1 below. However, the formula is rather intractable. It involves the Witt polynomials (as used to define operations on the ring of Witt vectors) and gives the $B_{p^m k}$ only as the components of a Witt vector with known 'ghost' components. The results of [10] and [11] give no explicit information about the modules $B_{p^m k}$ except that, as already mentioned, $B_{p^m k}$ is isomorphic to a direct summand of $V^{\otimes p^m k}$. In this chapter we shall give much more precise information.

As a motivating example, we consider the case where $k = 2$ and $p = 3$, as in Chapter 5. It is well known and easily verified that $V^{\otimes 2} \cong L_2(V) \oplus S_2(V)$, where $L_2(V)$ is the second Lie power of V and $S_2(V)$ is the symmetric square of V. The recursive formula from [11] gives $B_2 \cong L_2(V)$ and $3B_6 \oplus B_2^{\otimes 3} \cong L_2(V^{\otimes 3})$, where $3B_6$ denotes the direct sum of 3 isomorphic copies of B_6. It can be shown (see, for example, Chapter 5) that

$$B_6 \cong L_2(V) \otimes S_2(V) \otimes S_2(V). \tag{6.1.1}$$

Examples like this suggested that, in general, $B_{p^m k}$ is isomorphic to a direct sum of tensor products of direct summands of $V^{\otimes k}$, although we had no *a priori* reason to suspect this. In this chapter we will prove this fact.

We shall see that $V^{\otimes k} \cong \bigoplus_{d|k} \phi(d) U_{k,d}$, for certain modules $U_{k,d}$ indexed by the divisors d of k, where ϕ denotes Euler's function and $\phi(d)U_{k,d}$ denotes the direct sum of $\phi(d)$ isomorphic copies of $U_{k,d}$. Thus

$$V^{\otimes p^m k} \cong (V^{\otimes k})^{\otimes p^m} \cong \bigoplus_{\lambda \in \Lambda} U_{k,\lambda(1)} \otimes \cdots \otimes U_{k,\lambda(p^m)},$$

where Λ is an index set of cardinality k^{p^m} and for each $\lambda \in \Lambda$ there corresponds a p^m-tuple $(\lambda(1), \ldots, \lambda(p^m))$ of divisors of k. (Two elements $\lambda, \lambda' \in \Lambda$ may define the same p^m-tuple.) Our main result, Theorem 6.4.2, states that there is a subset Λ_0 of Λ such that

$$B_{p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} U_{k,\lambda(1)} \otimes \cdots \otimes U_{k,\lambda(p^m)}.$$

This is, of course, much stronger than the statement that $B_{p^m k}$ is isomorphic to a direct summand of $V^{\otimes p^m k}$.

In Theorem 6.4.3 we shall obtain a version of Theorem 6.4.2 in which the modules $U_{k,d}$ are replaced by a set of modules indexed only by those divisors c of k such that c and k/c are coprime. This is a sharpening of Theorem 6.4.2 in the case where k is not square-free (that is, where k is divisible by the square of some prime).

The modules $U_{k,d}$ are of well-established interest. They are given by the eigenspaces of the action of a cycle of length k in the symmetric group $\mathrm{Sym}(k)$ of degree k acting by permutation of the factors of $V^{\otimes k}$. These modules have appeared repeatedly, in various guises, in the theory of free Lie algebras and the representation theory of groups. They correspond to the $K\mathrm{Sym}(k)$-modules induced from one-dimensional modules for the cyclic subgroup generated by a k-cycle, as considered in [45, Chapter 8].

Section 6.2 will set the scene by giving basic results about the $U_{k,d}$ and some related modules. The main results on the modules $B_{p^m k}$ are obtained in Section 6.4. However, the proofs of these results rest heavily on Section 6.3, which is largely combinatorial. We study certain polynomials defined in an arithmetic way and apply methods from elementary number theory, combinatorics and the theory of Witt vectors.

6.2 Summands of tensor powers

Let K be a field, G a group, and V a finite-dimensional (right) KG-module. The tensor algebra $T(V)$ is the free associative algebra on V, and the nth homogeneous component $T_n(V)$ may be identified with the nth tensor power of V, otherwise denoted by $V^{\otimes n}$. Recall that the free Lie algebra $L(V)$ may be regarded as embedded in $T(V)$.

We recall Theorem 5.1.1, the ‘Decomposition Theorem’, [10, Theorem 4.4].

Theorem 5.1.1. [10, Theorem 4.4]. *Let K be a field of prime characteristic p, G a group, and V a finite-dimensional KG-module. Let k be a positive integer not divisible by p. Then, for each non-negative integer m, there is a submodule $B_{p^m k}$ of $L_{p^m k}(V)$ such that $B_{p^m k}$ is isomorphic to a direct summand of $V^{\otimes p^m k}$ and*

$$L_{p^m k}(V) = L_{p^m}(B_k) \oplus L_{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L_1(B_{p^m k}).$$

For an arbitrary field K, let R_{KG} denote the Green ring (representation ring) of G over K. This is the ring spanned by the isomorphism classes of finite-dimensional (right) KG-modules with sum and product coming from the direct sum and tensor product of modules. (It has a $\mathbb{Z}$-basis consisting of the isomorphism classes of the finite-dimensional indecomposable KG-modules.) If V is any finite-dimensional KG-module we often write V for the corresponding element of R_{KG}. Thus, for modules V_1 and V_2, we have $V_1 = V_2$ in R_{KG} if and only if $V_1 \cong V_2$. Note that the tensor power $V^{\otimes n}$ may be written as V^n in R_{KG}.

The following result is part of [11, Theorem 4.2].

Theorem 6.2.1. [11, Theorem 4.2]. *Further to Theorem 5.1.1, the equation*

$$p^m B_{p^m k} + p^{m-1}(B_{p^{m-1}k})^p + \cdots + p(B_{pk})^{p^{m-1}} + (B_k)^{p^m} = L_k(V^{p^m}).$$

holds in the Green ring R_{KG} for every non-negative integer m.

Theorem 6.2.1 is the starting-point of our study of the modules $B_{p^m k}$. The equations of the theorem describe the $B_{p^m k}$ recursively in terms of the modules $L_k(V^{p^m})$, and the polynomials on the left-hand side of these equations may be recognised as the ‘Witt polynomials’ (see [50, Chapter II, Section 6]). Thus, to gain further information about the $B_{p^m k}$, we must expect to deal with Witt vectors.

Let p be a prime number. For any commutative ring R we write R^∞ for the set of all countably infinite ‘vectors’ $\mathbf{a} = (a_0, a_1, a_2, \dots)$ with $a_i \in R$ for $i \geq 0$.

In the present context these vectors are called 'Witt vectors'. For $\mathbf{a} \in R^\infty$, where $\mathbf{a} = (a_0, a_1, a_2, \dots)$, define $\gamma_p(\mathbf{a}) \in R^\infty$ by

$$\gamma_p(\mathbf{a}) = (b_0, b_1, b_2, \dots), \tag{6.2.1}$$

where, for $i \geq 0$,

$$b_i = p^i a_i + p^{i-1} a_{i-1}^p + \cdots + p a_1^{p^{i-1}} + a_0^{p^i}. \tag{6.2.2}$$

In the language of Witt vectors, b_0, b_1, $\dots$ are the 'ghost' components of $\mathbf{a}$.

The equations of Theorem 6.2.1 can now be written as

$$\gamma_p(B_k, B_{pk}, B_{p^2k}, \dots) = (L_k(V), L_k(V^p), L_k(V^{p^2}), \dots). \tag{6.2.3}$$

Our problem is to try to unravel B_k, B_{pk}, $\dots$ from this equation. Much of our effort will be devoted to writing the modules $L_k(V^{p^m})$ in the Green ring in a form that facilitates calculations with Witt vectors.

From now on in this section, we take K to be a field of arbitrary characteristic, G a group, and V a finite-dimensional KG-module. Furthermore, we take k to be a positive integer not divisible by char(K). Let E be the extension field of K obtained by adjoining (if necessary) a primitive kth root of unity ε, and let $\langle \varepsilon \rangle$ denote the cyclic group generated by ε, consisting of all kth roots of unity in E.

We shall often need to use the following elementary fact about a cyclic group of order k.

Proposition 6.2.2. *If x and y are elements of a cyclic group of order k with $|x| = |y|$ then there exists l prime to k such that $x^l = y$.*

Proof. Let x, y have order $d|k$ in $< a : a^k = 1 >$. Then $x = a^{kn/d}$, $y = a^{km/d}$ for some m and n coprime to d. Hence we see that x and y each generate $< a^{k/d} >$ and we must have that $x^r = y$ for some r coprime to d.

Next consider three sets of primes. Let $\mathfrak{A}$ denote the set of primes dividing d, $\mathfrak{B}$ the set of primes dividing k and r and let $\mathfrak{C}$ denote the set of primes dividing k, but not r or d. It is easy to see that these sets are disjoint and that their union $\mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C}$ is the set of all primes dividing k. Let q be the product of all primes in $\mathfrak{C}$. Since r is coprime to d, there exist integers s and t such that $sr + td = 1$. Set $l = r + qtd = r + q(1 - sr)$. Then we have that $x^l = x^{r+qtd} = x^r x^{qtd} = x^r = y$. We claim that l is coprime to k. Indeed, if $p \in \mathfrak{A}$, then $p|d$ and we have that $p|qtd$, but $p \nmid r$, since r is coprime to d. So $p \nmid l$ for all $p \in \mathfrak{A}$. If $p \in \mathfrak{B}$, we have that $p|r$ and $p|qsr$, yet $p \nmid q$,by definition of q. So $p \nmid l$ for all $p \in \mathfrak{B}$. Finally, let $p \in \mathfrak{C}$. We have that $p|qtd$, by definition of q, but $p \nmid r$, by definition of $\mathfrak{C}$. So $p \nmid l$ for all $p \in \mathfrak{C}$. Thus we have shown that if $p|k$ then $p \nmid l$ and, hence, l is coprime to k as required. □

Let V_E denote the EG-module $E \otimes V$ (with tensor product taken over K), and let $V_E^{\otimes k}$ denote the kth tensor power of V_E, identified with $E \otimes V^{\otimes k}$. The symmetric group $\mathrm{Sym}(k)$ acts on the right on $V_E^{\otimes k}$ by permuting the tensor factors and this action commutes with the right action of G. Let σ be the k-cycle $(1\ 2\ \ldots\ k)$ in $\mathrm{Sym}(k)$. Then we can write $V_E^{\otimes k}$ as a direct sum of σ-eigenspaces, namely

$$V_E^{\otimes k} = \bigoplus_{\xi \in \langle \varepsilon \rangle} (V_E^{\otimes k})_\xi, \tag{6.2.4}$$

where

$$(V_E^{\otimes k})_\xi = \{v \in V_E^{\otimes k} : v\sigma = \xi v\}.$$

(This can be proved from Maschke's theorem or by diagonalising the matrix representing the action of σ.) Clearly each $(V_E^{\otimes k})_\xi$ is an EG-submodule of $V_E^{\otimes k}$. Also, for l prime to k, σ and σ^l are conjugate in $\mathrm{Sym}(k)$, from which it follows that

$$(V_E^{\otimes k})_\xi \cong (V_E^{\otimes k})_{\xi'} \text{ when } |\xi| = |\xi'|. \tag{6.2.5}$$

Here $|\xi|$ denotes the (multiplicative) order of an element ξ of $\langle \varepsilon \rangle$.

In [5, Section 4] it was shown, with different notation, that, for each $\xi \in \langle \varepsilon \rangle$, there is a KG-submodule $(V^{\otimes k})_\xi$ of $V^{\otimes k}$ such that

$$E \otimes (V^{\otimes k})_\xi \cong (V_E^{\otimes k})_\xi. \tag{6.2.6}$$

(In the notation of [5], $(V^{\otimes k})_\xi$ corresponds to U^*_ξ.) By the Noether–Deuring theorem [15, (29.11)], two modules are isomorphic if they are isomorphic after field extension. Thus (6.2.4), (6.2.5) and (6.2.6) yield

$$V^{\otimes k} \cong \bigoplus_{\xi \in \langle \varepsilon \rangle} (V^{\otimes k})_\xi \tag{6.2.7}$$

and

$$(V^{\otimes k})_\xi \cong (V^{\otimes k})_{\xi'} \text{ when } |\xi| = |\xi'|. \tag{6.2.8}$$

For each divisor d of k, let $U_{k,d}$ denote a KG-module satisfying

$$U_{k,d} \cong (V^{\otimes k})_\xi, \text{ where } |\xi| = d. \tag{6.2.9}$$

Thus we may write (6.2.7) in the Green ring as

$$V^k = \sum_{d|k} \phi(d) U_{k,d}. \tag{6.2.10}$$

Lemma 6.2.3. *We have $U_{k,k} \cong L_k(V)$.*

Proof. By the Noether–Deuring theorem it suffices to obtain $E \otimes U_{k,k} \cong L_k(V_E)$. Since $|\varepsilon| = k$, we have $U_{k,k} \cong (V^{\otimes k})_\varepsilon$. Thus, by (6.2.6), $E \otimes U_{k,k} \cong (V_E^{\otimes k})_\varepsilon$. Hence it suffices to show that

$$(V_E^{\otimes k})_\varepsilon \cong L_k(V_E). \tag{6.2.11}$$

This holds by a result of Klyachko [33, Theorem].

A character-theoretic proof of (6.2.11) can be given in the following way. By the Noether–Deuring theorem, we may assume that E is infinite. Also, it is enough to prove (6.2.11) in the case where G is the general linear group $\mathrm{GL}(V_E)$. We can consider (formal) characters of modules as defined in [21], and, by [33, proof of Proposition 1], $(V_E^{\otimes k})_\varepsilon$ and $L_k(V_E)$ have the same character, just as in characteristic 0. Furthermore, $(V_E^{\otimes k})_\varepsilon$ is a direct summand of $V_E^{\otimes k}$, by (6.2.4), and, as is well known, $L_k(V_E)$, considered as a submodule of $V_E^{\otimes k}$ via our embedding λ, is also a direct summand of $V_E^{\otimes k}$, because $\mathrm{char}(E) \nmid k$ (see [17, Section 3.1], for example). However, direct summands of

$V_E^{\otimes k}$ with the same character are isomorphic (because they are tilting modules—see [17]). Thus (6.2.11) holds. □

Since the modules $(V^{\otimes k})_\xi$ or $U_{k,d}$ have a wider significance than for our purposes, we summarise a few facts about these modules. Taking K to contain a primitive kth root of unity we have $(V^{\otimes k})_\xi = V^{\otimes k} e_\xi$ where e_ξ is the idempotent of $K\mathrm{Sym}(k)$ defined, using a k-cycle σ, by

$$e_\xi = \frac{1}{k}\sum_{i=0}^{k-1} \xi^{-i}\sigma^i.$$

Indeed, it is easy to see that $V^{\otimes k} e_\xi \subseteq (V^{\otimes k})_\xi$, and since $V^{\otimes k} \cong \bigoplus_{\xi \in \langle \varepsilon \rangle} V^{\otimes k} e_\xi$, the result follows. Thus, under the Schur correspondence, $(V^{\otimes k})_\xi$ corresponds to $K\mathrm{Sym}(k)e_\xi$, namely the $K\mathrm{Sym}(k)$-module induced from the one-dimensional $K\langle\sigma\rangle$-module on which σ acts as multiplication by ξ. A formula for the character of this induced module is given in [46, 4.17 Lemma], and this module is important in the work of Kraśkiewicz and Weyman [36]: see also [45, Chapter 8].

We note also that the modules $(V^{\otimes k})_\xi$ are involved in the definition of Adams operations ψ^n on R_{KG}, as explained in [5], further to the work of Benson [3]. For example, by [5, (4.4)], $\psi^k(V) = \sum_{d|k} \mu(d) U_{k,d}$, where μ denotes the Möbius function.

We return to the needs of the present chapter. For each positive integer r and each divisor d of k, let $M_{k,d}^{(r)}$ denote a KG-module satisfying

$$M_{k,d}^{(r)} \cong \bigoplus_{\xi_1 \cdots \xi_r = \xi} (V^{\otimes k})_{\xi_1} \otimes \cdots \otimes (V^{\otimes k})_{\xi_r}, \tag{6.2.12}$$

where $|\xi| = d$ and where the sum is over all r-tuples $(\xi_1, \ldots, \xi_r)$ of elements of $\langle\varepsilon\rangle$ satisfying $\xi_1 \cdots \xi_r = \xi$. (It is easy to see that this sum is the same, up to isomorphism, for all $\xi \in \langle\varepsilon\rangle$ of order d.)

Lemma 6.2.4. *For $d \mid k$ we have $M_{k,d}^{(r)} \cong ((V^{\otimes r})^{\otimes k})_\xi$, where $\xi \in \langle\varepsilon\rangle$ has order d.*

Proof. By (6.2.6) and (6.2.12), $E \otimes M_{k,d}^{(r)}$ is isomorphic to the module defined in the same way over E as $M_{k,d}^{(r)}$ is defined over K. If $E \otimes M_{k,d}^{(r)} \cong ((V_E^{\otimes r})^{\otimes k})_\xi$ then, by

(6.2.6), we have $E \otimes M_{k,d}^{(r)} \cong E \otimes ((V^{\otimes r})^{\otimes k})_\xi$ and the required result follows by the Noether–Deuring theorem. Thus it is enough to prove the result over E and, to ease the notation, we may take $K = E$.

Let $M = \bigotimes_{(i,j)\in\Omega} V_{(i,j)}$, where $\Omega = \{(i,j) : 1 \leq i \leq k, 1 \leq j \leq r\}$, and $V_{(i,j)} \cong V$ for all (i,j). In the symmetric group on Ω let τ be the cycle

$$((1,1),(1,2),\ldots,(1,r),(2,1),(2,2),\ldots,(2,r),\ldots,(k,1),(k,2),\ldots,(k,r)).$$

Thus τ^r has order k and is a product of k-cycles, namely $\tau^r = \sigma_1 \cdots \sigma_r$, where, for $j = 1,\ldots,r$, we have $\sigma_j = ((1,j),(2,j),\ldots,(k,j))$. Writing M as the direct sum of τ^r-eigenspaces, we have

$$M = \bigoplus_{\xi\in\langle\varepsilon\rangle} M_\xi. \tag{6.2.13}$$

We may also write M in the form $M = N^{(1)} \otimes \cdots \otimes N^{(r)}$, where, for $j = 1,\ldots,r$,

$$N^{(j)} = V_{(1,j)} \otimes \cdots \otimes V_{(k,j)} \cong V^{\otimes k}.$$

Writing $N^{(j)}$ as the direct sum of σ_j-eigenspaces, we have $N^{(j)} = \bigoplus_{\xi\in\langle\varepsilon\rangle} N_\xi^{(j)}$. Thus

$$M = \bigoplus_{\xi_1,\ldots,\xi_r\in<\varepsilon>} (N_{\xi_1}^{(1)} \otimes \cdots \otimes N_{\xi_r}^{(r)}). \tag{6.2.14}$$

Since $\tau^r = \sigma_1 \cdots \sigma_r$, we have

$$N_{\xi_1}^{(1)} \otimes \cdots \otimes N_{\xi_r}^{(r)} \subseteq M_{\xi_1\cdots\xi_r}. \tag{6.2.15}$$

Therefore, by (6.2.13), (6.2.14) and (6.2.15),

$$M_\xi = \bigoplus_{\xi_1\cdots\xi_r=\xi} (N_{\xi_1}^{(1)} \otimes \cdots \otimes N_{\xi_r}^{(r)}), \tag{6.2.16}$$

for all $\xi \in \langle\varepsilon\rangle$. Since $N^{(j)} \cong V^{\otimes k}$, (6.2.12) and (6.2.16) give

$$M_\xi \cong M_{k,d}^{(r)}, \text{ where } |\xi| = d. \tag{6.2.17}$$

We now write M in the form

$$M = (V_{(1,1)} \otimes \cdots \otimes V_{(1,r)}) \otimes \cdots \otimes (V_{(k,1)} \otimes \cdots \otimes V_{(k,r)})$$

and note that τ^r permutes the k factors of M in a cycle of length k. Hence we have $M_\xi \cong ((V^{\otimes r})^{\otimes k})_\xi$, for all $\xi \in \langle \varepsilon \rangle$. The lemma now follows from (6.2.17). □

Corollary 6.2.5. *For every positive integer r, $M^{(r)}_{k,k} \cong L_k(V^{\otimes r})$.*

Proof. By Lemma 6.2.3 and (6.2.9), $L_k(V^{\otimes r}) \cong ((V^{\otimes r})^{\otimes k})_\varepsilon$. Thus the result follows from Lemma 6.2.4. □

By Corollary 6.2.5 and (6.2.12), the modules $L_k(V^r)$ can be expressed, in the Green ring, as polynomials with positive integer coefficients in the modules $U_{k,d}$. In the case where K has prime characteristic p, it follows from (6.2.3) that the modules $B_{p^m k}$ can be written in $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} R_{KG}$ as polynomials in the $U_{k,d}$ with coefficients from $\mathbb{Z}[1/p]$. However, it is not obvious that the latter polynomials have integer coefficients or that these coefficients are positive. Our main theorem will establish these facts.

Let T be a cyclic group of order k generated by an element t, and recall that $\langle \varepsilon \rangle$ is also a cyclic group of order k. Let $R_{KG}T$ be the group ring of T with coefficients in R_{KG} and let Φ be the element of $R_{KG}T$ defined by

$$\Phi = (V^{\otimes k})_1 t^0 + (V^{\otimes k})_\varepsilon t^1 + \cdots + (V^{\otimes k})_{\varepsilon^{k-1}} t^{k-1}. \tag{6.2.18}$$

By (6.2.8), the coefficient of t^i in Φ is equal to the coefficient of t^j whenever $|t^i| = |t^j|$. For each divisor d of k, let s_d be the sum of all elements of T of order d. Then

$$\Phi = \sum_{d|k} U_{k,d} s_d, \tag{6.2.19}$$

and Φ is fixed by every automorphism of $R_{KG}T$ that fixes coefficients in R_{KG} and maps t to t^l for some l prime to k. Clearly, for every positive integer r, Φ^r is fixed by these same automorphisms, and hence the coefficient of t^i in Φ^r is equal to the coefficient of t^j whenever $|t^i| = |t^j|$. We write $[\Phi^r]_k$ to denote the coefficient of t (or any element of order k) in Φ^r.

By (6.2.18), the coefficient of t in Φ^r is

$$\sum_{\xi_1 \cdots \xi_r = \varepsilon} (V^{\otimes k})_{\xi_1} \cdots (V^{\otimes k})_{\xi_r},$$

Hence, by (6.2.12) and Corollary 6.2.5,

$$[\Phi^r]_k = L_k(V^r). \tag{6.2.20}$$

In order to obtain further information about the modules $L^k(V^r)$ we shall study the properties of the coefficients $[\Phi^r]_k$. We do this in the next section by working in a suitable polynomial ring.

6.3 Witt vectors and polynomials

We begin this section by developing some methods for dealing with Witt vectors.

Let p be a prime number and let R be a commutative ring with identity in which p is not a zero-divisor. We write R^∞ for the set of all Witt vectors over R, as in Section 6.2, and we define $\gamma_p : R^\infty \to R^\infty$ by means of (6.2.1) and (6.2.2). For $\mathbf{b} \in R^\infty$, there can be at most one element $\mathbf{a}$ of R^∞ such that $\gamma_p(\mathbf{a}) = \mathbf{b}$. Furthermore, if p is a unit of R, there exists $\mathbf{a}$ such that $\gamma_p(\mathbf{a}) = \mathbf{b}$ and the components of $\mathbf{a}$ can be obtained recursively from (6.2.2).

In the following lemma we regard R^∞ as a ring under the operations of R taken componentwise.

Lemma 6.3.1. *Let $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}' \in R^\infty$, where $\gamma_p(\mathbf{a}) = \mathbf{b}$ and $\gamma_p(\mathbf{a}') = \mathbf{b}'$. Let S be the subring of R generated by the components of $\mathbf{a}$ and $\mathbf{a}'$. Let $\mathbf{b}''$ be any element of the subring of R^∞ generated by $\mathbf{b}$ and $\mathbf{b}'$ under the componentwise operations. Then there exists $\mathbf{a}'' \in S^\infty$ such that $\gamma_p(\mathbf{a}'') = \mathbf{b}''$.*

Proof. This is an immediate consequence of a theorem of Witt [59], for which we refer to [50, Theorem II.6]. □

Lemma 6.3.2. *Let k be a positive integer not divisible by p. Then there exists $\mathbf{c} \in \mathbb{Z}^\infty$ such that*

$$\gamma_p(\mathbf{c}) = (1, k^{p-1}, k^{p^2-1}, \dots). \tag{6.3.1}$$

Proof. Let $\mathbf{c}$ be the element of $(\mathbb{Z}[1/p])^\infty$ satisfying (6.3.1), where $\mathbf{c} = (c_0, c_1, \dots)$. We prove by induction that $c_i \in \mathbb{Z}$ for all i. This shows that $\mathbf{c} \in \mathbb{Z}^\infty$.

By (6.3.1), we have

$$p^i c_i + p^{i-1} c_{i-1}^p + \cdots + p c_1^{p^{i-1}} + c_0^{p^i} = k^{p^i - 1}$$

for all $i \geq 0$. Thus $c_0 = 1 \in \mathbb{Z}$. Also, for $i \geq 1$, we have

$$p^i c_i = k^{p^i - 1} - (p^{i-1} c_{i-1}^p + \cdots + p c_1^{p^{i-1}} + 1).$$

To prove that $c_i \in \mathbb{Z}$ it suffices to show that the right-hand side is congruent to 0 modulo p^i. By Euler's theorem, $a^{\phi(p^j)} \equiv 1 \pmod{p^j}$ for every positive integer j and every integer a not divisible by p. Since $\phi(p^j) = p^j - p^{j-1}$, it follows that $a^{p^j} \equiv a^{p^{j-1}} (\mathrm{mod} p^j)$, for every integer a. Thus

$$\begin{aligned} &k^{p^i - 1} - (p^{i-1} c_{i-1}^p + \cdots + p c_1^{p^{i-1}} + 1) \\ &\qquad \equiv k^{p^i - 1} - (p^{i-1} c_{i-1} + \cdots + p c_1^{p^{i-2}} + 1) \pmod{p^i}. \end{aligned}$$

However, by (6.3.1) $p^{i-1} c_{i-1} + \cdots + p c_1^{p^{i-2}} + 1 = k^{p^{i-1} - 1}$. Thus

$$k^{p^i - 1} - (p^{i-1} c_{i-1}^p + \cdots + p c_1^{p^{i-1}} + 1) \equiv k^{p^i - 1} - k^{p^{i-1} - 1} \pmod{p^i}.$$

Now,

$$k(k^{p^i - 1} - k^{p^{i-1} - 1}) = k^{p^i} - k^{p^{i-1}} \equiv 0 \pmod{p^i}.$$

Since k is not divisible by p, the result follows. □

Corollary 6.3.3. *Let k be a positive integer not divisible by p. Then there exists $\mathbf{d} \in (\mathbb{Z}[1/k])^\infty$ such that $\gamma_p(\mathbf{d}) = (k^{-1}, k^{-1}, k^{-1}, \dots)$.*

Proof. With componentwise multiplication, we have

$$(k^{-1}, k^{-1}, k^{-1}, \dots) = (k^{-1}, k^{-p}, k^{-p^2}, \dots)(1, k^{p-1}, k^{p^2-1}, \dots).$$

However, $\gamma_p(k^{-1}, 0, 0, \dots) = (k^{-1}, k^{-p}, k^{-p^2}, \dots)$. Hence the result follows from Lemmas 6.3.1 and 6.3.2. □

From now on we consider polynomial rings $\mathbb{Z}[\mathcal{U}]$ and $\mathbb{Q}[\mathcal{U}]$, where $\mathcal{U}$ is a set of indeterminates. We state a consequence of the preceding results in a form suited for application later in this section.

Proposition 6.3.4. *Let* $\mathbf{b} \in (\mathbb{Z}[\mathcal{U}])^\infty$ *where* $\mathbf{b} = (b_0, b_1, \dots)$*. Suppose that there exist* $r_1, \dots, r_n \in \mathbb{Z}$*,* $g_1, \dots, g_n \in \mathbb{Z}[\mathcal{U}]$*, and a positive integer* k *not divisible by* p*, such that*

$$b_i = \frac{1}{k}(r_1 g_1^{p^i} + \cdots + r_n g_n^{p^i}),$$

for all $i \geq 0$*. Then there exists* $\mathbf{a} \in (\mathbb{Z}[\mathcal{U}])^\infty$ *such that* $\gamma_p(\mathbf{a}) = \mathbf{b}$*.*

Proof. Since $\mathbf{b} \in (\mathbb{Z}[\mathcal{U}])^\infty$, there exists $\mathbf{a} \in (\mathbb{Z}[1/p][\mathcal{U}])^\infty$ such that $\gamma_p(\mathbf{a}) = \mathbf{b}$. However, by hypothesis, $\mathbf{b}$ belongs to the subring of $(\mathbb{Z}[1/k][\mathcal{U}])^\infty$ generated, componentwise, by $(k^{-1}, k^{-1}, k^{-1}, \dots)$ and the elements $(g_j, g_j^p, g_j^{p^2}, \dots)$ for $j = 1, \dots, n$. Also, $\gamma_p(g_j, 0, 0, \dots) = (g_j, g_j^p, g_j^{p^2}, \dots)$. Thus, by Lemma 6.3.1 and Corollary 6.3.3, there exists $\mathbf{e} \in (\mathbb{Z}[1/k][\mathcal{U}])^\infty$ such that $\gamma_p(\mathbf{e}) = \mathbf{b}$. Hence $\mathbf{a} = \mathbf{e}$ and so

$$\mathbf{a} \in (\mathbb{Z}[1/p][\mathcal{U}])^\infty \cap (\mathbb{Z}[1/k][\mathcal{U}])^\infty = (\mathbb{Z}[\mathcal{U}])^\infty.$$

□

Any element of $\mathbb{Q}[\mathcal{U}]$ may be written as a sum, with rational coefficients, of monomials in the elements of $\mathcal{U}$. For $a, b \in \mathbb{Q}[\mathcal{U}]$, we write $a \preccurlyeq b$ if every coefficient in a is less than or equal to the corresponding coefficient in b, that is, if $b - a$ has only non-negative coefficients.

Proposition 6.3.5. *Let* $\mathbf{b} = (b_0, b_1, \dots) \in (\mathbb{Z}[\mathcal{U}])^\infty$*. Suppose that* $b_i \succcurlyeq 0$ *and*

$$b_i^{p^j} \preccurlyeq b_{i+j} \tag{6.3.2}$$

for all $i, j \geq 0$*. Let* $\mathbf{a}$ *be the element of* $(\mathbb{Z}[1/p][\mathcal{U}])^\infty$ *satisfying* $\gamma_p(\mathbf{a}) = \mathbf{b}$*, where* $\mathbf{a} = (a_0, a_1, \dots)$*. Then* $a_0 = b_0 \succcurlyeq 0$ *and, for* $m \geq 1$*,*

$$0 \preccurlyeq p^m a_m \preccurlyeq b_m - b_0^{p^m}. \tag{6.3.3}$$

Proof. For all $i \geq 0$, write $d_i = b_i - b_0^{p^i}$. Thus $d_i \succcurlyeq 0$, by (6.3.2). Hence, for all $i, j \geq 0$,

$$d_i^{p^j} + b_0^{p^{i+j}} \preccurlyeq (d_i + b_0^{p^i})^{p^j} = b_i^{p^j} \preccurlyeq b_{i+j}.$$

Therefore

$$d_i^{p^j} \preccurlyeq d_{i+j}. \tag{6.3.4}$$

Since $\gamma_p(\mathbf{a}) = \mathbf{b}$, we have $a_0 = b_0 \succcurlyeq 0$. It remains to prove (6.3.3) for $m \geq 1$, and this may be written as $0 \preccurlyeq p^m a_m \preccurlyeq d_m$. Since $\gamma_p(\mathbf{a}) = \mathbf{b}$, and $a_0 = b_0$, we have $pa_1 = b_1 - b_0^p = d_1$. Thus (6.3.3) is true for $m = 1$. We use induction on m. Since $\gamma_p(\mathbf{a}) = \mathbf{b}$ and $a_0 = b_0$,

$$\begin{aligned} p^{m+1}a_{m+1} &= b_{m+1} - p^m a_m^p - p^{m-1}a_{m-1}^{p^2} - \cdots - pa_1^{p^m} - b_0^{p^{m+1}} \\ &= d_{m+1} - p^m a_m^p - p^{m-1}a_{m-1}^{p^2} - \cdots - pa_1^{p^m}. \end{aligned} \tag{6.3.5}$$

Also, by the inductive hypothesis, $0 \preccurlyeq a_{m-i} \preccurlyeq p^{-(m-i)}d_{m-i}$ for $i = 0, \ldots, m-1$. Thus

$$0 \preccurlyeq p^{m-i}a_{m-i}^{p^{i+1}} \preccurlyeq p^{(m-i)(1-p^{i+1})}d_{m-i}^{p^{i+1}}$$

for $i = 0, \ldots, m-1$. Hence, by (6.3.5),

$$d_{m+1} \succcurlyeq p^{m+1}a_{m+1} \succcurlyeq d_{m+1} - p^{m(1-p)}d_m^p - p^{(m-1)(1-p^2)}d_{m-1}^{p^2} - \cdots - p^{(1-p^m)}d_1^{p^m}.$$

Therefore, by (6.3.4),

$$\begin{aligned} d_{m+1} \succcurlyeq p^{m+1}a_{m+1} &\succcurlyeq \left(1 - \frac{p^m}{p^{mp}} - \frac{p^{m-1}}{p^{(m-1)p^2}} - \cdots - \frac{p}{p^{p^m}}\right) d_{m+1} \\ &\succcurlyeq \left(1 - \frac{1}{p^m} - \frac{1}{p^{m-1}} - \cdots - \frac{1}{p}\right) d_{m+1} \succcurlyeq 0. \end{aligned}$$

This completes the induction. □

From now on in this section, let k be a positive integer and let $\mathcal{U}$ be a set of indeterminates indexed by the divisors of k, namely, $\mathcal{U} = \{u_d : d \mid k\}$.

As in Section 6.2, let T be a cyclic group of order k generated by an element t. We consider the group ring $\Gamma = \mathbb{Z}[\mathcal{U}]T$. This consists of all elements of the form $g = g_0 t^0 + \cdots + g_{k-1}t^{k-1}$, with $g_i \in \mathbb{Z}[\mathcal{U}]$ for $i = 0, \ldots, k-1$.

For each positive integer l prime to k there is an automorphism of Γ that fixes all coefficients in $\mathbb{Z}[\mathcal{U}]$ and maps t to t^l. Let Γ^* be the subring of Γ consisting of those elements fixed by all such automorphisms. For $g = \sum g_i t^i \in \Gamma$, we have $g \in \Gamma^*$ if and only if $g_i = g_j$ whenever $|t^i| = |t^j|$. For $g \in \Gamma^*$ and each divisor d of k, we write $[g]_d$ to denote g_i where $|t^i| = d$. As in Section 6.2, let s_d be the sum of all elements of T of order d. Thus, for $g \in \Gamma^*$, we have

$$g = \sum_{d|k} [g]_d s_d, \tag{6.3.6}$$

where $[g]_d \in \mathbb{Z}[\mathcal{U}]$ for each divisor d of k.

Let ω be a primitive kth root of unity in $\mathbb{C}$. Thus $\langle\omega\rangle$ is a cyclic group of order k consisting of all complex kth roots of unity. For each non-negative integer j there is a homomorphism $\alpha_j : \Gamma \to \mathbb{C}[\mathcal{U}]$ that fixes all coefficients in $\mathbb{Z}[\mathcal{U}]$ and maps t to ω^j. For $g \in \Gamma$ we usually write $g(\omega^j)$ instead of $g\alpha_j$, because we think of α_j as the substitution $t \mapsto \omega^j$. It is easily verified that,

$$s_d(\omega^j) = \rho_d(j), \tag{6.3.7}$$

where $\rho_d(j)$ denotes Ramanujan's sum, namely the sum of the jth powers of all complex primitive dth roots of unity. Note that $\rho_d(j) \in \mathbb{Z}$ (by [26, Theorem 271], for example). Thus α_j restricts to a homomorphism $\alpha_j : \Gamma^* \to \mathbb{Z}[\mathcal{U}]$. Indeed, for $g \in \Gamma^*$, (6.3.6) and (6.3.7) give

$$g(\omega^j) = \sum_{d|k} \rho_d(j)[g]_d. \tag{6.3.8}$$

We now show that (6.3.8) allows $[g]_k$ to be written in terms of the elements $g(\omega^j)$, by means of the Möbius function μ.

Lemma 6.3.6. *For all $g \in \Gamma^*$,*

$$k[g]_k = \sum_{e|k} \mu(e) g(\omega^{k/e}).$$

Proof. By (6.3.8), we have

$$\sum_{e|k} \mu(e) g(\omega^{k/e}) = \sum_{e|k} \mu(e) \sum_{d|k} \rho_d(k/e)[g]_d.$$

However, by [26, Theorem 271],

$$\rho_d(k/e) = \sum_{m|(d,k/e)} \mu(d/m)m,$$

where the sum is over all m such that $m \mid d$ and $m \mid (k/e)$. Therefore

$$\sum_{e|k} \mu(e) g(\omega^{k/e}) = \sum_{e|k} \sum_{d|k} \sum_{m|(d,k/e)} \mu(e)\mu(d/m)m[g]_d.$$

Altering the order of summation gives

$$\sum_{e|k} \mu(e) g(\omega^{k/e}) = \sum_{d|k} \sum_{m|d} \sum_{e|(k/m)} \mu(e)\mu(d/m)m[g]_d.$$

However, $\sum_{e|(k/m)} \mu(e) = 0$ unless $m = k$. Also, if $m = k$ we must have $d = k$. Therefore

$$\sum_{e|k} \mu(e) g(\omega^{k/e}) = k[g]_k.$$

□

Imitating the description of Φ in (6.2.19), we let $f \in \Gamma$ be defined by

$$f = \sum_{d|k} u_d s_d. \tag{6.3.9}$$

Thus $f \in \Gamma^*$ and $f^r \in \Gamma^*$ for every positive integer r. Recall the definition of $\preccurlyeq$ given before Proposition 6.3.5. Then, for all $d \mid k$ and all r, we have

$$[f^r]_d \in \mathbb{Z}[\mathcal{U}] \text{ and } [f^r]_d \succcurlyeq 0. \tag{6.3.10}$$

Also, by Lemma 6.3.6,

$$[f^r]_k = \frac{1}{k} \sum_{e|k} \mu(e)(f(\omega^{k/e}))^r,$$

where $f(\omega^{k/e}) \in \mathbb{Z}[\mathcal{U}]$ for all e. Here, division by k is possible within $\mathbb{Z}[\mathcal{U}]$ because $[f^r]_k \in \mathbb{Z}[\mathcal{U}]$. Thus, by (6.3.8), we obtain

$$[f^r]_k = \frac{1}{k}\sum_{e|k} \mu(e) \left(\sum_{d|k} \rho_d(k/e) u_d\right)^r. \tag{6.3.11}$$

We can now prove the key result of this section.

Theorem 6.3.7. *Let k be a positive integer and let p be a prime number not dividing k. Let $\mathcal{U} = \{u_d : d \mid k\}$, T a cyclic group of order k, and $f \in \mathbb{Z}[\mathcal{U}]T$, as defined in* (6.3.9)*. Then there exist elements $h_0, h_1, h_2, \dots$ of $\mathbb{Z}[\mathcal{U}]$ satisfying*

$$\gamma_p(h_0, h_1, h_2, \dots) = ([f]_k, [f^p]_k, [f^{p^2}]_k, \dots) \tag{6.3.12}$$

and for all $m \geq 0$,

$$0 \preccurlyeq p^m h_m \preccurlyeq \left(\sum_{d|k} \phi(d) u_d\right)^{p^m},$$

where ϕ denotes Euler's function.

Proof. The first statement follows from (6.3.10), (6.3.11) and Proposition 6.3.4. For the second statement we wish to apply Proposition 6.3.5 with $b_i = [f^{p^i}]_k$ for all i. Thus we need to verify the hypotheses of Proposition 6.3.5. For all $i \geq 0$, we have $[f^{p^i}]_k \in \mathbb{Z}[\mathcal{U}]$ and $[f^{p^i}]_k \succcurlyeq 0$ by (6.3.10). Also, we may write

$$f^{p^i} = [f^{p^i}]_k t + \sum_{x \in T} a_x x,$$

where $a_x \in \mathbb{Z}[\mathcal{U}]$ and $a_x \succcurlyeq 0$ for all $x \in T$. Therefore, for all $j \geq 0$,

$$f^{p^{i+j}} = (f^{p^i})^{p^j} = ([f^{p^i}]_k)^{p^j} t^{p^j} + \sum_{x \in T} b_x x,$$

where $b_x \in \mathbb{Z}[\mathcal{U}]$ and $b_x \succcurlyeq 0$ for all $x \in T$. However, t^{p^j} has order k. Thus we obtain $([f^{p^i}]_k)^{p^j} \preccurlyeq [f^{p^{i+j}}]_k$ for all $i, j \geq 0$, and so the hypotheses of Proposition 6.3.5 are satisfied. By (6.3.3), we have $0 \preccurlyeq p^m h_m \preccurlyeq [f^{p^m}]_k$, for all $m \geq 0$.

Recall that $\alpha_0 : \Gamma^* \to \mathbb{Z}[\mathcal{U}]$ is the homomorphism that fixes coefficients in $\mathbb{Z}[\mathcal{U}]$ and maps t to 1. Clearly $f\alpha_0 = \sum_{d|k} \phi(d)u_d$. Thus

$$f^{p^m}\alpha_0 = \left(\sum_{d|k} \phi(d)u_d\right)^{p^m}.$$

However, since $f^{p^m} = \sum_{d|k}[f^{p^m}]_d s_d$, we have $f^{p^m}\alpha_0 = \sum_{d|k} \phi(d)[f^{p^m}]_d$. Thus

$$[f^{p^m}]_k \preccurlyeq f^{p^m}\alpha_0 = \left(\sum_{d|k} \phi(d)u_d\right)^{p^m}.$$

Hence we obtain the second statement of the theorem. □

We shall now move towards a result that is sharper than Theorem 6.3.7 in the case where k is not square-free. Let $\tilde{k}$ denote the product of the (distinct) prime divisors of k. We write $c \parallel k$ to denote that $c \mid k$ and that c and k/c are coprime. Thus $c \parallel k$ if and only if $c \mid k$ and c is a product of maximal prime-power factors of k. We say that c is a total divisor of k. Each divisor d of k may be written uniquely in the form $d = d^*d'$ where $d^* \parallel k$ and $d' \mid k/\tilde{k}$. Here d^* is the largest divisor of d such that $d^* \parallel k$ and $d' = d/d^*$. Note that the sets $\{u_d : d^* = c\}$ form a partition of $\mathcal{U}$, as c ranges over the divisors of k such that $c \parallel k$. For each such c, write

$$w_c = \sum_{d:d^*=c} \phi(d/c)u_d = u_c + \sum_{\substack{d:d^*=c \\ d\neq c}} \phi(d/c)u_d, \tag{6.3.13}$$

and set $\mathcal{W} = \{w_c : c \parallel k\}$. By (6.3.13), the subring of $\mathbb{Z}[\mathcal{U}]$ generated by $\mathcal{W}$ may be identified with the polynomial ring $\mathbb{Z}[\mathcal{W}]$. Thus we can take $\mathbb{Z}[\mathcal{W}] \subseteq \mathbb{Z}[\mathcal{U}]$ and $\mathbb{Q}[\mathcal{W}] \subseteq \mathbb{Q}[\mathcal{U}]$. Let $\preccurlyeq_{\mathcal{W}}$ be the relation on $\mathbb{Q}[\mathcal{W}]$ defined analogously to $\preccurlyeq$ on $\mathbb{Q}[\mathcal{U}]$, but using coefficients of monomials in elements of $\mathcal{W}$.

Lemma 6.3.8. *(i)Every element x of T may be written uniquely in the form $x = x^*x'$, such that, for $|x| = d$, we have $|x^*| = d^*$ and $|x'| = d'$.*
(ii) Let $d \mid k$, and let y be an element of T of order d^*. Then the number of elements x of T satisfying $|x| = d$ and $x^* = y$ is $\phi(d/d^*)$.

Proof. (i) It is straightforward to check that if x^* and x' exist, we must have that $x^* = x^m$ and $x' = x^n$ for some $m, n \in \mathbb{N}$. Since $d = d^*d'$, where d^* and d' are coprime, we may write $sd^* + td' = 1$ for some integers s and t. Set $x^* = x^{td'}$ and $x' = x^{sd^*}$. Then we have that $x^*x' = x^{td'}x^{sd^*} = x^{sd^*+td'} = x$ and it is easy to see that $|x^*| = d^*$ and $|x'| = d'$. Indeed, we have that $(x^*)^{d^*} = (x^{td'})^{d^*} = 1$ and $(x')^{d'} = (x^{sd^*})^{d'} = 1$. Now, suppose that $(x^{td'})^l = 1$ for some $l|d^*$. Then we must have that $d|td'l$, since x has order d. Now from $d = d^*d'$ it follows that $d^*|tl$. However, t and d^* are coprime, hence we must have that $d^*|l$. Similarly we find that $|x^{sd^*}| = d'$. It now follows easily that x^* and x' do not depend upon the choice of s and t and are, in fact, uniquely determined.

(ii) For $x \in T$, write $x = x^*x'$, as in (i). We have that $|x| = d$ and $x^* = y$ if and only $y^{-1}x$ has order d/d^*. Thus the number of such elements x is $\phi(d/d^*)$. □

Lemma 6.3.9. *Let $d \mid k$ and $e \mid \tilde{k}$. Then $\rho_d(k/e) = \phi(d/d^*)\rho_{d^*}(k/e)$.*

Proof. We apply Lemma 6.3.8 to $\langle\omega\rangle$ rather than T. For $x \in \langle\omega\rangle$, write $x = x^*x'$, as in Lemma 6.3.8 (i). Then

$$\rho_d(k/e) = \sum_{x:|x|=d} x^{k/e} = \sum_{x:|x|=d} (x^*)^{k/e}(x')^{k/e}.$$

However, $(x')^{k/e} = 1$, because k/e is divisible by $k/\tilde{k}$. Hence

$$\rho_d(k/e) = \sum_{x:|x|=d} (x^*)^{k/e}.$$

For each element y of $\langle\omega\rangle$ of order d^*, there are, by Lemma 6.3.8 (ii), exactly $\phi(d/d^*)$ elements x of order d such that $x^* = y$. Thus

$$\rho_d(k/e) = \sum_{x:|x|=d} (x^*)^{k/e} = \phi(d/d^*) \sum_{y:|y|=d^*} y^{k/e} = \phi(d/d^*)\rho_{d^*}(k/e).$$

This is the required result. □

Suppose that $e \mid \tilde{k}$. Then, by Lemma 6.3.9 and (6.3.13),

$$\begin{aligned}\sum_{d|k} \rho_d(k/e)u_d &= \sum_{c\|k} \sum_{d:d^*=c} \phi(d/c)\rho_c(k/e)u_d \\ &= \sum_{c\|k} \rho_c(k/e)w_c. \end{aligned} \tag{6.3.14}$$

Note that $e \mid \tilde{k}$ holds if and only if $e \mid k$ and $\mu(e) \neq 0$. Thus, by (6.3.11) and (6.3.14),

$$[f^r]_k = \frac{1}{k} \sum_{e|k} \mu(e) \left(\sum_{c\|k} \rho_c(k/e)w_c \right)^r, \tag{6.3.15}$$

for every positive integer r.

Let $\iota : \mathbb{Q}[\mathcal{W}] \to \mathbb{Q}[\mathcal{U}]$ be the inclusion homomorphism given, for $c \parallel k$, by

$$w_c\iota = w_c = u_c + \sum_{\substack{d:d^*=c,\\ d\neq c}} \phi(d/c)u_d,$$

and let $\kappa : \mathbb{Q}[\mathcal{U}] \to \mathbb{Q}[\mathcal{W}]$ be the homomorphism given by $u_d\kappa = w_d$, if $d \parallel k$, and $u_d\kappa = 0$, otherwise. Thus $\iota\kappa$ is the identity on $\mathbb{Q}[\mathcal{W}]$.

By (6.3.15), $[f^r]_k \in \mathbb{Q}[\mathcal{W}]$. This is enough for the following theorem. However, $[f^r]_k\iota \in \mathbb{Z}[\mathcal{U}]$, by (6.3.10), and so, by applying κ, we have $[f^r]_k \in \mathbb{Z}[\mathcal{W}]$.

Theorem 6.3.10. *Further to Theorem 6.3.7, let $\mathcal{W} = \{w_c : c \parallel k\}$, where w_c is defined by* (6.3.13)*. Then we have $h_m \in \mathbb{Z}[\mathcal{W}]$ and*

$$0 \preccurlyeq_{\mathcal{W}} p^m h_m \preccurlyeq_{\mathcal{W}} \left(\sum_{c\|k} \phi(c)w_c \right)^{p^m},$$

for all $m \geq 0$.

Proof. By (6.3.15), we have $[f]_k, [f^p]_k, [f^{p^2}]_k, \cdots \in \mathbb{Q}[\mathcal{W}]$. It follows, by (6.3.12), that $h_m \in \mathbb{Q}[\mathcal{W}]$ for all m. Let ι and κ be defined as above, where $\iota\kappa$ is the identity on $\mathbb{Q}[\mathcal{W}]$. By Theorem 6.3.7, $h_m\iota \in \mathbb{Z}[\mathcal{U}]$ and

$$0 \preccurlyeq p^m(h_m\iota) \preccurlyeq \left(\sum_{d|k} \phi(d)u_d \right)^{p^m}.$$

Applying κ, we find that $h_m \in \mathbb{Z}[\mathcal{W}]$ and

$$0 \preccurlyeq_{\mathcal{W}} p^m h_m \preccurlyeq_{\mathcal{W}} \left(\sum_{c\|k} \phi(c)w_c \right)^{p^m}.$$

□

6.4 Main results

We continue with the notation of Sections 6.2 and 6.3. In particular, for each divisor d of k, s_d is the sum of the elements of T of order d, $\Phi = \sum_{d|k} U_{k,d}s_d$, $f = \sum_{d|k} u_d s_d$, and $\mathcal{U} = \{u_d : d \mid k\}$.

Let $\chi : \mathbb{Z}[\mathcal{U}] \to R_{KG}$ be the homomorphism given by the substitution $u_d \mapsto U_{k,d}$ for all d. This extends to a homomorphism $\chi : \mathbb{Z}[\mathcal{U}]T \to R_{KG}T$, fixing the elements of T, and we have $f\chi = \Phi$. Hence, by (6.2.20), for every positive integer r,

$$[f^r]_k\chi = [\Phi^r]_k = L_k(V^r). \qquad (6.4.1)$$

Applying χ to (6.3.11) and using (6.4.1), we obtain the following result in R_{KG}.

Proposition 6.4.1. *For every positive integer r,*

$$L_k(V^r) = \frac{1}{k}\sum_{e|k} \mu(e) \left(\textstyle\sum_{d|k} \rho_d(k/e)U_{k,d}\right)^r.$$

This can be compared with a result concerning Adams operations in the Green ring that follows from [5, Theorem 6.1]:

$$L_k(V^r) = \frac{1}{k}\sum_{e|k} \mu(e)(\psi^e(V^{k/e}))^r.$$

Indeed, for every divisor e of k, it can be shown that

$$\psi^e(V^{k/e}) = \sum_{d|k} \rho_d(k/e)U_{k,d}. \qquad (6.4.2)$$

Suppose now that K has prime characteristic p, and let $h_0, h_1, \ldots$ be the elements of $\mathbb{Z}[\mathcal{U}]$ given by Theorem 6.3.7. Thus, by (6.3.12) and (6.4.1),

$$\gamma_p(h_0\chi, h_1\chi, h_2\chi \ldots) = (L_k(V), L_k(V^p), L_k(V^{p^2}) \ldots).$$

By comparison with (6.2.3), we obtain

$$h_m\chi = B_{p^m k}, \text{ for all } m \geq 0. \qquad (6.4.3)$$

Also, by (6.2.10),

$$\left(\sum_{d|k}\phi(d)u_d\right)^{p^m}\chi = V^{p^m k}. \tag{6.4.4}$$

However, we may write

$$\left(\sum_{d|k}\phi(d)u_d\right)^{p^m} = \sum_{\lambda\in\Lambda} u_{\lambda(1)}\cdots u_{\lambda(p^m)},$$

where Λ is an index set (of cardinality k^{p^m}) and for each $\lambda\in\Lambda$ there corresponds a p^m-tuple $(\lambda(1),\ldots,\lambda(p^m))$ of divisors of k. Notice that two elements $\lambda,\lambda'\in\Lambda$ may define the same p^m-tuple. Thus, by (6.4.4),

$$V^{\otimes p^m k}\cong\bigoplus_{\lambda\in\Lambda}U_{k,\lambda(1)}\otimes\cdots\otimes U_{k,\lambda(p^m)}. \tag{6.4.5}$$

The inequality for $p^m h_m$ in Theorem 3.7 implies that

$$0\preccurlyeq h_m\preccurlyeq\left(\sum_{d|k}\phi(d)u_d\right)^{p^m}.$$

Hence there is a subset Λ_0 of Λ such that

$$h_m=\sum_{\lambda\in\Lambda_0}u_{\lambda(1)}\cdots u_{\lambda(p^m)}. \tag{6.4.6}$$

Our first main result now follows from (6.4.3) by applying χ to (6.4.6).

Theorem 6.4.2. *Let K be a field of prime characteristic p, G a group, and V a finite-dimensional KG-module. Let k be a positive integer not divisible by p and let m be a non-negative integer. Write*

$$V^{\otimes p^m k}\cong\bigoplus_{\lambda\in\Lambda}U_{k,\lambda(1)}\otimes\cdots\otimes U_{k,\lambda(p^m)},$$

as in (6.4.5). *Let $B_{p^m k}$ be the module given by Theorem 5.1.1. Then there exists a subset Λ_0 of Λ such that*

$$B_{p^m k}\cong\bigoplus_{\lambda\in\Lambda_0}U_{k,\lambda(1)}\otimes\cdots\otimes U_{k,\lambda(p^m)}.$$

By Theorem 6.4.2, $B_{p^m k}$ is isomorphic to a direct summand of $V^{\otimes p^m k}$ of a very specific form. Also, we see that $B_{p^m k}$ may be written in the Green ring as a polynomial in the modules $U_{k,d}$. The polynomial has positive integer coefficients and is homogeneous of degree p^m. This polynomial is, of course, the polynomial h_m of Theorem 6.3.7. Thus it depends only on k, p and m.

We shall now see how Theorem 6.4.2 can be sharpened when k is not square-free. As in Section 6.3 let $\tilde{k}$ denote the product of the prime divisors of k and, for $d \mid k$, we write $d = d^* d'$ where $d^* \parallel k$ and $d' \mid k/\tilde{k}$.

Recall from Section 6.2 that ε is a primitive kth root of unity in an extension field of K. Let Θ be the set of all elements θ of $\langle \varepsilon \rangle$ such that $|\theta|$ and $k/|\theta|$ are coprime. Every element ξ of $\langle \varepsilon \rangle$ may be written uniquely in the form $\xi = \xi^* \xi'$, as in Lemma 6.3.8 (i), where $\xi^* \in \Theta$. For each $\theta \in \Theta$, define

$$W_\theta = \bigoplus_{\xi : \xi^* = \theta} (V^{\otimes k})_\xi. \tag{6.4.7}$$

Thus, by (6.2.7), we have

$$V^{\otimes k} \cong \bigoplus_{\theta \in \Theta} W_\theta. \tag{6.4.8}$$

For each c such that $c \parallel k$, let $W_{k,c}$ denote a module isomorphic to W_θ, where $|\theta| = c$. (It is easy to see that $W_\theta \cong W_{\theta'}$ when $|\theta| = |\theta'|$.) Thus, in the Green ring,

$$V^k = \sum_{c \parallel k} \phi(c) W_{k,c}. \tag{6.4.9}$$

Suppose that $c \parallel k$ and $|\theta| = c$. If $\xi^* = \theta$ then the order of ξ is some number d satisfying $d^* = c$. Also, by Lemma 6.3.8 (ii), for each d such that $d^* = c$, the number of elements ξ of order d satisfying $\xi^* = \theta$ is $\phi(d/c)$. Thus we may write (6.4.7) in the Green ring as

$$W_{k,c} = \sum_{d : d^* = c} \phi(d/c) U_{k,d}. \tag{6.4.10}$$

Let $\mathcal{W} = \{w_c : c \parallel k\}$, as in Section 6.3. Then, by (6.3.13) and (6.4.10), we have that $w_c \chi = W_{k,c}$ for all c. Thus, by (6.4.9),

$$\left(\textstyle\sum_{c \parallel k} \phi(c) w_c\right)^{p^m} \chi = V^{p^m k}. \tag{6.4.11}$$

However, we may write

$$\left(\sum_{c\|k} \phi(c)w_c\right)^{p^m} = \sum_{\delta\in\Delta} w_{\delta(1)} \cdots w_{\delta(p^m)},$$

where Δ is an index set and for each $\delta \in \Delta$ there is a corresponding a p^m-tuple $(\delta(1), \ldots, \delta(p^m))$ of total divisors of k. Notice that two elements $\delta, \delta' \in \Delta$ may define the same p^m-tuple. Thus, by (6.4.11),

$$V^{\otimes p^m k} \cong \bigoplus_{\delta\in\Delta} W_{k,\delta(1)} \otimes \cdots \otimes W_{k,\delta(p^m)}. \qquad (6.4.12)$$

Our second main result now follows from Theorem 6.3.10 in the same way as Theorem 6.4.2 follows from Theorem 6.3.7.

Theorem 6.4.3. *Further to Theorem 6.4.2, write*

$$V^{\otimes p^m k} \cong \bigoplus_{\delta\in\Delta} W_{k,\delta(1)} \otimes \cdots \otimes W_{k,\delta(p^m)},$$

as in (6.4.12)*. Then there exists a subset Δ_0 of Δ such that*

$$B_{p^m k} \cong \bigoplus_{\delta\in\Delta_0} W_{k,\delta(1)} \otimes \cdots \otimes W_{k,\delta(p^m)}.$$

Theorem 6.4.3 expresses $B_{p^m k}$, up to isomorphism, in terms of modules $W_{k,c}$ indexed by the divisors c of k satisfying $c \parallel k$. Such a divisor c is determined uniquely by the set of prime divisors of k that divide c. Thus, in effect, the modules $W_{k,c}$ are indexed by the subsets of the set of all prime divisors of k.

6.5 Examples

We begin with a simple example to illustrate how the modules B_{pk} may be calculated up to isomorphism, when k is prime.

Example. Let p and k be distinct primes and let K be a field of characteristic p. We shall find B_{pk} as an element of R_{KG}.

Since k is a prime the only modules $U_{k,d}$ are $U_{k,k}$ and $U_{k,1}$ and, by (6.2.10),

$$V^k = (k-1)U_{k,k} + U_{k,1}.$$

Also, by Lemma 6.2.3, $U_{k,k} = L_k(V)$ in R_{KG}. By Proposition 6.4.1

$$\begin{aligned} L_k(V^p) &= \frac{1}{k}\left[(U_{k,1} + (k-1)U_{k,k})^p - (U_{k,1} - U_{k,k})^p\right] \\ &= \frac{1}{k}\sum_{i=0}^{p}\binom{p}{i}[(k-1)^i - (-1)^i]U_{k,1}^{p-i}U_{k,k}^i. \end{aligned}$$

Thus, by (6.2.3), $B_k = L_k(V) = U_{k,k}$ and

$$B_{pk} = \frac{1}{p}(L_k(V^p) - U_{k,k}^p) = \sum_{i=0}^{p} m_i U_{k,1}^{p-i}U_{k,k}^i,$$

where $m_i \in \mathbb{Z}$ and

$$m_i = \begin{cases} 0 & \text{for } i = 0, \\ \frac{1}{pk}\binom{p}{i}((k-1)^i - (-1)^i) & \text{for } 0 < i < p, \\ \frac{1}{pk}((k-1)^p - (-1)^p) - \frac{1}{p} & \text{for } i = p. \end{cases}$$

In the case where $k = 2$ and $p = 3$ we obtain $B_6 = U_{2,1}^2 U_{2,2}$. In this case it is easily verified that $U_{2,1} = S_2(V)$ and $U_{2,2} = L_2(V)$. Thus we obtain (6.1.1).

Similarly, in the case where $k = 3$ and $p = 2$, we obtain $B_6 = U_{3,1}U_{3,3}$. Again it is easily verified that $U_{3,1} = A_3(V)$ and $U_{3,3} = L_3(V)$. Thus we have proven our conjecture of Chapter 5 in the case where V is a finite-dimensional module.

In fact, for k a prime power, we have the following interesting result.

Theorem 6.5.1. *Let p and q be distinct primes, let $k = q^s$ for some $s \geq 0$ and let K be a field of characteristic p. Let G be a group, V a finite-dimensional KG-module and let m be a non-negative integer. Then we may write each of the modules $B_{p^m k}$, up to isomorphism, as a direct sum of tensor products of summands of $V^{\otimes k}$, where $L_k(V)$ occurs as a tensor factor. That is, for $k = q^s$, we may write*

$$B_{p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} U_{k,\lambda(1)} \otimes \cdots \otimes U_{k,\lambda(p^m)},$$

where for each $\lambda \in \Lambda_0$, the corresponding p^m-tuple is of the form

$$(\lambda(1), \dots, \lambda(i-1), k, \lambda(i), \dots \lambda(p^m-1)),$$

for some $1 \leq i \leq p^m$.

Proof. Since $k = q^s$ is a prime power, the only modules $U_{k,d}$ are $U_{k,1}, U_{k,q}, \dots, U_{k,q^s}$. So, by Theorem 6.4.2, we have that in the Green ring,

$$B_{p^m k} = \sum c_n U_{k,1}^{n_0} U_{k,q}^{n_1} \cdots U_{k,q^s}^{n_s}, \tag{6.5.1}$$

where the sum ranges over all compositions $n = (n_0, n_1, \dots, n_s)$ of p^m, and the c_n are non-negative integers.

Now, by Proposition 6.4.1 we have that, in the Green ring,

$$L_k(V^{p^m}) = \frac{1}{k}\left(\left(\sum_{d|k} \rho_d(q^s) U_{k,d}\right)^{p^m} - \left(\sum_{d|k} \rho_d(q^{s-1}) U_{k,d}\right)^{p^m}\right).$$

Since $k = q^s$ we have that $\rho_d(q^s) = \rho_d(q^{s-1})$ for all $d|k$ with $d \neq k$, Hence, we may write

$$\begin{aligned} L_k(V^{p^m}) &= \frac{1}{k}\left((A + \rho_k(q^s)U_{k,k})^{p^m} - \left(A + \rho_k(q^{s-1})U_{k,k}\right)^{p^m}\right) \\ &= \frac{1}{k}\sum_{i=0}^{p^m} \binom{p^m}{i} (\rho_k(q^s)^i - \rho_k(q^{s-1})^i) A^{p^m-i} U_{k,k}^i, \end{aligned} \tag{6.5.2}$$

where $A = \sum_{d|k, d\neq k} \rho_d(q^s) U_{k,d} = \sum_{d|k,d\neq k} \rho_d(q^{s-1}) U_{k,d}$. It is easy to see that the coefficient of A^{p^m} in (6.5.2) is zero, giving that $U_{k,k} \cong L_k(V)$ occurs, up to isomorphism, as a tensor factor of $L_k(V^{\otimes p^m})$.

We now prove, by induction on m, that, in the Green ring, each $B_{p^m k}$ also permits a tensor factor of $L_k(V)$. It is easy to see that this holds for $m = 0$, since $B_k \cong L_k(V)$. Suppose it holds for all non-negative integers $j < m$. By Theorem 6.2.1 we may write

$$p^m B_{p^m k} = L_k(V^{p^m}) - \left[p^{m-1}(B_{p^{m-1}k})^p + \dots + p(B_{pk})^{p^{m-1}} + (B_k)^{p^m}\right] \tag{6.5.3}$$

in the Green ring. Notice that, by induction, each term on the right-hand side of (6.5.3) permits a tensor factor of $L_k(V)$. By comparing coefficients and using the fact that the coefficients occurring in the decomposition (6.5.1) of $B_{p^m k}$ are non-negative integers, we see that $B_{p^m k}$ also permits a tensor factor of $L_k(V)$. □

It is not true in general that $L_k(V)$ is a tensor factor of $B_{p^m k}$. Indeed, for $p = 5$, $k = 6$, it is easy to check that $U_{6,2} \otimes U_{6,3}^{\otimes 4}$ occurs, up to isomorphism, as a direct summand of B_{30}.

6.6 Infinite-dimensional modules

Throughout this chapter we have assumed V to be a finite-dimensional KG-module. We might now wonder what happens in the case where V is infinite-dimensional. As we have seen in Chapter 5, for a field of characteristic 3 we have the following KG-module decomposition,

$$L_6(V) \cong L_3(L_2(V)) \oplus [L_2(V) \otimes S_2(V) \otimes S_2(V)],$$

for arbitrary KG-modules V. This example suggests that Theorems 5.1.1 and 6.4.2 hold not only for finite-dimensional modules, but may generalise to *arbitrary* modules.

The modules $B_{p^m k}$ of Theorem 5.1.1 have thus far only been defined in the case where V is finite-dimensional. Recently, R. M. Bryant has shown that Theorems 5.1.1 and 6.2.1 of Bryant and Schocker (see [10] and [11]) can in fact be extended to arbitrary modules, by a fairly standard set of arguments. We reproduce the proofs of these generalised results, with kind permission, in the form of a preprint of R. M. Bryant included in Appendix B. We summarise Bryant's results below, and shall refer to the generalised decomposition theorem as the 'Bryant-Schocker Decomposition Theorem' from now on.

Theorem 6.6.1. The Bryant-Schocker Decomposition Theorem

Let K be a field of positive characteristic p, G a group and V a KG-module. Let k

be a positive integer not divisible by p. Then for each non-negative integer m there is a submodule $B_{p^m k}$ *of* $L_{p^m k}(V)$ *such that* $B_{p^m k}$ *is isomorphic to a direct summand of* $V^{\otimes p^m k}$ *and*

$$L_{p^m k}(V) = L_{p^m}(B_k) \oplus L_{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L_1(B_{p^m k}).$$

Moreover, each $B_{p^m k}$ *is isomorphic to a direct summand of* $L_k(V^{\otimes p^m})$ *and can be written as* $B_{p^m k} = V^{\otimes p^m k}\hat{b}_{p^m k}$*, for some* $\hat{b}_{p^m k} \in K\mathrm{Sym}(p^m k)$*.*

In fact, by using idempotents in the symmetric group algebra, it is possible to reinterpret all of our main results in the more general setting where V is allowed to have infinite dimension. By defining modules $U_{k,d}$ to be certain direct summands of $V^{\otimes k}$ given by idempotents $\hat{u}_{k,d} \in K\mathrm{Sym}(p^m k)$, Bryant goes on to show that a version of our Theorem 6.4.2 holds for modules of arbitrary dimension.

Theorem 6.6.2. *Let K be a field of prime characteristic p, G a group, and V a KG-module. Let k be a positive integer not divisible by p and let m be a non-negative integer. Write*

$$V^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda} U_{k,\lambda(1)} \otimes \cdots \otimes U_{k,\lambda(p^m)},$$

where Λ *is the index set of §6.4 and the modules* $U_{k,d}$ *are defined by* $U_{k,d} = V^{\otimes k}\hat{u}_{k,d}$ *for some idempotents* $\hat{u}_{k,d} \in K\mathrm{Sym}(k)$ *(see Appendix B). Let* $B_{p^m k}$ *be the module given by Theorem 6.6.1. Then there exists a subset* Λ_0 *of* Λ *such that*

$$B_{p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} U_{k,\lambda(1)} \otimes \cdots \otimes U_{k,\lambda(p^m)}.$$

It should be noted that the subset Λ_0 in the above theorem is completely independent of the choice of the module V. It can also be shown in this general setting that, $U_{k,k} = V^{\otimes k}\hat{u}_{k,k} \cong L_k(V)$ (Appendix B, Proposition B4.5) and hence it is easy to see that a version of Theorem 6.5.1 follows easily from Theorem 6.6.2.

Theorem 6.6.3. *Let p and q be distinct primes, let* $k = q^s$ *for some* $s \geq 0$ *and let K be a field of characteristic p. Let G be a group, V a KG-module and let m be a non-negative integer. Then we may write each of the modules* $B_{p^m k}$*, up to isomorphism,* *as a direct sum of tensor products of summands of* $V^{\otimes k}$*, where* $L_k(V)$ *occurs as a tensor factor.*

In the following chapter we shall refer to the modules $B_{p^i k}$ with $k \neq 1$ as the *Bryant-Schocker modules* for $L_{p^m k}(V)$.

Chapter 7

Torsion in groups and homology of Lie powers

7.1 Introduction

Let G be a group given by a free presentation $G = F/R$. We denote by $\gamma_c R$ the cth term of the lower central series of R:

$$R = \gamma_1 R \trianglerighteq \gamma_2 R \trianglerighteq \cdots \trianglerighteq \gamma_c R \trianglerighteq \cdots ,$$

where $\gamma_c R = [\gamma_{c-1} R, R]$. The quotient $F/[\gamma_c R, F]$ with $c \geq 2$, is a free central extension,

$$1 \to \gamma_c R/[\gamma_c R, F] \to F/[\gamma_c R, F] \to F/\gamma_c R \to 1, \tag{7.1.1}$$

of $F/\gamma_c R$, which is in turn an extension,

$$1 \to R/\gamma_c R \to F/\gamma_c R \to F/R \to 1, \tag{7.1.2}$$

of $G = F/R$ with free nilpotent kernel $R/\gamma_c R$. While (7.1.2) is always torsion-free [52], elements of finite order may occur in the centre of (7.1.1). We recall the work of Gupta [23] mentioned in Chapter 1, who considered a special case of this phenomenon, namely the case where $c = 2$ and $R = F'$.

In the 1970's and 1980's, various features of the free central extensions (7.1.1) were studied by a number of authors, with a focus on the mysterious occurrence of torsion in the central quotient $\gamma_c R/[\gamma_c R, F]$ and its connection with homology. We shall see that the study of Lie rings is closely connected to this problem. The lower central quotients $\gamma_c R/\gamma_{c+1} R$ each carry the structure of a $\mathbb{Z}G$-module with action induced by conjugation in F. Trivialising the $\mathbb{Z}G$-action yields an isomorphism [2]

$$H_0(G, \gamma_c R/\gamma_{c+1} R) = (\gamma_c R/\gamma_{c+1} R) \otimes_G \mathbb{Z} \cong \gamma_c R/[\gamma_c R, F], \tag{7.1.3}$$

identifying the centre of (7.1.1) with a zero-dimensional homology group. Moreover, we have the following classical isomorphism between the lower central quotients of free groups and the homogeneous components of free Lie rings (see [51] for example):

$$\gamma_c R/\gamma_{c+1} R \cong L_c(R_{ab}). \tag{7.1.4}$$

Here $L_c(R_{ab})$ denotes the degree c homogeneous component of the free Lie ring $L(R_{ab})$ on the relation module, $R_{ab} = R/R'$, stemming from our original free presentation for G. The relation module R_{ab} is the abelianisation of R. It is a $\mathbb{Z}G$-module with the action of G given via conjugation in F. Now (7.1.3) and (7.1.4) yield the isomorphism

$$\gamma_c R/[\gamma_c R, F] \cong H_0(G, L_c(R_{ab})) = L_c(R_{ab}) \otimes_G \mathbb{Z}. \tag{7.1.5}$$

This isomorphism has been used as the starting point of several successful attempts to study the quotient $\gamma_c R/[\gamma_c R, F]$. A series of papers [37], [38], [39], [40] by Kuz'min focused upon the case $c = 2$, whilst papers by Thomson [56], Zerck [60] and Stöhr [54], [55] addressed the general case $c \geq 2$. The main achievements of all these papers with respect to the quotient $\gamma_c R/[\gamma_c R, F]$ were the following two results. The first, bounding the exponent of the (possibly trivial) torsion subgroup $\mathrm{t}(\gamma_c R/[\gamma_c R, F])$, is that

$$4\ \mathrm{t}(\gamma_2 R/[\gamma_2 R, F]) = 0 \quad \text{and} \quad c\ \mathrm{t}(\gamma_c R/[\gamma_c R, F]) = 0 \quad \text{for } c \geq 3. \tag{7.1.6}$$

This result was proven by Kuz'min [38] in the case $c = 2$ and later in full generality by Stöhr [54]. The second result gives a precise identification of the torsion subgroup in the case when $c = p$ a prime, under mild conditions on G (there should be no elements of order p in G). In this case

$$\mathrm{t}(\gamma_p R/[\gamma_p R, F]) \cong H_4(G, \mathbb{Z}_p). \tag{7.1.7}$$

A slightly weaker version of (7.1.7) in the case where $p = 2$ was given by Kuz'min [39]. The result for p an arbitrary prime is due to Stöhr [54]. Without the restrictions on the presence of torsion in G as required for (7.1.7), more complicated homological descriptions of the torsion subgroup $\mathrm{t}(\gamma_p R/[\gamma_p R, F])$ for p a prime can be found in [40] for $p = 2$ and in [27] for general p.

As far as the concrete identification of the torsion subgroup is concerned, (7.1.7) is where the boundary of our knowledge was at the end of the 1980's. The case where c is a prime was understood, and virtually nothing was known in the case where c is a composite number. In 1993 there was a single addition to this body of results. The case $c = 4$ was dealt with in [55], where it was shown that if G has no elements of order 2, then

$$\mathrm{t}(\gamma_4 R/[\gamma_4 R, F]) \cong H_6(G, \mathbb{Z}_2). \tag{7.1.8}$$

We summarise the results (7.1.6), (7.1.7) and (7.1.8) in Theorem 7.1.1 below for ease of reference.

Theorem 7.1.1. [54],[55]. *Let F be a non-cyclic free group, R a normal subgroup of F and let $c \geq 2$. Then, $\gamma_c R/[\gamma_c R, F]$ decomposes into the sum of a free abelian group and a torsion group $t_c = t(\gamma_c R/[\gamma_c R, F])$, where*

(i) $4t_2 = 0$ *and* $ct_c = 0$ *for* $c \geq 3$.

(ii) If $c = p$ is a prime and $G = F/R$ has no p-torsion, then $t_p \cong H_4(G, \mathbb{Z}_p)$.

(iii) If $c = 4$ and $G = F/R$ has no 2-torsion, then $t_4 \cong H_6(G, \mathbb{Z}_2)$.

This determines the torsion subgroup of $\gamma_c R/[\gamma_c R, F]$ up to isomorphism for c prime or equal to 4. The first case where the torsion subgroup t_c is unknown is when $c = 6$.

Theorem 7.1.1 tells us that the exponent of t_6 divides 6, however, we are unable to say whether the torsion subgroup contains any elements of order 2 or 3.

In this chapter we will add to the results in Theorem 7.1.1 by proving the following result.

Theorem 7.5.5. *Let F be a non-cyclic free group and R a normal subgroup of F such that the quotient $G = F/R$ has no elements of order 2 or 3. Then, $\gamma_6 R/[\gamma_6 R, F]$ is torsion-free.*

This theorem is the first addition to the results given in Theorem 7.1.1 since 1993. It has been made possible by recent results on modular Lie powers, in particular, the Bryant-Schocker Decomposition Theorem (see Chapter 6, Theorem 6.6.1). We shall use our knowledge of the decomposition of the sixth Lie power in characteristic 2 and 3 (see Chapters 5 and 6) along with (7.1.5) to obtain new information on the torsion subgroup t_6. Indeed, by (7.1.5) we see that the problem of determining the torsion subgroup of $\gamma_c R/[\gamma_c R, F]$ is equivalent to determining the torsion subgroup of $H_0(G, L_c(R_{ab}))$. We shall show that when G has no 2-torsion and no 3-torsion $H_0(G, L_6(R_{ab}))$ is torsion-free.

In fact, we can say much more. Using the results of Chapter 6 we shall see that:

(i) If G has no 2-torsion and no p-torsion then $H_0(G, L_{2p^m}(R_{ab}))$ has no p-torsion.

(ii) If G has no 3-torsion and no p-torsion then $H_0(G, L_{3p^m}(R_{ab}))$ has no p-torsion.

Of course, $c = 6$ is the only time that (i) and (ii) coincide, giving the fact that $H_0(G, L_6(R_{ab})) \cong \gamma_6/[\gamma_6 R, F]$ is torsion-free.

7.2 Homology and Lie powers

In this section we record a few basic results from homological algebra which we shall use repeatedly. We shall refer to [28] and [41] for proofs of these results and further reference.

We begin by recalling a few properties of the tensor product. Let K be a ring, A a right K-module and B a left K-module. As usual, we write $A \otimes_K B$ to denote the tensor product of A and B over K. If $K = \mathbb{Z}$, we often suppress this notation and write simply $A \otimes B$. Let $\{B_i\}_{i \in I}$ be a family of left K-modules. Then we have that $A \otimes_K \bigoplus_{i \in I} B_i \cong \bigoplus_{i \in I}(A \otimes_K B_i)$. Similarly, let $\{A_i\}_{i \in I}$ be a family of right K-modules. Then we have that $\left(\bigoplus_{i \in I} A_i\right) \otimes_K B \cong \bigoplus_{i \in I}(A_i \otimes_K B)$.

Next let $0 \to B' \to B \to B'' \to 0$, be an exact sequence of left K-modules. Then, for any right K-module A, the sequence $A \otimes_K B' \to A \otimes_K B \to A \otimes_K B'' \to 0$ of abelian groups, is exact.

Similarly, let $0 \to A' \to A \to A'' \to 0$, be an exact sequence of right K-modules. For any left K-module A, the sequence $A' \otimes_K B \to A \otimes_K B \to A'' \otimes_K B \to 0$ of abelian groups, is exact.

We say that $A \otimes_K -$ and $- \otimes_K B$ are additive, right-exact functors from the category of K-modules to the category of abelian groups.

A chain complex $\mathcal{C}$ over K is a sequence of K-modules

$$\mathcal{C} : \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots ,$$

such that $\partial_{n+1}\partial_n = 0$ for all $n \in \mathbb{Z}$. To each chain complex $\mathcal{C}$ we associate the graded K-module $H(\mathcal{C}) = \{H_n(\mathcal{C})\}$, where $H_n(\mathcal{C}) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$, for all $n \in \mathbb{Z}$, is called the nth homology module of $\mathcal{C}$. Let A be a K-module. We say that the chain complex

$$\mathcal{P} : \cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to 0,$$

is a projective resolution of A if each P_i is a projective K-module and the sequence

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0,$$

is exact.

Let A be a right K-module and B a left K-module. For $n \geq 0$, we define $\operatorname{Tor}_n^K(A, B)$ to be the abelian group

$$\operatorname{Tor}_n^K(A, B) = H_n(A \otimes_K \mathcal{P}),$$

where $\mathcal{P}$ is any projective resolution of B. Equivalently, we may define

$$\operatorname{Tor}_n^K(A, B) = H_n(\mathcal{P} \otimes_K B),$$

where $\mathcal{P}$ is any projective resolution of A (see [28] for further details). Notice that $\operatorname{Tor}_0^K(A, B) = A \otimes_K B$. Indeed, let $\mathcal{P}$ be a projective resolution of A. We consider the chain complex $\mathcal{P} \otimes_K B$,

$$\mathcal{P} \otimes_K B : \cdots \to P_{n+1} \otimes_K B \xrightarrow{\partial_{n+1}} P_n \otimes_K B \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} P_1 \otimes_K B \xrightarrow{\partial_1} P_0 \otimes_K B \xrightarrow{\partial_0} 0.$$

Since $- \otimes_K B$ is right-exact we have that the sequence

$$P_1 \otimes_K B \xrightarrow{\partial_1} P_0 \otimes_K B \xrightarrow{\alpha} A \otimes_K B \to 0$$

is exact. Thus

$$\begin{aligned} \operatorname{Tor}_0^K(A, B) &= H_0(\mathcal{P} \otimes_K B) \\ &= \ker \partial_0 / \operatorname{Im} \partial_1 \\ &= (P_0 \otimes_K B)/\ker \alpha \\ &= A \otimes_K B. \end{aligned}$$

Lemma 7.2.1. *Let G be a group and let N be a (right) $\mathbb{Z}G$-module. Let $\mathbb{Z}_p$ denote the integers modulo p. Then $\mathbb{Z}_p$is a trivial (left) $\mathbb{Z}G$-module and a trivial (left) $\mathbb{Z}_pG$-module. Let $M = N \otimes \mathbb{Z}_p$. Then, M may be considered as both a right $\mathbb{Z}G$-module and a right $\mathbb{Z}_pG$-module (via the diagonal action) and*

$$\operatorname{Tor}_n^{\mathbb{Z}G}(M, \mathbb{Z}) \cong \operatorname{Tor}_n^{\mathbb{Z}_pG}(M, \mathbb{Z}_p).$$

Proof. Let $\mathcal{P} : \cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to 0$ be a free resolution of $\mathbb{Z}$ considered as a trivial $\mathbb{Z}G$-module. We can construct a free resolution $\mathcal{Q}$ of $\mathbb{Z}_p$ as a trivial $\mathbb{Z}_pG$-module as follows:

$$\mathcal{Q} : \cdots \to P_{n+1} \otimes \mathbb{Z}_p \to P_n \otimes \mathbb{Z}_p \to \cdots \to P_1 \otimes \mathbb{Z}_p \to P_0 \otimes \mathbb{Z}_p \to 0.$$

It is easy to see that each $Q_n = P_n \otimes \mathbb{Z}_p$ is a free $\mathbb{Z}_pG$-module. It remains to check that the sequence

$$\cdots \to P_{n+1} \otimes \mathbb{Z}_p \to P_n \otimes \mathbb{Z}_p \to \cdots \to P_1 \otimes \mathbb{Z}_p \to P_0 \otimes \mathbb{Z}_p \to \mathbb{Z}_p \to 0$$

is exact. This follows from the fact that $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ is a split exact sequence of $\mathbb{Z}$-modules.

Let $M \otimes_{\mathbb{Z}G} \mathcal{P}$ and $M \otimes_{\mathbb{Z}_pG} \mathcal{Q}$ be given by

$$M \otimes_{\mathbb{Z}G} \mathcal{P} : \cdots \xrightarrow{\partial_{n+2}} M \otimes_{\mathbb{Z}G} P_{n+1} \xrightarrow{\partial_{n+1}} M \otimes_{\mathbb{Z}G} P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} M \otimes_{\mathbb{Z}G} P_1 \xrightarrow{\partial_1} M \otimes_{\mathbb{Z}G} P_0 \xrightarrow{\partial_0} 0$$

and

$$\begin{aligned} M \otimes_{\mathbb{Z}_pG} \mathcal{Q} : \cdots &\xrightarrow{\partial_{n+2}\otimes 1} M \otimes_{\mathbb{Z}_pG} P_{n+1} \otimes \mathbb{Z}_p \xrightarrow{\partial_{n+1}\otimes 1} M \otimes_{\mathbb{Z}_pG} P_n \otimes \mathbb{Z}_p \xrightarrow{\partial_n\otimes 1} \cdots \\ &\xrightarrow{\partial_2\otimes 1} M \otimes_{\mathbb{Z}_pG} P_1 \otimes \mathbb{Z}_p \xrightarrow{\partial_1\otimes 1} M \otimes_{\mathbb{Z}_pG} P_0 \otimes \mathbb{Z}_p \xrightarrow{\partial_0\otimes 1} 0. \end{aligned}$$

Then, by definition, we have that $\operatorname{Tor}_n^{\mathbb{Z}G}(M, \mathbb{Z}) = H_n(M \otimes_{\mathbb{Z}G} \mathcal{P}) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$ and $\operatorname{Tor}_n^{\mathbb{Z}_pG}(M, \mathbb{Z}_p) = H_n(M \otimes_{\mathbb{Z}_pG} \mathcal{Q}) = \ker(\partial_n \otimes 1) / \operatorname{Im}(\partial_{n+1} \otimes 1)$. Since $M = N \otimes \mathbb{Z}_p$, it is easily verified that an element $x \otimes a \in (M \otimes_{\mathbb{Z}_pG} P_n) \otimes \mathbb{Z}_p$ is contained in the kernel of $\partial_n \otimes 1$ if and only if x is contained in the kernel of ∂_n or $a = 0$. Similarly, $y \otimes a \in (M \otimes_{\mathbb{Z}_pG} P_n) \otimes \mathbb{Z}_p$ is contained in the image of $\partial_{n+1} \otimes 1$ if and only if y is contained in the image of ∂_{n+1}. Thus, we see that $\operatorname{Tor}_n^{\mathbb{Z}G}(M, \mathbb{Z}) \cong \operatorname{Tor}_n^{\mathbb{Z}_pG}(M, \mathbb{Z}_p)$. □

From now on we will write G-module to mean $\mathbb{Z}G$-module. When taking tensor products of G-modules, in order to simplify notation we shall write $A \otimes_G B$ to mean $A \otimes_{\mathbb{Z}G} B$. We shall also write $\operatorname{Tor}_n^G(A, B)$ to denote $\operatorname{Tor}_n^{\mathbb{Z}G}(A, B)$.

The nth homology group of G with coefficients in the right G-module A is defined by

$$H_n(G, A) = \operatorname{Tor}_n^G(A, \mathbb{Z}).$$

Given a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{7.2.1}$$

of G-modules, we define the long exact homology sequence of (7.2.1) to be the sequence

$$\begin{aligned} \cdots \to H_n(G,B) \to \quad & H_n(G,C) && \to H_{n-1}(G,A) \to \cdots \\ \cdots \to \quad & H_1(G,C) && \to A \otimes_G \mathbb{Z} \to B \otimes_G \mathbb{Z} \to C \otimes_G \mathbb{Z} \to 0. \end{aligned}$$

Proposition 7.2.2. *Let P be a projective G-module. Then*

$$H_n(G,P) = 0 \qquad \forall n \geq 1$$

Proof. Since P is a projective G-module we have that

$$\mathcal{P} : \cdots \to 0 \to \cdots \to 0 \to P \to 0,$$

is a projective resolution of P. Now, $H_n(G,P) = \operatorname{Tor}_n^G(P,\mathbb{Z}) = H_n(\mathcal{P} \otimes_G \mathbb{Z})$, where

$$\mathcal{P} \otimes_G \mathbb{Z} : \cdots \xrightarrow{\partial_n} 0 \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} 0 \xrightarrow{\partial_1} P \otimes_G \mathbb{Z} \xrightarrow{\partial_0} 0.$$

So,

$$\begin{aligned} H_n(G,P) &= H_n(\mathcal{P} \otimes_G \mathbb{Z}) \\ &= \ker \partial_n / \operatorname{Im} \partial_{n+1} \\ &= 0 \quad \text{for } n \geq 1, \end{aligned}$$

as required. □

We shall also need the following result about Lie powers and symmetric powers of KG-modules, due to Stöhr [54].

Theorem 7.2.3. [54] *Let G be a group without c-torsion, K a commutative ring with 1 and let A be a free KG-module. Then $S_c(A)$ and $L_c(A)$ are free KG-modules.*

An easy corollary of Theorem 7.2.3 is that, with the same restrictions on G, we can say that a Lie power of a projective KG-module is projective.

Corollary 7.2.4. *Let G be a group without c-torsion, K a commutative ring with* 1 *and let B be a projective KG-module. Then $L_c(B)$ is a projective KG-module.*

Proof. If B is projective then we must have that B is the direct summand of a free module A. That is, $A = B \oplus B'$ for some KG-module B'. Hence, there exists a projection $\varrho : A \twoheadrightarrow B$. This induces a projection $\varrho_c : L_c(A) \twoheadrightarrow L_c(B)$ given by

$$[x_1, x_2, \ldots, x_c] \mapsto [x_1\varrho, x_2\varrho, \ldots, x_c\varrho].$$

Thus, $L_c(B)$ is a direct summand of $L_c(A)$. Since A is a free KG-module, we have that $L_c(A)$ is also free, by Theorem 7.2.3, and hence we have that $L_c(B)$ is a projective KG-module. □

7.3 Homology of the Bryant-Schocker modules

Recall the Bryant-Schocker Decomposition Theorem (Theorem 6.6.1). We shall show that the Bryant-Schocker modules $B_{p^m k}$ are homologically trivial for $V = R_{ab} \otimes \mathbb{Z}_p$; the relation module reduced modulo p.

Lemma 7.3.1. *Let G be a group with free presentation given by $G = F/R$ and let p be a prime. Let R_{ab} denote the relation module stemming from this free presentation. Then*

$$H_n(G, L_k((R_{ab})^{\otimes m}) \otimes \mathbb{Z}_p) = 0$$

for all $n \geq 1$, $m \geq 1$, $k \geq 2$ with $p \nmid k$.

Proof. Let $\lambda_k : L_k((R_{ab})^{\otimes m}) \to ((R_{ab})^{\otimes m})^{\otimes k}$ denote the canonical embedding of $L_k((R_{ab})^{\otimes m})$ into $((R_{ab})^{\otimes m})^{\otimes k}$ and let $\pi_k : ((R_{ab})^{\otimes m})^{\otimes k} \to L_k((R_{ab})^{\otimes m})$ denote the natural projection of the kth tensor power of $(R_{ab})^{\otimes m}$ onto the kth Lie power of $(R_{ab})^{\otimes m}$ (see Chapter 4). Recall that the tensor power $((R_{ab})^{\otimes m})^{\otimes k}$ is a module for the symmetric group $\mathrm{Sym}(k)$. By identifying $((R_{ab})^{\otimes m})^{\otimes k}$ with $(R_{ab})^{\otimes km}$, each permutation in $\mathrm{Sym}(k)$ may be regarded as an element of $\mathrm{Sym}(km)$ acting on $(R_{ab})^{\otimes km}$. The important point about the action of $\mathrm{Sym}(km)$ on $(R_{ab})^{\otimes km}$ is that the induced action of $\mathrm{Sym}(km)$ on the homology groups $H_n(G, (R_{ab})^{\otimes km} \otimes \mathbb{Z}_p)$ is trivial for all $n \geq 1$ (see [60, Proposition 3.2]). Hence, so is the action of $\mathrm{Sym}(k)$ on $H_n(G, ((R_{ab})^{\otimes m})^{\otimes k} \otimes \mathbb{Z}_p)$. Suppose that $k > 2$ and let

$$\zeta_k = (1\ 2\ 3) + (1\ 3\ 2) - (1\ 2) = 1 + (1\ 2\ 3) + (1\ 3\ 2) - 1 - (1\ 2) \in \mathbb{Z}\mathrm{Sym}(k).$$

Then $((R_{ab})^{\otimes m})^{\otimes k}\zeta_k \subseteq \ker \pi_k$, by the Jacobi identity and anticommutativity:

$$\begin{aligned}(a_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes a_k)\zeta_k\pi_k \quad &= \quad [a_1, a_2, a_3, \ldots, a_k] + [a_3, a_1, a_2, \ldots, a_k] \\ &\quad +[a_2, a_3, a_1, \ldots, a_k] - [a_1, a_2, a_3, \ldots, a_k] \\ &\quad -[a_2, a_1, a_3, \ldots, a_k] \\ &= \quad 0.\end{aligned}$$

Now, let $x \in H_n(G, L_k((R_{ab})^{\otimes m}) \otimes \mathbb{Z}_p)$ with $n \geq 1$. For a map ϕ of G-modules, we shall let $H(\phi)$ denote the induced map between the corresponding homology groups. Then, on one hand we have

$$xH(\lambda_k\zeta_k\pi_k \otimes 1) = 0,$$

since $L_k((R_{ab})^{\otimes m})\lambda_k\zeta_k \subseteq ((R_{ab})^{\otimes m})^{\otimes k}\zeta_k \subseteq \ker \pi_k$.

On the other hand, since $\mathrm{Sym}(k)$ acts trivially on $H_n(G, ((R_{ab})^{\otimes m})^{\otimes k} \otimes \mathbb{Z}_p)$, for all $n \geq 1$, we have

$$xH(\lambda_k\zeta_k\pi_k \otimes 1) = xH(\lambda_k\pi_k \otimes 1) = kx.$$

Therefore, we must have that every element of $H_n(G, L_k((R_{ab})^{\otimes m}) \otimes \mathbb{Z}_p)$ with $n \geq 1$ is annihilated by k. Since $p \nmid k$, we must have that $H_n(G, L_k((R_{ab})^{\otimes m}) \otimes \mathbb{Z}_p) = 0$ for all $n \geq 1$ and $k > 2$.

For $k = 2$, we perform a similar argument with $\zeta_2 = 1 + (1\ 2)$. Then we have that $((R_{ab})^{\otimes m})^{\otimes 2}\zeta_2 \subseteq \ker \pi_2$ by anticommutativity:

$$(a_1 \otimes a_2)\zeta_2\pi_2 \quad = \quad [a_1, a_2] + [a_2, a_1] = 0.$$

Now, let $x \in H_n(G, L_2((R_{ab})^{\otimes m}) \otimes \mathbb{Z}_p)$, with $n \geq 1$. Then, on one hand we have

$$xH(\lambda_2\zeta_2\pi_2 \otimes 1) = 0$$

since $L_2((R_{ab})^{\otimes m})\lambda_2\zeta_2 \subseteq ((R_{ab})^{\otimes m})^{\otimes 2}\zeta_2 \subseteq \ker \pi_2$. On the other hand, since Sym(2) acts trivially on $H_n(G, ((R_{ab})^{\otimes m})^{\otimes 2} \otimes \mathbb{Z}_p)$, for all $n \geq 1$, we have that

$$xH(\lambda_2\zeta_2\pi_2 \otimes 1) = xH(2\lambda_2\pi_2 \otimes 1) = 4x.$$

Therefore, we must have that every element of $H_n(G, L_k((R_{ab})^{\otimes m}) \otimes \mathbb{Z}_p)$ with $n \geq 1$ is annihilated by 4. Since $p \neq 2$ we must have that $H_n(G, L_2((R_{ab})^{\otimes m}) \otimes \mathbb{Z}_p) = 0$ for all $n \geq 1$. This completes the proof. □

Corollary 7.3.2. *Let G be a group with free presentation given by $G = F/R$ and let p be a prime. Let R_{ab} denote the relation module stemming from this free presentation. Then for any $m \geq 0$, $k \geq 2$ with $p \nmid k$ the Bryant-Schocker modules for $L(R_{ab} \otimes \mathbb{Z}_p)$ have trivial homology in all positive dimensions:*

$$H_n(G, B_{p^m k}) = 0 \quad \textit{for all}\, n \geq 1.$$

Proof. By the Bryant-Schocker Decomposition Theorem (Theorem 6.6.1), we have that each $B_{p^m k}$ is isomorphic to a direct summand of $L_k((R_{ab})^{\otimes p^m} \otimes \mathbb{Z}_p)$, which is homologically trivial in all positive dimensions, by Lemma 7.3.1. The result follows immediately. □

7.4 Some useful exact sequences

In this section we record some useful exact sequences of KG-modules, including the augmentation sequence and the relation sequence. We also give a six-term exact sequence of G-modules due to R. Stöhr, which we shall use to prove that the second Lie power of the relation module reduced modulo 3 is projective as a $\mathbb{Z}_3G$-module (Lemma 7.4.1). We next describe a split short exact sequence of KG-modules involving the third Lie power, in the case where 3 is invertible in K. Finally, we shall produce a five-term exact sequence of KG-modules, which we shall use to prove that the third Lie power of the relation module reduced modulo 2 is projective as a $\mathbb{Z}_2G$-module (Lemma 7.4.4). By the results of Chapter 5, Lemma 7.4.1 also provides an alternative proof of the fact that the Bryant-Schocker module $B_6^{(3)}$ (in characteristic 3) is homologically trivial. In fact, we shall see that this module is projective.

7.4.1 The augmentation sequence

Let G be a group, $\mathbb{Z}G$ the integral group ring. We define a map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ by

$$\sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} \alpha_g,$$

called the augmentation of $\mathbb{Z}G$. It is easy to check that ε is a homomorphism of rings. The kernel of ε, which we shall denote by IG, is called the augmentation ideal of $\mathbb{Z}G$. The augmentation ideal is a $\mathbb{Z}$-free G-module (the action of G given by multiplication) with generating set $\{s - 1 : s \in S\}$, where S is a generating set of G. The augmentation sequence is then simply the following short exact sequence of G-modules

$$0 \to IG \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0, \tag{7.4.1}$$

where $\mathbb{Z}$ is the trivial module.

7.4.2 The relation sequence

Let G be a group with free presentation $1 \to R \to F \xrightarrow{\rho} G \to 1$ and let $R_{ab} = R/R'$ denote the relation module stemming from this presentation. The action of G on R_{ab} is given by conjugation. Let $P = IF \otimes_G \mathbb{Z}G$. Then P is a G-module via the G-action given by

$$(u \otimes h)g = u \otimes hg \quad \forall g, h \in G, \forall u \in IF.$$

The module P is G-free on the set $\{(x-1)\otimes 1 : x \in X\}$, where X is a free generating set of F. The following short exact sequence of G-modules is referred to as the relation sequence

$$0 \to R_{ab} \xrightarrow{\mu} P \xrightarrow{\sigma} IG \to 0, \tag{7.4.2}$$

where the embedding μ, called the Magnus embedding, and the epimorphism σ are given by

$$\mu : rR' \mapsto (r-1)\otimes 1,\ \forall r \in R \quad \text{and} \quad \sigma : (x-1)\otimes 1 \mapsto x\rho - 1,\ \forall x \in X.$$

See [28, Chapter VI §6] for further reference.

7.4.3 A six-term exact sequence

Let G be a group with free presentation $1 \to R \to F \xrightarrow{\rho} G \to 1$ and let R_{ab}, P and IG be defined as above. Then there exists the following six-term exact sequence

$$0 \to L_2(R_{ab}) \to L_2(P) \to IG \otimes P \to S_2\mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z}_2 \to 0. \tag{7.4.3}$$

This sequence is due to R. Stöhr [54]. Notice that, provided G has no 2-torsion, the four middle terms are free G-modules. This follows from the fact that $\mathbb{Z}G$ and P are free G-modules, and, by Theorem 7.2.3, we have that the second Lie and symmetric powers of a free G-module are again free.

Lemma 7.4.1. *If G has no 2-torsion then $L_2(R_{ab}) \otimes \mathbb{Z}_p$ is a projective $\mathbb{Z}_pG$-module for all primes $p \neq 2$.*

Proof. Reduction of (7.4.3) modulo p turns this sequence into the five-term exact sequence

$$0 \to L_2(R_{ab}) \otimes \mathbb{Z}_p \to L_2(P) \otimes \mathbb{Z}_p \to IG \otimes P \otimes \mathbb{Z}_p \to S_2(\mathbb{Z}G) \otimes \mathbb{Z}_p \to \mathbb{Z}_pG \to 0.$$

Since the four terms to the right of $L_2(R_{ab}) \otimes \mathbb{Z}_p$ are all free $\mathbb{Z}_pG$-modules, it follows immediately that $L_2(R_{ab}) \otimes \mathbb{Z}_p$ is a projective $\mathbb{Z}_pG$-module. □

We have already seen that the Bryant-Schocker modules are homologically trivial in all positive dimensions. By imposing additional conditions on G, we may apply Lemma 7.4.1 to show that the Bryant-Schocker modules for $L_{2p^m}(R_{ab} \otimes \mathbb{Z}_p)$ are projective as $\mathbb{Z}_pG$-modules.

Corollary 7.4.2. *Let G be a group with no 2-torsion and free presentation $G = F/R$. Let R_{ab} be the relation module stemming this presentation. Then the Bryant-Schocker modules B_{2p^i} for $L_{2p^m}(R_{ab} \otimes \mathbb{Z}_p)$ are projective $\mathbb{Z}_pG$-modules and, hence, have trivial homology in all positive dimensions.*

Proof. We recall from Chapter 6 that each of the modules B_{2p^i} $(0 \leq i \leq m)$ permits $L_2(R_{ab} \otimes \mathbb{Z}_p)$ as a tensor factor (by Theorem 6.6.3). By Lemma 7.4.1 we have that $L_2(R_{ab} \otimes \mathbb{Z}_p)$ is a projective $\mathbb{Z}_pG$-module. Hence, each B_{2p^i} is a projective $\mathbb{Z}_pG$-module and, thus, has trivial homology in all positive dimensions. □

7.4.4 A split short exact sequence involving the third Lie power

Let G be a group, K a commutative ring with 1 and let V be a K-free right KG-module. It is well known (see [25, Corollary 3.1]) that the third Lie power fits into the short exact sequence

$$0 \to L_3(V) \to V \otimes S_2(V) \to S_3(V) \to 0, \tag{7.4.4}$$

where the maps are given by

$$\begin{aligned} [a_1, a_2, a_3] &\mapsto a_1 \otimes (a_2 \circ a_3) - a_2 \otimes (a_1 \circ a_3) \\ \text{and} \quad a_1 \otimes (a_2 \circ a_3) &\mapsto a_1 \circ a_2 \circ a_3. \end{aligned}$$

Moreover, if 3 is invertible in K, we have that this sequence splits via the map $S_3(V) \to V \otimes S_2(V)$ given by,

$$a_1 \circ a_2 \circ a_3 \mapsto \frac{1}{3}(a_1 \otimes (a_2 \circ a_3) + a_2 \otimes (a_1 \circ a_3) + a_3 \otimes (a_1 \circ a_2)).$$

Thus we have the direct sum decomposition

$$V \otimes S_2(V) \cong L_3(V) \oplus S_3(V).$$

7.4.5 A five-term exact sequence

Let K be a commutative ring with 1 and let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{7.4.5}$$

be a short exact sequence of K-free KG-modules. Consider the sequence

$$0 \to L_3(A) \xrightarrow{d_1} B \otimes S_2(A) \xrightarrow{d_2} S_3(B) \xrightarrow{d_3} B \otimes S_2(C) \xrightarrow{d_4} (C \otimes S_2(C))/S_3(C) \to 0, \tag{7.4.6}$$

where the maps are given as follows:

(i) $d_1 : L_3(A) \to B \otimes S_2(A)$ is given by

$$[a_1, a_2, a_3] \mapsto a_1\alpha \otimes (a_2 \circ a_3) - a_2\alpha \otimes (a_1 \circ a_3),$$

(ii) $d_2 : B \otimes S_2(A) \to S_3(B)$ is given by

$$b_1 \otimes (a_2 \circ a_3) \mapsto b_1 \circ a_2\alpha \circ a_3\alpha,$$

(iii) $d_3 : S_3(B) \to B \otimes S_2(C)$ is given by

$$b_1 \circ b_2 \circ b_3 \mapsto b_1 \otimes (b_2\beta \circ b_3\beta) + b_2 \otimes (b_1\beta \circ b_3\beta) + b_3 \otimes (b_1\beta \circ b_2\beta),$$

and

(iv) $d_4 : B \otimes S_2(C) \to (C \otimes S_2(C))/S_3(C)$ is given by

$$b_1 \otimes (c_2 \circ c_3) \mapsto b_1\beta \otimes (c_2 \circ c_3) + S_3(C),$$

where $S_3(C)$ is considered as a submodule of $C \otimes S_2(C)$ via the embedding

$$c_1 \circ c_2 \circ c_3 \mapsto c_1 \otimes (c_2 \circ c_3) + c_2 \otimes (c_1 \circ c_3) + c_3 \otimes (c_1 \circ c_2).$$

It is easy to check that the maps d_1, d_2, d_3 and d_4 are KG-module homomorphisms. We also see that the above sequence is a chain complex of KG-modules. Indeed, we have that

$$\begin{aligned}[a_1, a_2, a_3] &\overset{d_1}{\mapsto} a_1\alpha \otimes (a_2 \circ a_3) - a_2\alpha \otimes (a_1 \circ a_3) \\ &\overset{d_2}{\mapsto} a_1\alpha \circ a_2\alpha \circ a_3\alpha - a_2\alpha \circ a_1\alpha \circ a_3\alpha = 0,\end{aligned}$$

by definition of the symmetric product,

$$\begin{aligned}b_1 \otimes (a_2 \circ a_3) &\overset{d_2}{\mapsto} b_1 \circ a_2\alpha \circ a_3\alpha \\ &\overset{d_3}{\mapsto} b_1 \otimes (a_2\alpha\beta \circ a_3\alpha\beta) + a_2\alpha \otimes (b_1\beta \circ a_3\alpha\beta) + a_3\alpha \otimes (b_1\beta \circ a_2\alpha\beta) \\ &= 0,\end{aligned}$$

since $\alpha\beta = 0$, and

$$\begin{aligned}b_1 \circ b_2 \circ b_3 &\overset{d_3}{\mapsto} b_1 \otimes (b_2\beta \circ b_3\beta) + b_2 \otimes (b_1\beta \circ b_3\beta) + b_3 \otimes (b_1\beta \circ b_2\beta) \\ &\overset{d_4}{\mapsto} b_1\beta \otimes (b_2\beta \circ b_3\beta) b_2\beta \otimes (b_1\beta \circ b_3\beta) b_3\beta \otimes (b_1\beta \circ b_2\beta) + S_3(C) \\ &= 0,\end{aligned}$$

since $b_1\beta \otimes (b_2\beta \circ b_3\beta) b_2\beta \otimes (b_1\beta \circ b_3\beta) b_3\beta \otimes (b_1\beta \circ b_2\beta) \in S_3(C)$. In fact, we have that this sequence is exact.

Lemma 7.4.3. *The five-term sequence* (7.4.6) *is exact. Moreover, if* 3 *is invertible in* K*, we have that the final term of this sequence is isomorphic to* $L_3(C)$*.*

Proof. Since A, B and C are K-free we have that the sequence (7.4.5) splits over K and, hence, $B \cong A \oplus C$ as K-modules. We also have that the short exact sequence $0 \to L_3(A) \to A \otimes S_2(A) \to S_3(A) \to 0$ splits over K (see section 7.4.4), so that $A \otimes S_2(A) \cong L_3(A) \oplus S_3(A)$. Now,

$$B \otimes S_2(A) \cong [A \otimes S_2(A)] \oplus [C \otimes S_2(A)] \cong L_3(A) \oplus S_3(A) \oplus [C \otimes S_2(A)]. \quad (7.4.7)$$

So we have that d_1 maps $L_3(A)$ isomorphically onto the first direct summand of $B \otimes S_2(A)$ in (7.4.7). For the symmetric cube, $S_3(B)$, we have the direct sum decomposition,

$$S_3(B) \cong S_3(A) \oplus [C \otimes S_2(A)] \oplus [S_2(C) \otimes A] \oplus S_3(C), \qquad (7.4.8)$$

and it is easy to see that d_2 maps the second and third summands of $B \otimes S_2(A)$ in (7.4.7) isomorphically onto the first and second summands of $S_3(B)$ in (7.4.8).

Next, the tensor product $B \otimes S_2(A)$ has direct sum decomposition

$$B \otimes S_2(C) \cong [A \otimes S_2(C)] \oplus [C \otimes S_2(C)] \cong [A \otimes S_2(C)] \oplus S_3(C) \oplus L_3(C), \qquad (7.4.9)$$

and we see that d_3 maps the third and fourth summands of $S_3(B)$ in (7.4.8) isomorphically onto the first and second summands of $B \otimes S_2(C)$ in (7.4.9). To summarise, we see that we have the following picture:

$$\begin{array}{ccccccccc}
L_3(A) & \xrightarrow{d_1} & B \otimes S_2(A) & \xrightarrow{d_2} & S_3(B) & \xrightarrow{d_3} & B \otimes S_2(C) & \xrightarrow{d_4} & (C \otimes S_2(C))/S_3(C) \\
\hline
L_3(A) & \xrightarrow{\sim} & L_3(A) & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
 & & S_3(A) & \xrightarrow{\sim} & S_3(A) & \rightarrow & 0 & \rightarrow & 0 \\
 & & C \otimes S_2(A) & \xrightarrow{\sim} & C \otimes S_2(A) & \rightarrow & 0 & \rightarrow & 0 \\
 & & & & S_2(C) \otimes A & \xrightarrow{\sim} & A \otimes S_2(C) & \rightarrow & 0 \\
 & & & & S_3(C) & \xrightarrow{\sim} & S_3(C) & \rightarrow & 0 \\
 & & & & & & L_3(C) & \xrightarrow{\sim} & L_3(C)
\end{array}$$

where $(C \otimes S_2(C))/S_3(C) \cong L_3(C)$ as KG-modules, whenever 3 is invertible in K (see Section 7.4.4). □

Lemma 7.4.4. *If G has no 3-torsion then $L_3(R_{ab}) \otimes \mathbb{Z}_p$ is a projective $\mathbb{Z}_pG$-module for all primes $p \neq 3$.*

Proof. Applying Lemma 7.4.3 to the relation and augmentation sequences, both reduced modulo p, gives the following exact sequences:

$$\begin{aligned} 0 \rightarrow L_3(R_{ab}) \otimes \mathbb{Z}_p \;&\rightarrow\; P \otimes S_2(R_{ab}) \otimes \mathbb{Z}_p \rightarrow S_3(P) \otimes \mathbb{Z}_p \\ &\rightarrow\; P \otimes S_2(IG) \otimes \mathbb{Z}_p \rightarrow L_3(IG) \otimes \mathbb{Z}_p \rightarrow 0, \qquad (7.4.10) \end{aligned}$$

and

$$
\begin{aligned}
0 \to L_3(IG) \otimes \mathbb{Z}_p \ &\to \ \mathbb{Z}G \otimes S_2(IG) \otimes \mathbb{Z}_p \to S_3(\mathbb{Z}G) \otimes \mathbb{Z}_p \\
&\to \ \mathbb{Z}G \otimes S_2(\mathbb{Z}) \otimes \mathbb{Z}_p \ \to L_3(\mathbb{Z}) \otimes \mathbb{Z}_p \to 0. \qquad (7.4.11)
\end{aligned}
$$

In (7.4.11), we have $\mathbb{Z}G \otimes S_2(\mathbb{Z}) \otimes \mathbb{Z}_p \cong \mathbb{Z}_pG$ and $L_3(\mathbb{Z}) \otimes \mathbb{Z}_p = 0$. Moreover, since the right hand term in (7.4.10) coincides with the left-hand term in (7.4.11), the two sequences can be combined to give a single seven-term exact sequence:

$$
\begin{aligned}
0 \to L_3(R_{ab}) \otimes \mathbb{Z}_p \ &\to \ P \otimes S_2(R_{ab}) \otimes \mathbb{Z}_p \to S_3(P) \otimes \mathbb{Z}_p \\
&\to \ P \otimes S_2(IG) \ \otimes \mathbb{Z}_p \to \mathbb{Z}G \otimes S_2(IG) \otimes \mathbb{Z}_p \\
&\qquad \to S_3(\mathbb{Z}G) \otimes \mathbb{Z}_p \to \mathbb{Z}_pG \to 0 \quad (7.4.12)
\end{aligned}
$$

of $\mathbb{Z}_pG$-modules. Since P and $\mathbb{Z}G$ are free $\mathbb{Z}G$-modules, it follows that the second, fourth and fifth terms in this sequence are free $\mathbb{Z}_pG$-modules. Moreover, by Theorem 7.2.3 it follows that the two symmetric cubes are also free $\mathbb{Z}_pG$-modules. Now, since all terms to the right of $L_3(R_{ab}) \otimes \mathbb{Z}_p$ are $\mathbb{Z}_pG$-free, we must have that $L_3(R_{ab}) \otimes \mathbb{Z}_p$ is a projective $\mathbb{Z}_pG$-module. □

We may apply Lemma 7.4.4 to show that the Bryant-Schocker modules B_{3p^i} for $L_{3p^m}(R_{ab} \otimes \mathbb{Z}_p)$ are projective as $\mathbb{Z}_pG$-modules.

Corollary 7.4.5. *Let G be a group with no 3-torsion and free presentation $G = F/R$. Let R_{ab} be the relation module stemming this presentation. Then the Bryant-Schocker modules B_{3p^i} for $L_{3p^m}(R_{ab} \otimes \mathbb{Z}_p)$ are projective $\mathbb{Z}_pG$-modules and, hence, have trivial homology in all positive dimensions.*

Proof. We recall from Chapter 6 that each B_{3p^i} $(0 \leq i \leq m)$ permits $L_3(R_{ab} \otimes \mathbb{Z}_p)$ as a tensor factor (by Theorem 6.6.3). By Lemma 7.4.4 we have that $L_3(R_{ab} \otimes \mathbb{Z}_p)$ is a projective $\mathbb{Z}_pG$-module. Hence, each B_{3p^i} is a projective $\mathbb{Z}_pG$-module and, thus, has trivial homology in all positive dimensions. □

7.5 Main results

We now come to the main results of this chapter.

Proposition 7.5.1. *Let p be a prime, $p \neq 2$. Let F be a non-cyclic free group and R a normal subgroup of F such that the quotient $G = F/R$ has no elements of order p or order 2. Then,*

$$H_n(G, L_{2p^m}(R_{ab} \otimes \mathbb{Z}_p)) = 0 \quad \textit{for all } n \geq 1, m \geq 0.$$

Proof. By the Bryant-Schocker Decomposition Theorem we have that for all $n \geq 0$

$$\begin{aligned} H_n(G, L_{2p^m}(R_{ab} \otimes \mathbb{Z}_p)) = \quad & H_n(G, L_{p^m}(B_2)) \oplus H_n(G, L_{p^{m-1}}(B_{2p})) \\ \oplus \quad & H_n(G, L_{p^{m-2}}(B_{2p^2})) \oplus \cdots \oplus H_n(G, B_{2p^m}), \end{aligned} \tag{7.5.1}$$

where the B_{2p^i} $(0 \leq i \leq m)$ denote the Bryant-Schocker modules for $L_{2p^m}(R_{ab} \otimes \mathbb{Z}_p)$. Since G has no 2-torsion, we may apply Corollary 7.4.2 to deduce that the Bryant-Schocker modules B_{2p^i} are all projective as $\mathbb{Z}_pG$-modules.

Next, since G has no p-torsion we have, by Lemma 7.2.4, that each $L_{p^{m-i}}(B_{2p^i})$ is also a projective $\mathbb{Z}_pG$-module. Thus, each $L_{p^{m-i}}(B_{2p^i})$ has zero homology in all positive dimensions and we see that every summand on the right hand side of (7.5.1) is zero. The result follows. □

Proposition 7.5.2. *Let p be a prime, $p \neq 3$. Let F be a non-cyclic free group and R a normal subgroup of F such that the quotient $G = F/R$ has no elements of order p or order 3. Then,*

$$H_n(G, L_{3p^m}(R_{ab} \otimes \mathbb{Z}_p)) = 0 \quad \textit{for all } n \geq 1, m \geq 0.$$

Proof. By the Bryant-Schocker Decomposition Theorem we have that for all $n \geq 0$

$$\begin{aligned} H_n(G, L_{3p^m}(R_{ab} \otimes \mathbb{Z}_p)) = \quad & H_n(G, L_{p^m}(B_3)) \oplus H_n(G, L_{p^{m-1}}(B_{3p})) \\ \oplus \quad & H_n(G, L_{p^{m-2}}(B_{3p^2})) \oplus \cdots \oplus H_n(G, B_{3p^m}), \end{aligned} \tag{7.5.2}$$

where the B_{3p^i} $(0 \leq i \leq m)$ denote the Bryant-Schocker modules for $L_{3p^m}(R_{ab} \otimes \mathbb{Z}_p)$. Since G has no 3-torsion, we may apply Corollary 7.4.5 to deduce that the Bryant-Schocker modules B_{3p^i} are all projective as $\mathbb{Z}_pG$-modules.

Next, since G has no p-torsion we have, by Lemma 7.2.4, that each $L_{p^{m-i}}(B_{3p^i})$ is also a projective $\mathbb{Z}_pG$-module. Thus, each $L_{p^{m-i}}(B_{3p^i})$ has zero homology in all positive dimensions and we see that every summand on the right hand side of (7.5.2) is zero. The result follows. □

Lemma 7.5.3. *Let p be a prime, $p \neq 2$. Let F be a non-cyclic free group and R a normal subgroup of F, such that the quotient $G = F/R$ has no elements of order p or order 2. Then, the exponent of the torsion subgroup of $\gamma_{2p^m}R/[\gamma_{2p^m}R, F]$ divides 2, for all $m \geq 0$.*

Proof. By Theorem 7.1.1 we have that the exponent of the torsion subgroup of $\gamma_{2p^m}R/[\gamma_{2p^m}R, F]$ divides $2p^m$. By (7.1.5), it is enough to show that $L_{2p^m}(R_{ab}) \otimes_G \mathbb{Z}$ has no p-torsion. Consider the short exact sequence

$$0 \to L_{2p^m}(R_{ab}) \xrightarrow{p} L_{2p^m}(R_{ab}) \to L_{2p^m}(R_{ab}) \otimes \mathbb{Z}_p \to 0, \tag{7.5.3}$$

where p denotes multiplication by p and the second map is simply reduction modulo p. The long exact homology sequence of (7.5.3) is given by

$$\begin{aligned} \cdots \to H_1(G, L_{2p^m}(R_{ab})) &\to H_1(G, L_{2p^m}(R_{ab}) \otimes \mathbb{Z}_p) \xrightarrow{\alpha} L_{2p^m}(R_{ab}) \otimes_G \mathbb{Z} \\ &\xrightarrow{p\otimes 1} L_{2p^m}(R_{ab}) \otimes_G \mathbb{Z} \to L_{2p^m}(R_{ab}) \otimes_G \mathbb{Z}_p \to 0. \end{aligned}$$

Hence, we see that $L_{2p^m}(R_{ab}) \otimes_G \mathbb{Z}$ has no p-torsion if and only if $\ker(p \otimes 1) = \operatorname{Im} \alpha = 0$. The result now follows by Proposition 7.5.1. □

Lemma 7.5.4. *Let p be a prime, $p \neq 3$. Let F be a non-cyclic free group and R a normal subgroup of F, such that the quotient $G = F/R$ has no elements of order p or order 3. Then, the exponent of the torsion subgroup of $\gamma_{3p^m}R/[\gamma_{3p^m}R, F]$ divides 3, for all $m \geq 0$.*

Proof. By Theorem 7.1.1 we have that the exponent of the torsion subgroup of $\gamma_{3p^m}R/[\gamma_{3p^m}R, F]$ divides $3p^m$. By (7.1.5), it is enough to show that $L_{3p^m}(R_{ab}) \otimes_G \mathbb{Z}$ has no p-torsion. Consider the short exact sequence

$$0 \to L_{3p^m}(R_{ab}) \xrightarrow{p} L_{3p^m}(R_{ab}) \to L_{3p^m}(R_{ab}) \otimes \mathbb{Z}_p \to 0, \tag{7.5.4}$$

where p denotes multiplication by p and the second map is simply reduction modulo p. The long exact homology sequence of (7.5.4) is given by

$$\begin{aligned} \cdots \to H_1(G, L_{3p^m}(R_{ab})) \quad &\to \quad H_1(G, L_{3p^m}(R_{ab}) \otimes \mathbb{Z}_p) \xrightarrow{\alpha} L_{3p^m}(R_{ab}) \otimes_G \mathbb{Z} \\ &\xrightarrow{p\otimes 1} \quad L_{3p^m}(R_{ab}) \otimes_G \mathbb{Z} \quad \to L_{3p^m}(R_{ab}) \otimes_G \mathbb{Z}_p \to 0. \end{aligned}$$

Hence, we see that $L_{3p^m}(R_{ab})\otimes_G\mathbb{Z}$ has no p-torsion if and only if $\ker(p\otimes 1) = \operatorname{Im}\alpha = 0$. The result now follows by Proposition 7.5.2. □

Theorem 7.5.5. *Let F be a non-cyclic free group and R a normal subgroup of F such that the quotient $G = F/R$ has no elements of order 2 or 3. Then, $\gamma_6 R/[\gamma_6 R, F]$ is torsion-free.*

Proof. This follows immediately from Lemmas 7.5.3 and 7.5.4. □

Example. Let F be a free group of rank d. We consider the case where $R = F'$, that is, when $G = F/F'$ is free abelian. In this case the quotient $F/[\gamma_c F', F]$ with $c \geq 2$ is the free centre-by-(nilpotent of class $(c-1)$)-by abelian group of the same rank as F. We have seen that if torsion occurs in $F/[\gamma_c F', F]$ then it occurs in the centre $\gamma_c F'/[\gamma_c F', F]$. By (7.1.7) we see that for $c = p$ a prime the torsion subgroup $t(\gamma_p F'/[\gamma_p F', F])$ is isomorphic to $H_4(G, \mathbb{Z}_p)$. Also, by (7.1.8) we have that the torsion subgroup $t(\gamma_4 F'/[\gamma_4 F', F]) \cong H_6(G, \mathbb{Z}_2)$. Now, for G free abelian, we have that the homology group $H_k(G, \mathbb{Z}_p)$ is an elementary abelian p-group of rank $\binom{d}{k}$ (we follow the convention that if $d < k$ then $\binom{d}{k} = 0$ and hence $H_k(G, \mathbb{Z}_p) = 0$).

Thus, we see that if the rank of F is large enough, we find torsion elements in $\gamma_c F'/[\gamma_c F', F]$ for c prime or equal to 4. However, by Theorem 7.5.5 we have the remarkable result that $\gamma_6 F'/[\gamma_6 F', F]$ is, in fact, *always* torsion-free.

Chapter 8

Further Work

In this chapter we briefly outline a few problems which we would recommend for further research.

8.1 Standard tableaux

In Chapter 3 we proved that for every partition λ of $n > 6$ there exists a standard tableau with major index coprime to n. This was sufficient for our needs. However, by the Kraśkiewicz and Weyman theorem (Theorem 3.1.3), it is known that for every such partition there exists a standard tableau with major index congruent to 1 (or any other fixed positive integer coprime to n) modulo n. It would be nice to be able to give a proof of this fact.

Here is a proof for rectangular partitions. We first notice that swapping the entries i and $i+1$ in a standard tableau T of shape λ gives rise to a new standard tableau T' of shape λ, provided that the entries i and $i+1$ do not occur in the same row or column. We shall call such an operation a switch and denote this by $T \xrightarrow{(i,i+1)} T'$.

Now, let $\lambda = (m^k)$ be a rectangular partition of $n = mk$ with $m > 2$. Let T be the standard tableau of shape λ obtained by filling in the entries row by row from left to right. We define standard tableaux $T_1, T_2, \ldots, T_k$ by a sequence of switches as follows:

$$\begin{array}{rcl} T & \xrightarrow{(m,m+1)} & T_1, \\ T_1 & \xrightarrow{(2m,2m+1)} & T_2, \\ T_2 & \xrightarrow{(3m,3m+1)} & T_3, \\ & \vdots & \\ T_{k-2} & \xrightarrow{((k-1)m,(k-1)m+1)} & T_{k-1}, \\ T_{k-1} & \xrightarrow{((k-1)m+1,(k-1)m+2)} & T_k. \end{array}$$

Notice that if $m = 2$ we cannot perform the final switch, as the entries $(k-1)m+1$ and $(k-1)m+2$ appear in the same column. (Also, we have that $(k-1)m+2 = n$, which cannot be moved in any switch on a rectangular partition.) It is straight-forward to check that

$$\begin{array}{rcl} \mathrm{maj}(T) & = & \frac{mk(k-1)}{2} \\ \mathrm{maj}(T_1) & = & \mathrm{maj}(T) + m \\ \mathrm{maj}(T_2) & = & \mathrm{maj}(T_1) + 2m = \mathrm{maj}(T) + 3m \\ & \vdots & \\ \mathrm{maj}(T_{k-1}) & = & \mathrm{maj}(T_{k-2}) + (k-1)m = \mathrm{maj}(T) + \frac{mk(k-1)}{2} \equiv 0 \bmod mk \\ \mathrm{maj}(T_k) & = & \mathrm{maj}(T_{k-1}) + 1 \equiv 1 \bmod mk. \end{array}$$

For $m = 2$ let T be the standard tableau of shape (2^k) obtained by filling in the entries column by column from top to bottom. We define standard tableaux T_1, T_2, T_3 and T_4 by the following sequence of switches:

$$T \xrightarrow{(k,k+1)} T_1 \xrightarrow{(k-1,k)} T_2 \xrightarrow{(k+1,k+2)} T_3 \xrightarrow{(k-2,k-1)} T_4.$$

Notice that for these switches to make sense we require that $k > 3$, since 1 cannot be moved in any switch. It is then easy to check that

$$\begin{aligned}
\text{maj}(T) &= 2k(k-1) \\
\text{maj}(T_1) &= 2k(k-2)+k \\
\text{maj}(T_2) &= 2k(k-2)+k+1 = \text{maj}(T)+3m \\
\text{maj}(T_3) &= 2k(k-2) \\
\text{maj}(T_4) &= 2k(k-2)+1 \equiv 1 \bmod 2k.
\end{aligned}$$

It may be possible to prove the non-rectangular case using switches, however, for more complicated shapes a more detailed analysis of the switching operation would be required.

8.2 Torsion

In Chapter 7 we proved that (i) for p a prime, $p \neq 2$, if $G = F/R$ has no elements of order p or order 2, then, for all $m \geq 0$, the exponent of the torsion subgroup of $\gamma_{2p^m}R/[\gamma_{2p^m}R, F]$ divides 2 and (ii) for p a prime, $p \neq 3$, if $G = F/R$ has no elements of order p or order 3, then, for all $m \geq 0$, the exponent of the torsion subgroup of $\gamma_{3p^m}R/[\gamma_{3p^m}R, F]$ divides 3. It is therefore natural to ask

(i) Can elements of order 2 occur in $\gamma_{2p^m}R/[\gamma_{2p^m}R, F]$?

(ii) Can elements of order 3 occur in $\gamma_{3p^m}R/[\gamma_{3p^m}R, F]$?

So far we have seen that elements of finite order can occur in $\gamma_c R/[\gamma_c R, F]$ for $c = 4$ or c a prime, whilst $\gamma_6 R/[\gamma_6 R, F]$ is always torsion free. We might wonder if $\gamma_c R/[\gamma_c R, F]$ is always torsion free, for c not a prime power. It would be nice to see how far the methods developed in Chapter 7 can be developed to answer questions such as this.

Appendix A

Preimages

A.1 Introduction

Let K be a field of characteristic 3, G a group and V a KG-module. In Section 5.3.1 we proved the existence of the following KG-module decomposition

$$L_6(V) = L_3(L_2(V)) \oplus B_6^{(3)},$$

where $B_6^{(3)} \cong L_2(V) \otimes S_2(V) \otimes S_2(V)$.

In order to assert the above isomorphism, it was shown that there exists a KG-module epimorphism $\psi_6^{(3)}$ of $L_6(V)$ onto $L_2(V) \otimes S_2(V) \otimes S_2(V)$ with kernel $L_3(L_2(V))$. In this appendix we verify that each of the maps listed in Lemma 5.3.6 is indeed a surjection.

Recall the family of KG-module homomorphisms $\psi_{H\sigma}$ where $H\sigma$ runs over all the cosets of H (see §5.3.1). Let $y = [a, b] \otimes c \circ d \otimes e \circ f$ be an arbitrary element of $L_2(V) \otimes S_2(V) \otimes S_2(V)$. In order to show that $\psi_{H\sigma}$ is a surjection, we need to find a preimage for y in $L_6(V)$.

The following pages give the preimages of $y = [a, b] \otimes c \circ d \otimes e \circ f$ for each of the maps $\psi_{H\sigma}$ listed in Lemma 5.3.6. We use the image line notation for permutations, as in §5.3.1

A.2 Results

For $\sigma^{-1} = 142356$ the preimage of $y = [a, b] \otimes c \circ d \otimes e \circ f$ under $\psi_{H\sigma}$ is given by

$-[a, f, e, d, c, b]$ $+[b, f, e, d, c, a]$ $+[[a, d, c, b], [e, f]]$
$-[[b, d, c, a], [e, f]]$ $+[[a, e, d, c], [b, f]]$ $+[[c, e, d, a], [b, f]]$
$+[[d, e, c, a], [b, f]]$ $-[[b, e, d, c], [a, f]]$ $-[[c, e, d, b], [a, f]]$
$-[[d, e, c, b], [a, f]]$ $+[[a, f, d, c], [b, e]]$ $+[[c, f, d, a], [b, e]]$
$+[[d, f, c, a], [b, e]]$ $-[[b, f, d, c], [a, e]]$ $-[[c, f, d, b], [a, e]]$
$-[[d, f, c, b], [a, e]]$ $+[[a, f, e, b], [c, d]]$ $-[[b, f, e, a], [c, d]]$
$+[[e, f, c, a], [b, d]]$ $-[[e, f, c, b], [a, d]]$ $+[[e, f, d, a], [b, c]]$
$-[[e, f, d, b], [a, c]]$ $+[[c, f, e, d], [a, b]]$ $+[[d, f, e, c], [a, b]]$
$+[[e, f, d, c], [a, b]]$ $+[[a, f, b], [c, e, d]]$ $+[[a, f, b], [d, e, c]]$
$-[[a, f, c], [b, e, d]]$ $-[[a, f, c], [d, e, b]]$ $-[[a, f, d], [b, e, c]]$
$-[[a, f, d], [c, e, b]]$ $+[[a, f, e], [c, d, b]]$ $+[[a, e, b], [c, f, d]]$
$+[[a, e, b], [d, f, c]]$ $-[[a, e, c], [b, f, d]]$ $-[[a, e, c], [d, f, b]]$
$-[[a, e, d], [b, f, c]]$ $-[[a, e, d], [c, f, b]]$ $-[[a, d, b], [c, f, e]]$
$+[[a, d, c], [e, f, b]]$ $-[[a, c, b], [d, f, e]]$ $-[[b, f, a], [c, e, d]]$
$-[[b, f, a], [d, e, c]]$ $+[[b, f, c], [d, e, a]]$ $+[[b, f, d], [c, e, a]]$
$-[[b, f, e], [c, d, a]]$ $-[[b, e, a], [c, f, d]]$ $-[[b, e, a], [d, f, c]]$
$+[[b, e, c], [d, f, a]]$ $+[[b, e, d], [c, f, a]]$ $+[[b, d, a], [c, f, e]]$
$-[[b, d, c], [e, f, a]]$ $+[[c, d, a], [e, f, b]]$ $-[[c, d, b], [e, f, a]]$
$+[[b, c, a], [d, f, e]]$

For $\sigma^{-1} = 145623$ the preimage of $y = [a,b] \otimes c \circ d \otimes e \circ f$ under $\psi_{H\sigma}$ is given by

$$
\begin{array}{lll}
-[a,f,e,d,c,b] & +[b,f,e,d,c,a] & +[[a,d,c,b],[e,f]] \\
-[[b,d,c,a],[e,f]] & -[[a,e,c,b],[d,f]] & +[[b,e,c,a],[d,f]] \\
-[[a,e,d,b],[c,f]] & +[[b,e,d,a],[c,f]] & +[[c,e,d,a],[b,f]] \\
+[[d,e,c,a],[b,f]] & -[[c,e,d,b],[a,f]] & -[[d,e,c,b],[a,f]] \\
-[[a,f,c,b],[d,e]] & +[[b,f,c,a],[d,e]] & -[[a,f,d,b],[c,e]] \\
+[[b,f,d,a],[c,e]] & +[[c,f,d,a],[b,e]] & +[[d,f,c,a],[b,e]] \\
-[[c,f,d,b],[a,e]] & -[[d,f,c,b],[a,e]] & +[[a,f,e,b],[c,d]] \\
-[[b,f,e,a],[c,d]] & +[[a,f,e,c],[b,d]] & -[[c,f,e,a],[b,d]] \\
+[[e,f,c,a],[b,d]] & -[[b,f,e,c],[a,d]] & +[[c,f,e,b],[a,d]] \\
-[[e,f,c,b],[a,d]] & +[[a,f,e,d],[b,c]] & -[[d,f,e,a],[b,c]] \\
+[[e,f,d,a],[b,c]] & -[[b,f,e,d],[a,c]] & +[[d,f,e,b],[a,c]] \\
-[[e,f,d,b],[a,c]] & +[[c,f,e,d],[a,b]] & +[[d,f,e,c],[a,b]] \\
+[[e,f,d,c],[a,b]] & -[[a,f,c],[b,e,d]] & +[[a,f,c],[d,e,b]] \\
-[[a,f,d],[b,e,c]] & +[[a,f,d],[c,e,b]] & +[[a,f,e],[c,d,b]] \\
-[[a,e,c],[b,f,d]] & +[[a,e,c],[d,f,b]] & -[[a,e,d],[b,f,c]] \\
+[[a,e,d],[c,f,b]] & +[[a,d,b],[e,f,c]] & +[[a,d,c],[e,f,b]] \\
+[[a,c,b],[e,f,d]] & -[[b,f,c],[d,e,a]] & -[[b,f,d],[c,e,a]] \\
-[[b,f,e],[c,d,a]] & -[[b,e,c],[d,f,a]] & -[[b,e,d],[c,f,a]] \\
-[[b,d,a],[e,f,c]] & -[[b,d,c],[e,f,a]] & +[[c,f,a],[d,e,b]] \\
-[[c,f,b],[d,e,a]] & +[[c,e,a],[d,f,b]] & -[[c,e,b],[d,f,a]] \\
+[[c,d,a],[e,f,b]] & -[[c,d,b],[e,f,a]] & -[[b,c,a],[e,f,d]]
\end{array}
$$

For $\sigma^{-1} = 241356$ the preimage of $y = [a, b] \otimes c \circ d \otimes e \circ f$ under $\psi_{H\sigma}$ is given by

$-[a, f, e, d, c, b]$ $+[b, f, e, d, c, a]$ $+[[a, d, c, b], [e, f]]$
$-[[b, d, c, a], [e, f]]$ $-[[a, e, c, b], [d, f]]$ $+[[b, e, c, a], [d, f]]$
$-[[a, e, d, b], [c, f]]$ $+[[b, e, d, a], [c, f]]$ $+[[c, e, d, a], [b, f]]$
$+[[d, e, c, a], [b, f]]$ $-[[c, e, d, b], [a, f]]$ $-[[d, e, c, b], [a, f]]$
$-[[a, f, c, b], [d, e]]$ $+[[b, f, c, a], [d, e]]$ $-[[a, f, d, b], [c, e]]$
$+[[b, f, d, a], [c, e]]$ $+[[c, f, d, a], [b, e]]$ $+[[d, f, c, a], [b, e]]$
$-[[c, f, d, b], [a, e]]$ $-[[d, f, c, b], [a, e]]$ $+[[a, f, e, b], [c, d]]$
$-[[b, f, e, a], [c, d]]$ $+[[a, f, e, c], [b, d]]$ $-[[c, f, e, a], [b, d]]$
$+[[e, f, c, a], [b, d]]$ $-[[b, f, e, c], [a, d]]$ $+[[c, f, e, b], [a, d]]$
$-[[e, f, c, b], [a, d]]$ $+[[a, f, e, d], [b, c]]$ $-[[d, f, e, a], [b, c]]$
$+[[e, f, d, a], [b, c]]$ $-[[b, f, e, d], [a, c]]$ $+[[d, f, e, b], [a, c]]$
$-[[e, f, d, b], [a, c]]$ $+[[c, f, e, d], [a, b]]$ $+[[d, f, e, c], [a, b]]$
$+[[e, f, d, c], [a, b]]$ $+[[a, f, b], [c, e, d]]$ $+[[a, f, b], [d, e, c]]$
$-[[a, f, c], [b, e, d]]$ $-[[a, f, c], [d, e, b]]$ $-[[a, f, d], [b, e, c]]$
$-[[a, f, d], [c, e, b]]$ $+[[a, f, e], [c, d, b]]$ $+[[a, e, b], [c, f, d]]$
$+[[a, e, b], [d, f, c]]$ $-[[a, e, c], [b, f, d]]$ $-[[a, e, c], [d, f, b]]$
$-[[a, e, d], [b, f, c]]$ $-[[a, e, d], [c, f, b]]$ $-[[a, d, b], [c, f, e]]$
$+[[a, d, c], [e, f, b]]$ $-[[a, c, b], [d, f, e]]$ $-[[b, f, a], [c, e, d]]$
$-[[b, f, a], [d, e, c]]$ $+[[b, f, c], [d, e, a]]$ $+[[b, f, d], [c, e, a]]$
$-[[b, f, e], [c, d, a]]$ $-[[b, e, a], [c, f, d]]$ $-[[b, e, a], [d, f, c]]$
$+[[b, e, c], [d, f, a]]$ $+[[b, e, d], [c, f, a]]$ $+[[b, d, a], [c, f, e]]$
$-[[b, d, c], [e, f, a]]$ $+[[c, d, a], [e, f, b]]$ $-[[c, d, b], [e, f, a]]$
$+[[b, c, a], [d, f, e]]$

For $\sigma^{-1} = 245613$ the preimage of $y = [a, b] \otimes c \circ d \otimes e \circ f$ under $\psi_{H\sigma}$ is given by

$$
\begin{array}{lll}
-[a,f,e,d,c,b] & +[b,f,e,d,c,a] & +[[a,d,c,b],[e,f]] \\
-[[b,d,c,a],[e,f]] & +[[a,e,d,c],[b,f]] & +[[c,e,d,a],[b,f]] \\
+[[d,e,c,a],[b,f]] & -[[b,e,d,c],[a,f]] & -[[c,e,d,b],[a,f]] \\
-[[d,e,c,b],[a,f]] & +[[a,f,d,c],[b,e]] & +[[c,f,d,a],[b,e]] \\
+[[d,f,c,a],[b,e]] & -[[b,f,d,c],[a,e]] & -[[c,f,d,b],[a,e]] \\
-[[d,f,c,b],[a,e]] & +[[a,f,e,b],[c,d]] & -[[b,f,e,a],[c,d]] \\
+[[e,f,c,a],[b,d]] & -[[e,f,c,b],[a,d]] & +[[e,f,d,a],[b,c]] \\
-[[e,f,d,b],[a,c]] & +[[c,f,e,d],[a,b]] & +[[d,f,e,c],[a,b]] \\
+[[e,f,d,c],[a,b]] & -[[a,f,c],[b,e,d]] & +[[a,f,c],[d,e,b]] \\
-[[a,f,d],[b,e,c]] & +[[a,f,d],[c,e,b]] & +[[a,f,e],[c,d,b]] \\
-[[a,e,c],[b,f,d]] & +[[a,e,c],[d,f,b]] & -[[a,e,d],[b,f,c]] \\
+[[a,e,d],[c,f,b]] & +[[a,d,b],[e,f,c]] & +[[a,d,c],[e,f,b]] \\
+[[a,c,b],[e,f,d]] & -[[b,f,c],[d,e,a]] & -[[b,f,d],[c,e,a]] \\
-[[b,f,e],[c,d,a]] & -[[b,e,c],[d,f,a]] & -[[b,e,d],[c,f,a]] \\
-[[b,d,a],[e,f,c]] & -[[b,d,c],[e,f,a]] & +[[c,f,a],[d,e,b]] \\
-[[c,f,b],[d,e,a]] & +[[c,e,a],[d,f,b]] & -[[c,e,b],[d,f,a]] \\
+[[c,d,a],[e,f,b]] & -[[c,d,b],[e,f,a]] & -[[b,c,a],[e,f,d]]
\end{array}
$$

For $\sigma^{-1} = 251634$ the preimage of $y = [a, b] \otimes c \circ d \otimes e \circ f$ under $\psi_{H\sigma}$ is given by

$$
\begin{array}{lll}
-[a, f, e, d, c, b] & +[b, f, e, d, c, a] & +[[a, d, c, b], [e, f]] \\
-[[b, d, c, a], [e, f]] & -[[a, e, c, b], [d, f]] & +[[b, e, c, a], [d, f]] \\
-[[a, e, d, b], [c, f]] & +[[b, e, d, a], [c, f]] & +[[c, e, d, a], [b, f]] \\
+[[d, e, c, a], [b, f]] & -[[c, e, d, b], [a, f]] & -[[d, e, c, b], [a, f]] \\
-[[a, f, c, b], [d, e]] & +[[b, f, c, a], [d, e]] & -[[a, f, d, b], [c, e]] \\
+[[b, f, d, a], [c, e]] & +[[c, f, d, a], [b, e]] & +[[d, f, c, a], [b, e]] \\
-[[c, f, d, b], [a, e]] & -[[d, f, c, b], [a, e]] & +[[a, f, e, b], [c, d]] \\
-[[b, f, e, a], [c, d]] & +[[a, f, e, c], [b, d]] & -[[c, f, e, a], [b, d]] \\
+[[e, f, c, a], [b, d]] & -[[b, f, e, c], [a, d]] & +[[c, f, e, b], [a, d]] \\
-[[e, f, c, b], [a, d]] & +[[a, f, e, d], [b, c]] & -[[d, f, e, a], [b, c]] \\
+[[e, f, d, a], [b, c]] & -[[b, f, e, d], [a, c]] & +[[d, f, e, b], [a, c]] \\
-[[e, f, d, b], [a, c]] & +[[c, f, e, d], [a, b]] & +[[d, f, e, c], [a, b]] \\
+[[e, f, d, c], [a, b]] & +[[a, f, e], [b, d, c]] & -[[a, f, e], [c, d, b]] \\
+[[a, d, b], [c, f, e]] & -[[a, d, b], [e, f, c]] & +[[a, d, c], [b, f, e]] \\
-[[a, d, c], [e, f, b]] & +[[a, c, b], [d, f, e]] & -[[a, c, b], [e, f, d]] \\
+[[b, f, e], [c, d, a]] & -[[b, d, a], [c, f, e]] & +[[b, d, a], [e, f, c]] \\
+[[b, d, c], [e, f, a]] & +[[c, f, a], [d, e, b]] & -[[c, f, b], [d, e, a]] \\
+[[c, e, a], [d, f, b]] & -[[c, e, b], [d, f, a]] & -[[c, d, a], [e, f, b]] \\
+[[c, d, b], [e, f, a]] & -[[b, c, a], [d, f, e]] & +[[b, c, a], [e, f, d]]
\end{array}
$$

Finally, for $\sigma^{-1} = 253416$ the preimage of $y = [a,b] \otimes c \circ d \otimes e \circ f$ under $\psi_{H\sigma}$ is given by

$$
\begin{array}{lll}
-[a,f,e,d,c,b] & +[b,f,e,d,c,a] & +[[a,d,c,b],[e,f]] \\
-[[b,d,c,a],[e,f]] & +[[a,e,d,c],[b,f]] & +[[c,e,d,a],[b,f]] \\
+[[d,e,c,a],[b,f]] & -[[b,e,d,c],[a,f]] & -[[c,e,d,b],[a,f]] \\
-[[d,e,c,b],[a,f]] & +[[a,f,d,c],[b,e]] & +[[c,f,d,a],[b,e]] \\
+[[d,f,c,a],[b,e]] & -[[b,f,d,c],[a,e]] & -[[c,f,d,b],[a,e]] \\
-[[d,f,c,b],[a,e]] & +[[a,f,e,b],[c,d]] & -[[b,f,e,a],[c,d]] \\
+[[e,f,c,a],[b,d]] & -[[e,f,c,b],[a,d]] & +[[e,f,d,a],[b,c]] \\
-[[e,f,d,b],[a,c]] & +[[c,f,e,d],[a,b]] & +[[d,f,e,c],[a,b]] \\
+[[e,f,d,c],[a,b]] & -[[a,f,b],[c,e,d]] & -[[a,f,b],[d,e,c]] \\
-[[a,f,c],[d,e,b]] & -[[a,f,d],[c,e,b]] & +[[a,f,e],[b,d,c]] \\
-[[a,f,e],[c,d,b]] & -[[a,e,b],[c,f,d]] & -[[a,e,b],[d,f,c]] \\
-[[a,e,c],[d,f,b]] & -[[a,e,d],[c,f,b]] & -[[a,d,b],[c,f,e]] \\
+[[a,d,c],[b,f,e]] & -[[a,d,c],[e,f,b]] & -[[a,c,b],[d,f,e]] \\
+[[b,f,a],[c,e,d]] & +[[b,f,a],[d,e,c]] & +[[b,f,c],[d,e,a]] \\
+[[b,f,d],[c,e,a]] & +[[b,f,e],[c,d,a]] & +[[b,e,a],[c,f,d]] \\
+[[b,e,a],[d,f,c]] & +[[b,e,c],[d,f,a]] & +[[b,e,d],[c,f,a]] \\
+[[b,d,a],[c,f,e]] & +[[b,d,c],[e,f,a]] & -[[c,f,a],[d,e,b]] \\
+[[c,f,b],[d,e,a]] & -[[c,e,a],[d,f,b]] & +[[c,e,b],[d,f,a]] \\
-[[c,d,a],[e,f,b]] & +[[c,d,b],[e,f,a]] & +[[b,c,a],[d,f,e]]
\end{array}
$$

Appendix B

Lie powers of infinite-dimensional modules, R. M. Bryant

We reproduce here, for reference purposes, an unpublished manuscript of R. M. Bryant. This document is a draft of a generalisation of two results of Bryant and Schocker, namely, the ‘Decomposition Theorem’ [10, Theorem 4.4] and a certain isomorphism of modules [11, Equation (4.5) of Theorem 4.2] (stated there in terms of the Green ring), to allow for infinite dimensional KG-modules V. Bryant also goes on to give a generalisation of our Theorem 6.4.2.

The work described in this appendix remains the intellectual property of R. M. Bryant and the article appears here in its original form. As such, we remark that there are a few notational differences between this appendix and the main text. In particular, we remind the reader to beware that in this appendix the action of the symmetric group of degree n on the nth tensor power is on the *left*. Our right action of the symmetric group is simply the opposite action to the one described here. Thus, some care needs to be taken when considering the idempotents obtained.

The article appears here with its own reference list labelled [B1], [B2], . . . , in order to avoid confusion with the references cited in the main text.

We thank Professor Bryant for his kind permission to reproduce this work here.

B.1 Introduction

Let G be a group and F a field. For any FG-module V, let $L(V)$ be the free Lie algebra on V (the free Lie algebra freely generated by any basis of V), and regard $L(V)$ as an FG-module by extending the action of G on V so that G acts on $L(V)$ by Lie algebra automorphisms. For each positive integer n, the nth homogeneous component $L^n(V)$ is a submodule of $L(V)$, called the nth Lie power of V.

In the case where V is finite-dimensional the modules $L^n(V)$ have been studied in considerable depth: see [B3] and the papers cited there. The results are best when F has characteristic 0, but there has recently been substantial progress in the case of prime characteristic p. In [B3], a general decomposition theorem was obtained which reduces the study of arbitrary Lie powers of V to the study of Lie powers of the form $L^{p^i}(B_r)$, where, for each r, B_r is a certain direct summand of the rth tensor power $V^{\otimes r}$. This is a reduction to Lie powers of p-power degree. Information about the isomorphism types of the modules B_r is given in [B4] and [B2].

Recently, Marianne Johnson and Ralph Stöhr have commenced a study of torsion in certain 'free central extensions' of groups and have found that they can make striking use of the decomposition theorem of [B3] and the results of [B4] and [B2] provided that these results are available for infinite-dimensional modules V (see [B9] and [B10]). The purpose of the present paper is to derive such results by utilising the results in the finite-dimensional case and some facts about modules for Schur algebras. One attractive consequence of the arguments here is a reformulation and sharpening of the previous results in terms of idempotents of the group algebras of the symmetric groups. This gives results that are uniform for all fields of the same characteristic.

Acknowledgement. I am grateful to Ralph Stöhr for suggesting this topic.

B.2 Preliminaries

All modules considered in this paper will be right modules, except for left modules arising from the action of symmetric groups on tensor powers, as explained below. We shall use the Schur algebras associated with the general linear group, as defined in [B6]. However, [B6] treats only infinite fields and uses left modules. Thus we give a self-contained summary of the basic facts (following the treatment in [B3, §2]).

Let F be a field. Let n and r be positive integers, and let $I(n,r)$ be the set of all ordered r-tuples $\mathbf{i} = (i_1, \ldots, i_r)$, where $i_1, \ldots, i_r \in \{1, \ldots, n\}$. Let $A_F(n,r)$ be the homogeneous component of degree r in the polynomial ring over F in n^2 commuting indeterminates c_{ij} $(1 \leq i, j \leq n)$. Thus $A_F(n,r)$ has an F-basis consisting of the monomials of degree r. For $\mathbf{i}, \mathbf{j} \in I(n,r)$, where $\mathbf{i} = (i_1, \ldots, i_r)$ and $\mathbf{j} = (j_1, \ldots, j_r)$, we write $c_{\mathbf{i},\mathbf{j}}$ for the monomial $c_{i_1 j_1} \cdots c_{i_r j_r}$. The $c_{\mathbf{i},\mathbf{j}}$ are not distinct (when $n, r > 1$) but they give a basis (with repetitions) of $A_F(n,r)$.

Let $S_F(n,r) = \mathrm{Hom}_F(A_F(n,r), F)$. Therefore $S_F(n,r)$ has a basis (with repetitions) consisting of the elements $\xi_{\mathbf{i},\mathbf{j}}$ (with $\mathbf{i}, \mathbf{j} \in I(n,r)$), where $\xi_{\mathbf{i},\mathbf{j}}(c_{\mathbf{i},\mathbf{j}}) = 1$ and $\xi_{\mathbf{i},\mathbf{j}}(c_{\mathbf{i}',\mathbf{j}'}) = 0$ if $c_{\mathbf{i}',\mathbf{j}'} \neq c_{\mathbf{i},\mathbf{j}}$. Multiplication in $S_F(n,r)$ may be defined as in [B6, §2.3]: for $\xi, \eta \in S_F(n,r)$,

$$(\xi\eta)(c_{\mathbf{i},\mathbf{j}}) = \sum_{\mathbf{k} \in I(n,r)} \xi(c_{\mathbf{i},\mathbf{k}})\eta(c_{\mathbf{k},\mathbf{j}}).$$

In this way $S_F(n,r)$ becomes an associative F-algebra with identity element. This is the Schur algebra of degree r. If E is an extension field of F then we usually identify $E \otimes_F S_F(n,r)$ with $S_E(n,r)$ in the obvious way.

For $g = (a_{ij}) \in \mathrm{GL}(n,F)$, define $\zeta_g \in S_F(n,r)$ by

$$\zeta_g(c_{i_1 j_1} \cdots c_{i_r j_r}) = a_{i_1 j_1} \cdots a_{i_r j_r} \in F.$$

Then (see [B6, §2.4]) the map $g \mapsto \zeta_g$ extends to an algebra homomorphism

$$F\mathrm{GL}(n,F) \longrightarrow S_F(n,r) \tag{B.2.1}$$

that is surjective if F is infinite. If U is a (right) $S_F(n,r)$-module then U may be regarded as a (right) $F\mathrm{GL}(n,F)$-module by means of (B.2.1), and such $F\mathrm{GL}(n,F)$-modules are called polynomial modules of degree r.

Let V be an n-dimensional F-space (that is, vector space over F) with basis $\{x_1,\ldots,x_n\}$ and consider the rth tensor power $V^{\otimes r}$. For $\mathbf{i} \in I(n,r)$, where $\mathbf{i} = (i_1,\ldots,i_r)$, write $x_{\mathbf{i}} = x_{i_1} \otimes \cdots \otimes x_{i_r} \in V^{\otimes r}$. Thus the elements $x_{\mathbf{i}}$ form a basis of $V^{\otimes r}$. With respect to the basis $\{x_1,\ldots,x_n\}$, the identity representation $\mathrm{GL}(n,F) \to \mathrm{GL}(n,F)$ gives V the structure of an $F\mathrm{GL}(n,F)$-module, called the 'natural' module. Hence $V^{\otimes r}$ becomes an $F\mathrm{GL}(n,F)$-module under the 'diagonal' action of $\mathrm{GL}(n,F)$. It is straightforward to check (see [B6, §2.6]) that $V^{\otimes r}$ is a polynomial module of degree r. Indeed, for all $\xi \in S_F(n,r)$, we have

$$x_{\mathbf{i}}\xi = \sum_{\mathbf{j}} \xi(c_{\mathbf{i},\mathbf{j}})x_{\mathbf{j}}. \tag{B.2.2}$$

In particular, with $r = 1$, V is an $S_F(n,1)$-module, called the natural module.

We take the symmetric group Σ_r to act on the right on $\{1,\ldots,r\}$. Then, if V is any F-space (of finite or infinite dimension), $V^{\otimes r}$ may be given the structure of a left $F\Sigma_r$-module by making Σ_r act on $V^{\otimes r}$ by 'place permutations'; that is, for $\sigma \in \Sigma_r$ and $v_1,\ldots,v_r \in V$, we take

$$\sigma(v_1 \otimes \cdots \otimes v_r) = v_{1\sigma} \otimes \cdots \otimes v_{r\sigma}. \tag{B.2.3}$$

It is easily seen that the action of $F\Sigma_r$ on $V^{\otimes r}$ is faithful when $\dim V \geq r$.

Let V be an F-space with basis $\{x_i : i \in I\}$, where I is some index set. The free associative algebra freely generated by $\{x_i : i \in I\}$ is denoted by $T(V)$ and, for each r, the rth homogeneous component is denoted by $T^r(V)$. However, we shall write products in $T(V)$ as tensor products so that $T(V)$ is thought of as the 'tensor algebra' on V and $T^r(V) = V^{\otimes r}$. Let $\psi \in \mathrm{End}_F(V)$. Then, since $T(V)$ is free on $\{x_i : i \in I\}$, there is a unique algebra endomorphism ψ^* of $T(V)$ such that $v\psi^* = v\psi$ for all $v \in V$. The restriction of ψ^* to $V^{\otimes r}$ gives $\psi^{\otimes r} \in \mathrm{End}_F(V^{\otimes r})$, and it is easy to verify that $\psi^{\otimes r}$ commutes with the action of σ on $V^{\otimes r}$ for all $\sigma \in \Sigma_r$.

Let G be a group and let V be an FG-module. Then, for each $g \in G$, the action of g on V is given by an (invertible) map $\psi_g \in \mathrm{End}_F(V)$. We can make $T(V)$ into an FG-module by taking the action of g to be the algebra automorphism ψ_g^*. Thus $V^{\otimes r}$ is a submodule on which g acts as $\psi_g^{\otimes r}$. (This is the 'diagonal' action we have already met.) Hence the actions of FG and $F\Sigma_r$ on $V^{\otimes r}$ commute; in other words, $V^{\otimes r}$ is an $(F\Sigma_r, FG)$-bimodule.

In particular, if V is the natural $F\mathrm{GL}(n, F)$-module, we see that $V^{\otimes r}$ is an $(F\Sigma_r, F\mathrm{GL}(n, F))$-bimodule. Indeed, from (B.2.2) and (B.2.3), it is straightforward to verify the stronger fact that $V^{\otimes r}$ is an $(F\Sigma_r, S_F(n, r))$-bimodule. For $u \in F\Sigma_r$, the right ideal $uF\Sigma_r$ of $F\Sigma_r$ may be regarded as a (right) $F\Sigma_r$-module.

Lemma B.2.1. *Let F be a field and r a positive integer. Let $e_1, \ldots, e_s, f_1, \ldots, f_t$ be idempotent elements of $F\Sigma_r$ such that there is an isomorphism*

$$e_1F\Sigma_r \oplus \cdots \oplus e_sF\Sigma_r \cong f_1F\Sigma_r \oplus \cdots \oplus f_tF\Sigma_r$$

of $F\Sigma_r$-modules. (These are 'external' direct sums: we do not assume that the ideals span their direct sums within $F\Sigma_r$.) Then, if A is an F-algebra and M is an $(F\Sigma_r, A)$-bimodule, there is an isomorphism of A-modules

$$e_1M \oplus \cdots \oplus e_sM \cong f_1M \oplus \cdots \oplus f_tM.$$

Proof. There is an isomorphism of A-modules $\alpha : F\Sigma_r \otimes_{F\Sigma_r} M \to M$ given by $u \otimes v \mapsto uv$ for all $u \in F\Sigma_r$, $v \in M$. Let e be an idempotent of $F\Sigma_r$. Since $eF\Sigma_r$ is a direct summand of $F\Sigma_r$, the A-module $eF\Sigma_r \otimes_{F\Sigma_r} M$ may be regarded as a direct summand (and hence submodule) of $F\Sigma_r \otimes_{F\Sigma_r} M$ and α restricts to give an isomorphism $eF\Sigma_r \otimes_{F\Sigma_r} M \cong eM$. Thus

$$\begin{aligned} e_1M \oplus \cdots \oplus e_sM &\cong (e_1F\Sigma_r \oplus \cdots \oplus e_sF\Sigma_r) \otimes_{F\Sigma_r} M \\ &\cong (f_1F\Sigma_r \oplus \cdots \oplus f_tF\Sigma_r) \otimes_{F\Sigma_r} M \cong f_1M \oplus \cdots \oplus f_tM, \end{aligned}$$

as required. ∎

The next lemma is a version of the well-known fact that the Schur functor has a 'right inverse': see [B6, §6]. However, we have found no reference that gives exactly what is needed here (with F arbitrary and $\dim V \geq r$), so we sketch a short self-contained proof.

Let F be a field and let n and r be positive integers, where $n \geq r$. Let $\phi : F\Sigma_r \to S_F(n,r)$ be the linear map satisfying $\sigma\phi = \xi_{(1,\dots,r),(1\sigma,\dots,r\sigma)}$ for all $\sigma \in \Sigma_r$, and write $\xi_1 = 1\phi$. It can be checked from the definition of multiplication in $S_F(n,r)$ that $(\sigma\tau)\phi = (\sigma\phi)(\tau\phi)$ for all $\sigma, \tau \in \Sigma_r$. If M is an $S_F(n,r)$-module then it is easily seen that $M\xi_1$ is invariant under $(F\Sigma_r)\phi$. Thus $M\xi_1$ becomes a right $F\Sigma_r$-module, denoted by $\mathrm{S}(M)$ (where S indicates the Schur functor).

Lemma B.2.2. *Let F be a field and r a positive integer. Let V be an F-space of finite dimension n, where $n \geq r$, and regard $V^{\otimes r}$ as an $(F\Sigma_r, S_F(n,r))$-bimodule. Then, for any right $F\Sigma_r$-module U, we have $\mathrm{S}(U \otimes_{F\Sigma_r} V^{\otimes r}) \cong U$.*

Proof. Let $\{x_1, \dots, x_n\}$ be a basis of V and let Z be the subspace of $V^{\otimes r}$ spanned by $\{x_{1\sigma} \otimes \cdots \otimes x_{r\sigma} : \sigma \in \Sigma_r\}$. Clearly there is an isomorphism of F-spaces $\theta : F\Sigma_r \to Z$ given by $\sigma\theta = x_{1\sigma} \otimes \cdots \otimes x_{r\sigma}$ for all $\sigma \in \Sigma_r$. It is easily checked that $\mathrm{S}(V^{\otimes r}) = (V^{\otimes r})\xi_1 = Z$ and that θ is an isomorphism of $(F\Sigma_r, F\Sigma_r)$-bimodules.

In particular, Z is injective as a left $F\Sigma_r$-module, hence a direct summand of $V^{\otimes r}$. Thus $U \otimes_{F\Sigma_r} Z$ is isomorphic to a direct summand W of $U \otimes_{F\Sigma_r} V^{\otimes r}$, where W is the subspace of $U \otimes_{F\Sigma_r} V^{\otimes r}$ spanned by $\{u \otimes z : u \in U, z \in Z\}$. Also,

$$\mathrm{S}(U \otimes_{F\Sigma_r} V^{\otimes r}) = (U \otimes_{F\Sigma_r} V^{\otimes r})\xi_1 = W, \tag{B.2.4}$$

since $(V^{\otimes r})\xi_1 = Z$. However, it is easily seen that $U \otimes_{F\Sigma_r} Z$ and W are isomorphic as right $F\Sigma_r$-modules. Therefore

$$W \cong U \otimes_{F\Sigma_r} Z \cong U \otimes_{F\Sigma_r} F\Sigma_r \cong U.$$

Thus the result follows from (B.2.4). □

Corollary B.2.3. *Let F, r and V be as in Lemma B.2.2. Let $e_1, \ldots, e_s, f_1, \ldots, f_t$ be idempotents of $F\Sigma_r$ such that there is an isomorphism of $S_F(n,r)$-modules*

$$e_1V^{\otimes r} \oplus \cdots \oplus e_sV^{\otimes r} \cong f_1V^{\otimes r} \oplus \cdots \oplus f_tV^{\otimes r}.$$

Then, if A is an F-algebra and M is an $(F\Sigma_r, A)$-bimodule, there is an isomorphism of A-modules

$$e_1M \oplus \cdots \oplus e_sM \cong f_1M \oplus \cdots \oplus f_tM.$$

Proof. Let $U_1 = e_1F\Sigma_r \oplus \cdots \oplus e_sF\Sigma_r$ and $U_2 = f_1F\Sigma_r \oplus \cdots \oplus f_tF\Sigma_r$. Then, as in the proof of Lemma B.2.1,

$$\begin{aligned} U_1 \otimes_{F\Sigma_r} V^{\otimes r} &\cong e_1V^{\otimes r} \oplus \cdots \oplus e_sV^{\otimes r} \\ &\cong f_1V^{\otimes r} \oplus \cdots \oplus f_tV^{\otimes r} \cong U_2 \otimes_{F\Sigma_r} V^{\otimes r}. \end{aligned}$$

Thus, by Lemma B.2.2, $U_1 \cong U_2$. Hence the result follows by Lemma B.2.1. □

Let F be a field and suppose that V is an F-space of finite dimension n. Let r be a positive integer, and regard $V^{\otimes r}$ as an $(F\Sigma_r, S_F(n,r))$-bimodule. Thus there are maps $\alpha : F\Sigma_r \to \mathrm{End}_F(V^{\otimes r})$ and $\beta : S_F(n,r) \to \mathrm{End}_F(V^{\otimes r})$, where $(F\Sigma_r)\alpha$ and $S_F(n,r)\beta$ are subalgebras of $\mathrm{End}_F(V^{\otimes r})$. Let $\mathrm{End}_{F\Sigma_r}(V^{\otimes r})$ and $\mathrm{End}_{S_F(n,r)}(V^{\otimes r})$ denote the centralizers in $\mathrm{End}_F(V^{\otimes r})$ of $(F\Sigma_r)\alpha$ and $S_F(n,r)\beta$, respectively. We require a version of 'Schur–Weyl duality'. However, this is usually stated only for infinite fields: see, for example, [B11, Theorem 1.2]. Thus we give the simple extra argument needed to deduce the result for an arbitrary field F.

Lemma B.2.4. *(Schur–Weyl duality). In the above notation,*

$$\mathrm{End}_{S_F(n,r)}(V^{\otimes r}) = (F\Sigma_r)\alpha, \quad \mathrm{End}_{F\Sigma_r}(V^{\otimes r}) = S_F(n,r)\beta.$$

Proof. Let E be an infinite extension field of F and write $V_E = E \otimes_F V$. We identify $V_E^{\otimes r}$ with $E \otimes_F V^{\otimes r}$. The analogues of α and β over E are

$$\alpha_E : E\Sigma_r \longrightarrow \mathrm{End}_E(V_E^{\otimes r}), \quad \beta_E : S_E(n,r) \longrightarrow \mathrm{End}_E(V_E^{\otimes r}).$$

Identifying $\mathrm{End}_E(V_E^{\otimes r})$ with $E \otimes_F \mathrm{End}_F(V^{\otimes r})$ we find that

$$(E\Sigma_r)\alpha_E = E \otimes_F (F\Sigma_r)\alpha, \quad S_E(n,r)\beta_E = E \otimes_F S_F(n,r)\beta. \tag{B.2.5}$$

Clearly,

$$(F\Sigma_r)\alpha \subseteq \mathrm{End}_{S_F(n,r)}(V^{\otimes r}), \quad S_F(n,r)\beta \subseteq \mathrm{End}_{F\Sigma_r}(V^{\otimes r}). \tag{B.2.6}$$

However, by [B11, Theorem 1.2],

$$\mathrm{End}_{S_E(n,r)}(V_E^{\otimes r}) = (E\Sigma_r)\alpha_E, \quad \mathrm{End}_{E\Sigma_r}(V_E^{\otimes r}) = S_E(n,r)\beta_E. \tag{B.2.7}$$

By (B.2.7) and (B.2.5), we have $\mathrm{End}_{S_E(n,r)}(V_E^{\otimes r}) = E \otimes_F (F\Sigma_r)\alpha$. However,

$$\mathrm{End}_{S_E(n,r)}(V_E^{\otimes r}) \cong E \otimes_F \mathrm{End}_{S_F(n,r)}(V^{\otimes r}),$$

by [B5, (29.5)], since $S_E(n,r)$ may be identified with $E \otimes_F S_F(n,r)$. Hence

$$E \otimes_F \mathrm{End}_{S_F(n,r)}(V^{\otimes r}) \cong E \otimes_F (F\Sigma_r)\alpha.$$

Therefore, from (B.2.6) and consideration of dimension, $\mathrm{End}_{S_F(n,r)}(V^{\otimes r}) = (F\Sigma_r)\alpha$. A similar argument gives the result for $\mathrm{End}_{F\Sigma_r}(V^{\otimes r})$. □

We require some background from [B3], and for this we follow [B3, §2] with only minor variations of notation and terminology.

Let F be a field and recall that, for any F-space V, we write $T(V)$ for the tensor algebra on V. Let $F(\infty)$ denote an F-space with a countably infinite basis $\{x_1, x_2, \dots\}$ and, for each positive integer n, let $F(n)$ denote the subspace of $F(\infty)$ with basis $\{x_1, \dots, x_n\}$. Then, with the obvious identifications,

$$T(F(1)) \subseteq T(F(2)) \subseteq \cdots \subseteq T(F(\infty)).$$

For positive integers n_1 and n_2, where $n_1 \leq n_2$, define $\pi_{n_2,n_1} \in \mathrm{End}_F(F(n_2))$ by

$$x_i\pi_{n_2,n_1} = \begin{cases} x_i & \text{for} \quad i \in \{1, \dots, n_1\}, \\ 0 & \text{for} \quad i \in \{n_1+1, \dots, n_2\}. \end{cases}$$

This extends to an endomorphism of $T(F(n_2))$ with image $T(F(n_1))$, and the restriction of this to $F(n_2)^{\otimes r}$ gives $\pi^{\otimes r}_{n_2,n_1} \in \mathrm{End}_F(F(n_2)^{\otimes r})$ with image $F(n_1)^{\otimes r}$.

For each n we regard $F(n)$ as the natural $S_F(n,1)$-module, so that $F(n)^{\otimes r}$ is an $S_F(n,r)$-module. Suppose that $\{W(n) : n \in \mathbb{N}\}$ is a family of modules such that, for all n, $W(n)$ is an $S_F(n,r)$-submodule of $F(n)^{\otimes r}$ and $W(n_2)\pi^{\otimes r}_{n_2,n_1} = W(n_1)$ for all n_1 and n_2 with $n_1 \le n_2$. Then we say that the family $\{W(n) : n \in \mathbb{N}\}$ is a *uniform submodule family* of $\{F(n)^{\otimes r} : n \in \mathbb{N}\}$.

Lemma B.2.5. *Let $\{W(n) : n \in \mathbb{N}\}$ be a uniform submodule family of $\{F(n)^{\otimes r} : n \in \mathbb{N}\}$ such that, for some $m \ge r$, $W(m)$ is a direct summand of $F(m)^{\otimes r}$. Then there exists an idempotent e of $F\Sigma_r$ such that $W(n) = eF(n)^{\otimes r}$ for all n.*

Proof. By the hypothesis on $W(m)$, there is an idempotent $S_F(m,r)$-module homomorphism $\rho : F(m)^{\otimes r} \to F(m)^{\otimes r}$ with image $W(m)$. By Lemma B.2.4, there exists $e \in F\Sigma_r$ such that $u\rho = eu$ for all $u \in F(m)^{\otimes r}$. Thus $W(m) = eF(m)^{\otimes r}$. Also, since ρ is idempotent and $F\Sigma_r$ acts faithfully on $F(m)^{\otimes r}$, e is an idempotent.

Now consider the family of modules $\{eF(n)^{\otimes r} : n \in \mathbb{N}\}$. Since $\pi^{\otimes r}_{n_2,n_1}$ commutes with the action of $F\Sigma_r$, we have $(eF(n_2)^{\otimes r})\pi^{\otimes r}_{n_2,n_1} = eF(n_1)^{\otimes r}$, for all n_1 and n_2 with $n_1 \le n_2$. Thus $\{eF(n)^{\otimes r}\}$ is a uniform submodule family of $\{F(n)^{\otimes r}\}$. Since $W(m) = eF(m)^{\otimes r}$, we may apply $\pi^{\otimes r}_{m,r}$ to obtain $W(r) = eF(r)^{\otimes r}$. Therefore, by [B3, Lemma 2.5], $W(n) = eF(n)^{\otimes r}$ for all n. $\square$

B.3 Lie powers

Let V be a vector space over a field F with basis $\{x_i : i \in I\}$. The tensor algebra $T(V)$ has the structure of a Lie algebra over F under the multiplication given by $[u,v] = u\otimes v - v\otimes u$ for all $u, v \in T(V)$, and, by a theorem of Witt, the Lie subalgebra generated by $\{x_i : i \in I\}$ is a free Lie algebra, freely generated by $\{x_i : i \in I\}$, which we denote by $L(V)$. For each positive integer r, we write

$$L^r(V) = L(V) \cap T^r(V) = L(V) \cap V^{\otimes r}.$$

If G is a group and V is an FG-module then, as seen in §B.2, $T(V)$ is an FG-module, and it is easily verified that $L(V)$ and $L^r(V)$ are submodules. We call the module $L^r(V)$ the rth Lie power of V. If V has finite dimension n and is regarded as the natural $S_F(n,1)$-module then $L^r(V)$ is an $S_F(n,r)$-submodule of $V^{\otimes r}$ (see [B3, §2]).

In [B3, §2, page 177] it was observed that if B is a subspace of $L^s(V)$, for some s, then the Lie subalgebra of $L(V)$ generated by B may be identified with the free Lie algebra $L(B)$, and then, for all r, $L^r(B)$ is a subspace of $L^{rs}(V)$. We make such identifications in the statement of the 'Decomposition Theorem' from [B3].

Theorem B.3.1. [B3, Theorem 4.4]. *Let F be a field of prime characteristic p. Let G be a group and V a finite-dimensional FG-module. For each positive integer r there is a submodule B_r of $L^r(V)$ such that B_r is a direct summand of $V^{\otimes r}$ and, for $m \geq 0$ and k not divisible by p,*

$$L^{p^m k}(V) = L^{p^m}(B_k) \oplus L^{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L^1(B_{p^m k}). \tag{B.3.1}$$

We shall extend this to modules V that may be infinite-dimensional.

Let p be a prime and let $\mathbb{F}_p$ be a field of p elements. Let k be a positive integer not divisible by p. By [B3, Theorem 4.2], for each positive integer s there exists a uniform submodule family $\{B_{sk}(n) : n \in \mathbb{N}\}$ of $\{\mathbb{F}_p(n)^{\otimes sk} : n \in \mathbb{N}\}$ such that, for all n, $B_{sk}(n) \subseteq L^{sk}(\mathbb{F}_p(n))$, $B_{sk}(n)$ is a direct summand of $\mathbb{F}_p(n)^{\otimes sk}$, and there is an equality of subspaces of $L(\mathbb{F}_p(n))$,

$$\begin{aligned} &L^k(\mathbb{F}_p(n)) \oplus L^{2k}(\mathbb{F}_p(n)) \oplus L^{3k}(\mathbb{F}_p(n)) \oplus \cdots \\ &\qquad = L(B_k(n)) \oplus L(B_{2k}(n)) \oplus L(B_{3k}(n)) \oplus \cdots . \end{aligned} \tag{B.3.2}$$

Therefore, for all $m \geq 0$, $\{B_{p^m k}(n)\}$ is a uniform submodule family of $\{\mathbb{F}_p(n)^{\otimes p^m k}\}$ such that, for all n,

$$B_{p^m k}(n) \subseteq L^{p^m k}(\mathbb{F}_p(n)) \tag{B.3.3}$$

and $B_{p^m k}(n)$ is a direct summand of $\mathbb{F}_p(n)^{\otimes p^m k}$. Furthermore, from (B.3.2), by comparing terms of degree $p^m k$ within $L(\mathbb{F}_p(n))$, we have

$$L^{p^m k}(\mathbb{F}_p(n)) = L^{p^m}(B_k(n)) \oplus L^{p^{m-1}}(B_{pk}(n)) \oplus \cdots \oplus L^1(B_{p^m k}(n)). \tag{B.3.4}$$

Lemma B.2.5, applied to the families $\{B_{p^m k}(n)\}$, gives the following result.

Lemma B.3.2. *For each $m \geq 0$ there exists an idempotent $b_{p^m k}$ of $\mathbb{F}_p\Sigma_{p^m k}$ such that, for all n,*

$$B_{p^m k}(n) = b_{p^m k}\left(\mathbb{F}_p(n)^{\otimes p^m k}\right). \tag{B.3.5}$$

We now ignore module structure and regard $\mathbb{F}_p(n)$ simply as an $\mathbb{F}_p$-space. Thus, by (B.3.3), (B.3.4) and (B.3.5), if W is any finite-dimensional $\mathbb{F}_p$-space, we have

$$b_{p^m k}W^{\otimes p^m k} \subseteq L^{p^m k}(W) \tag{B.3.6}$$

and

$$L^{p^m k}(W) = L^{p^m}(b_k W^{\otimes k}) \oplus L^{p^{m-1}}(b_{pk}W^{\otimes pk}) \oplus \cdots \oplus L^1(b_{p^m k}W^{\otimes p^m k}). \tag{B.3.7}$$

Let F be any extension field of $\mathbb{F}_p$. By tensoring the terms of (B.3.6) and (B.3.7) with F we find that (B.3.6) and (B.3.7) hold for any finite-dimensional F-space W. We now generalise to arbitrary dimension.

Proposition B.3.3. *Let p be a prime and let $\mathbb{F}_p$ be a field of p elements. Let k be a positive integer not divisible by p. Then, for each $m \geq 0$, there is an idempotent $b_{p^m k}$ of $\mathbb{F}_p\Sigma_{p^m k}$ such that if F is any extension field of $\mathbb{F}_p$ and V is any F-space (of finite or infinite dimension) we have*

$$b_{p^m k}V^{\otimes p^m k} \subseteq L^{p^m k}(V) \tag{B.3.8}$$

and

$$L^{p^m k}(V) = L^{p^m}(b_k V^{\otimes k}) \oplus L^{p^{m-1}}(b_{pk}V^{\otimes pk}) \oplus \cdots \oplus L^1(b_{p^m k}V^{\otimes p^m k}). \tag{B.3.9}$$

Proof. We use the idempotents $b_{p^m k}$ of Lemma B.3.2. Suppose that (B.3.8) does not hold. Then there exist $v_1, \ldots, v_{p^m k} \in V$ such that $b_{p^m k}(v_1 \otimes \cdots \otimes v_{p^m k}) \notin L^{p^m k}(V)$. Let W be the subspace of V spanned by $v_1, \ldots, v_{p^m k}$. Then $b_{p^m k}W^{\otimes p^m k} \not\subseteq L^{p^m k}(W)$, contrary to (B.3.6) over the field F. Thus (B.3.8) holds.

Write S_V for the subspace of $L(V)$ defined by

$$S_V = L^{p^m}(b_k V^{\otimes k}) + L^{p^{m-1}}(b_{pk} V^{\otimes pk}) + \cdots + L^1(b_{p^m k} V^{\otimes p^m k}).$$

If the sum on the right is not a direct sum then there exists a finite-dimensional subspace W of V such that the corresponding sum S_W is not a direct sum, contrary to (B.3.7) over F. Similarly, if $L^{p^m k}(V) \not\subseteq S_V$ then there exists a finite-dimensional subspace W of V such that $L^{p^m k}(W) \not\subseteq S_W$, again contrary to (B.3.7) over F. By (B.3.8), $S_V \subseteq L^{p^m k}(V)$. Thus (B.3.9) holds. □

In the rest of this paper, whenever F is a field of characteristic p, we assume that F is an extension field of $\mathbb{F}_p$ (rather than a field isomorphic to $\mathbb{F}_p$) so that the idempotents of Proposition B.3.3 are available. This is essentially a notational issue. We can now derive our first main result.

Theorem B.3.4. *Theorem B.3.1 holds for an FG-module V of arbitrary dimension, where we take $B_{p^m k} = b_{p^m k} V^{\otimes p^m k}$ for all $m \geq 0$ and all k not divisible by p.*

Proof. Since $V^{\otimes p^m k}$ is an $(F\Sigma_{p^m k}, FG)$-bimodule, $B_{p^m k}$ is an FG-submodule of $V^{\otimes p^m k}$. It is a direct summand, since $b_{p^m k}$ is an idempotent, and, by (B.3.8), it is a submodule of $L^{p^m k}(V)$. Finally, (B.3.9) gives (B.3.1). □

It is easily seen, by induction, that the modules $B_{p^m k}$ satisfying (B.3.1), for a given finite-dimensional module V, are determined uniquely up to isomorphism: thus we may take them to be the modules $b_{p^m k} V^{\otimes p^m k}$. Some information about the isomorphism types was given in [B4]. In stating the result of interest here we use the notation $\bigoplus^r B$ for the direct sum of r copies of an F-space B.

Theorem B.3.5. [B4, Theorem 4.2]. *In the notation of Theorem B.3.1, for each $m \geq 0$ there is an isomorphism of FG-modules*

$$(\textstyle\bigoplus^{p^m} B_{p^m k}) \oplus (\bigoplus^{p^{m-1}} B_{p^{m-1}k}^{\otimes p}) \oplus \cdots \oplus (\bigoplus^{p} B_{pk}^{\otimes p^{m-1}}) \oplus B_k^{\otimes p^m} \cong L^k(V^{\otimes p^m}). \quad \text{(B.3.10)}$$

We shall extend this to modules V that may be infinite-dimensional, taking B_i to be $b_i V^{\otimes i}$ for all i. However, we need some preliminary facts.

Let k be any positive integer and let ℓ_k be the element of $\mathbb{Z}\Sigma_k$ defined, using the cycles $(2,1)$, $(3,2,1)$, $\ldots$, $(k,\ldots,2,1)$ of Σ_k, by

$$\ell_k = (1-(k,\ldots,2,1))\cdots(1-(3,2,1))(1-(2,1)).$$

Let V be a vector space over a field F, and interpret ℓ_k as an element of $F\Sigma_k$. Then it is well known and straightforward to verify that

$$\ell_k(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = [\cdots[[v_1, v_2], v_3], \ldots, v_k], \tag{B.3.11}$$

for all $v_1, v_2, \ldots, v_k \in V$. It follows that $L^k(V) = \ell_k V^{\otimes k}$.

Suppose, for the moment, that $\operatorname{char} F = 0$. Then, by (B.3.11) and the Dynkin–Specht–Wever criterion for Lie elements [B7, Theorem V.8], we have

$$\ell_k^2(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = k\ell_k(v_1 \otimes v_2 \otimes \cdots \otimes v_k).$$

However, $F\Sigma_k$ acts faithfully on $V^{\otimes k}$ when $\dim V \geq k$. Hence $\ell_k^2 = k\ell_k$.

Now let F be any field such that $\operatorname{char} F \nmid k$. Thus $1/k$ exists in the prime subfield $\mathbb{F}$ of F and we can define an idempotent ω_k of $\mathbb{F}\Sigma_k$ (sometimes called the 'Dynkin idempotent') by $\omega_k = (1/k)\ell_k$. Then, for any F-space V,

$$L^k(V) = \omega_k V^{\otimes k}. \tag{B.3.12}$$

Let r and s be positive integers, and partition the set $\{1, \ldots, rs\}$ as

$$\{1,\ldots,s\} \cup \{s+1,\ldots,2s\} \cup \cdots \cup \{(r-1)s+1,\ldots,rs\}. \tag{B.3.13}$$

This gives, in an obvious way, an embedding $\mu : \Sigma_s \times \cdots \times \Sigma_s \to \Sigma_{rs}$, where the direct product has r factors. Let $\delta : \Sigma_s \to \Sigma_s \times \cdots \times \Sigma_s$ be the 'diagonal' embedding given by $\sigma\delta = (\sigma, \ldots, \sigma)$ for all $\sigma \in \Sigma_s$. Then, by composition of δ and μ, we obtain an embedding $\lambda : \Sigma_s \longrightarrow \Sigma_{rs}$. Consider the partition

$$\{1, s+1, \ldots, (r-1)s+1\} \cup \{2, s+2, \ldots, (r-1)s+2\} \cup \cdots \cup \{s, 2s, \ldots, rs\}, \tag{B.3.14}$$

with s parts. It is easily checked that, for all $\sigma \in \Sigma_s$, $\sigma\lambda$ is the permutation that permutes these s parts according to σ, while preserving the order of the numbers within each part.

Let F be any field and identify $F(\Sigma_s \times \cdots \times \Sigma_s)$ with $F\Sigma_s \otimes_F \cdots \otimes_F F\Sigma_s$. Then we may extend μ and λ linearly to obtain embeddings

$$\mu : F\Sigma_s \otimes \cdots \otimes F\Sigma_s \longrightarrow F\Sigma_{rs}, \quad \lambda : F\Sigma_s \longrightarrow F\Sigma_{rs}.$$

For $u, u_1, \ldots, u_r \in F\Sigma_s$ we write $u_1 \# \cdots \# u_r$ to denote $(u_1 \otimes \cdots \otimes u_r)\mu$ and $u^{\#r}$ for $u \# \cdots \# u$. Also, for $u \in F\Sigma_s$, we write $u^{[r]}$ to denote $u\lambda$. (In general, when u is a proper linear combination of group elements, $u^{[r]} \neq u^{\#r}$.) Note that, if $u, u_1, \ldots, u_r$ are idempotents of $F\Sigma_s$ then $u_1 \# \cdots \# u_r$, $u^{\#r}$ and $u^{[r]}$ are idempotents of $F\Sigma_{rs}$.

Let G be a group and let V be an FG-module. As usual, we index the factors of $V^{\otimes rs}$ by the set $\{1, \ldots, rs\}$: more formally, $V^{\otimes rs}$ is identified with $\bigotimes_{i=1}^{rs} V_i$ where, for each i, there is a fixed isomorphism from V_i to V. Let Σ_{rs} act on $V^{\otimes rs}$ by place permutations. Now consider $V^{\otimes r} \otimes \cdots \otimes V^{\otimes r}$, that is $(V^{\otimes r})^{\otimes s}$, where the rs factors V are indexed by $\{1, \ldots, rs\}$ in the order given by the partition (B.3.14). Let Σ_{rs} act on $(V^{\otimes r})^{\otimes s}$ by place permutations according to this indexing. It follows that $V^{\otimes rs}$ and $(V^{\otimes r})^{\otimes s}$ are isomorphic as $(F\Sigma_{rs}, FG)$-bimodules. Note that $F\Sigma_s$ also acts on $(V^{\otimes r})^{\otimes s}$ by place permutations, with $V^{\otimes r}$ instead of V in the usual construction. Then, for $w_1, \ldots, w_s \in V^{\otimes r}$ and $\sigma \in \Sigma_s$ we find that

$$(\sigma\lambda)(w_1 \otimes \cdots \otimes w_s) = w_{1\sigma} \otimes \cdots \otimes w_{s\sigma} = \sigma(w_1 \otimes \cdots \otimes w_s).$$

Thus $\sigma\lambda$ and σ act in the same way on $(V^{\otimes r})^{\otimes s}$. Hence, for $u \in F\Sigma_s$, $u^{[r]}$ and u act in the same way on $(V^{\otimes r})^{\otimes s}$.

Suppose that k is a positive integer not divisible by char F. By (B.3.12), $L^k(V^{\otimes r}) = \omega_k((V^{\otimes r})^{\otimes k})$. Thus (taking $s = k$ in the above analysis), we have $L^k(V^{\otimes r}) = \omega_k^{[r]}((V^{\otimes r})^{\otimes k})$. Hence we obtain an FG-module isomorphism

$$L^k(V^{\otimes r}) \cong \omega_k^{[r]} V^{\otimes rk}. \tag{B.3.15}$$

We next identify $V^{\otimes rs}$ with $V^{\otimes s} \otimes \cdots \otimes V^{\otimes s}$, that is $(V^{\otimes s})^{\otimes r}$, where the rs factors V are indexed in natural order, as given by (B.3.13). We also take $F\Sigma_s$ to act on $V^{\otimes s}$ by place permutations. Then, for $u_1, \ldots, u_r \in F\Sigma_s$, we find that

$$(u_1 \# \cdots \# u_r)(V^{\otimes s})^{\otimes r} = (u_1 V^{\otimes s}) \otimes \cdots \otimes (u_r V^{\otimes s}).$$

Thus there is an FG-module isomorphism

$$(u_1 \# \cdots \# u_r) V^{\otimes rs} \cong (u_1 V^{\otimes s}) \otimes \cdots \otimes (u_r V^{\otimes s}). \tag{B.3.16}$$

In particular, for $u \in F\Sigma_s$,

$$u^{\# r} V^{\otimes rs} \cong (u V^{\otimes s})^{\otimes r}. \tag{B.3.17}$$

Suppose that V is a finite-dimensional FG-module, as in Theorem B.3.5, where we take $B_i = b_i V^{\otimes i}$ for all i. Thus, by (B.3.10), (B.3.15) and (B.3.17), we have

$$\begin{aligned}(\textstyle\bigoplus^{p^m} b^{\#1}_{p^m k} V^{\otimes p^m k}) \oplus (\textstyle\bigoplus^{p^{m-1}} b^{\#p}_{p^{m-1}k} V^{\otimes p^m k}) \oplus \cdots \\ \cdots \oplus (\textstyle\bigoplus^{p} b^{\#p^{m-1}}_{pk} V^{\otimes p^m k}) \oplus b^{\#p^m}_{k} V^{\otimes p^m k} \cong \omega^{[p^m]}_k V^{\otimes p^m k}.\end{aligned} \tag{B.3.18}$$

Let W be an F-space of finite dimension n, where $n \geq p^m k$, and regard W as the natural $S_F(n,1)$-module. Let E be an infinite extension field of F. Thus we may regard $E \otimes_F W$ as the natural $S_E(n,1)$-module or, equivalently, the natural $E\mathrm{GL}(n,E)$-module. By (B.3.18) there is an isomorphism of $E\mathrm{GL}(n,E)$-modules

$$(\textstyle\bigoplus^{p^m} b^{\#1}_{p^m k}(E \otimes W)^{\otimes p^m k}) \oplus \cdots \oplus b^{\#p^m}_{k} (E \otimes W)^{\otimes p^m k} \cong \omega^{[p^m]}_k (E \otimes W)^{\otimes p^m k}. \tag{B.3.19}$$

Since E is infinite, this is an isomorphism of $S_E(n, p^m k)$-modules (see [B6, §2.4]). The spaces $b^{\#p^{m-i}}_{p^i k} W^{\otimes p^m k}$ and $\omega^{[p^m]}_k W^{\otimes p^m k}$ are $S_F(n, p^m k)$-modules, and we make the identifications $S_E(n, p^m k) = E \otimes S_F(n, p^m k)$,

$$b^{\#p^{m-i}}_{p^i k}(E \otimes W)^{\otimes p^m k} = E \otimes b^{\#p^{m-i}}_{p^i k} W^{\otimes p^m k}, \quad \omega^{[p^m]}_k (E \otimes W)^{\otimes p^m k} = E \otimes \omega^{[p^m]}_k W^{\otimes p^m k}.$$

Hence, by (B.3.19) and the Noether–Deuring theorem [B5, (29.11)], there is an $S_F(n, p^m k)$-module isomorphism

$$(\textstyle\bigoplus^{p^m} b^{\#1}_{p^m k} W^{\otimes p^m k}) \oplus \cdots \oplus b^{\#p^m}_{k} W^{\otimes p^m k} \cong \omega^{[p^m]}_k W^{\otimes p^m k}. \tag{B.3.20}$$

We can now derive our second main result.

Theorem B.3.6. *Theorem B.3.5 holds for an FG-module V of arbitrary dimension, where we take $B_{p^i k} = b_{p^i k} V^{\otimes p^i k}$ for all $i \geq 0$.*

Proof. By (B.3.20) and Corollary B.2.3, there is an isomorphism of the form (B.3.18) for arbitrary V. Thus, by (B.3.15) and (B.3.17), we obtain (B.3.10) for arbitrary V. □

B.4 Decomposition of the modules $B_{p^m k}$

Let F be a field and let k be a positive integer not divisible by char F. Let E be the field obtained from F by adjoining (if necessary) a primitive kth root of unity ϵ, and let $\langle\epsilon\rangle$ denote the multiplicative group generated by ϵ, consisting of all kth roots of unity in E. For $\xi \in \langle\epsilon\rangle$ write $|\xi|$ for the multiplicative order of ξ.

Let V be an F-space and write $V_E = E \otimes_F V$. Let σ_k be the k-cycle $(1, 2, \ldots, k)$ of Σ_k, and, for each $\xi \in \langle\epsilon\rangle$, let e_ξ be the element of $E\Sigma_k$ defined by

$$e_\xi = \frac{1}{k} \sum_{i=0}^{k-1} \xi^{-i} \sigma_k^i. \tag{B.4.1}$$

It is easy to verify that e_ξ is an idempotent of $E\Sigma_k$ and that $e_\xi V_E^{\otimes k}$ is the ξ-eigenspace of $V_E^{\otimes k}$ under the action of σ_k. Thus $V_E^{\otimes k} = \bigoplus_{\xi \in \langle\epsilon\rangle} e_\xi V_E^{\otimes k}$.

If A is an E-algebra such that $V_E^{\otimes k}$ is an $(E\Sigma_k, A)$-bimodule, then each $e_\xi V_E^{\otimes k}$ is an A-submodule of $V_E^{\otimes k}$. For l prime to k, σ_k and σ_k^l are conjugate in Σ_k. It follows easily that there is an isomorphism of A-modules

$$e_\xi V_E^{\otimes k} \cong e_{\xi'} V_E^{\otimes k} \text{ when } |\xi| = |\xi'|. \tag{B.4.2}$$

Now suppose that G is a group and that V is a finite-dimensional FG-module. As shown in [B2, §2], by means of [B1], there exist FG-modules $(V^{\otimes k})_\xi$ such that

$$E \otimes_F (V^{\otimes k})_\xi \cong e_\xi V_E^{\otimes k}, \tag{B.4.3}$$

$V^{\otimes k} \cong \bigoplus_{\xi \in \langle \epsilon \rangle} (V^{\otimes k})_\xi$, and $(V^{\otimes k})_\xi \cong (V^{\otimes k})_{\xi'}$ when $|\xi| = |\xi'|$. As in [B2], for each (positive) divisor d of k, let $U_{k,d}$ denote an FG-module satisfying

$$U_{k,d} \cong (V^{\otimes k})_\xi \text{ when } |\xi| = d. \tag{B.4.4}$$

Theorem B.4.1. [B2, Theorem 4.2]. *Let F be a field of prime characteristic p, G a group, and V a finite-dimensional FG-module. Let k be a positive integer not divisible by p and let m be a non-negative integer. Let $B_{p^m k}$ be the module given by Theorem B.3.1 and, for each divisor d of k, let $U_{k,d}$ be the module of (B.4.4). Then there is an index set Λ_0, and, for each $\lambda \in \Lambda_0$, a p^m-tuple $(\lambda(1), \ldots, \lambda(p^m))$ of divisors of k, such that*

$$B_{p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} U_{k,\lambda(1)} \otimes \cdots \otimes U_{k,\lambda(p^m)}.$$

The purpose of this section is to generalise this theorem to modules V that are allowed to be infinite-dimensional. We begin with a general result.

Lemma B.4.2. *Let k be a positive integer and let E be a field with prime field $\mathbb{F}$. Let w be an idempotent of $E\Sigma_k$. Then there exists an idempotent w_0 of $\mathbb{F}\Sigma_k$ such that the ideals $wE\Sigma_k$ and $w_0 E\Sigma_k$ of $E\Sigma_k$ are isomorphic as $E\Sigma_k$-modules.*

Proof. Suppose that P is a principal indecomposable $\mathbb{F}\Sigma_k$-module and let R denote the radical of $\mathbb{F}\Sigma_k$. Thus P/PR is irreducible (see [B5, (54.11)]). We have

$$E \otimes_{\mathbb{F}} (P/PR) \cong (E \otimes_{\mathbb{F}} P)/(E \otimes_{\mathbb{F}} P)(E \otimes_{\mathbb{F}} R).$$

Since $\mathbb{F}$ is a splitting field for Σ_k (see [B8, Theorem 11.5]), $E \otimes_{\mathbb{F}} (P/PR)$ is irreducible. Hence $(E \otimes_{\mathbb{F}} P)/(E \otimes_{\mathbb{F}} P)(E \otimes_{\mathbb{F}} R)$ is irreducible. However, $E \otimes_{\mathbb{F}} R$ is the radical of $E \otimes_{\mathbb{F}} \mathbb{F}\Sigma_k$ (see [B5, (29.22)]). It follows that $E \otimes_{\mathbb{F}} P$ is indecomposable.

In the rest of the proof we identify $E \otimes_{\mathbb{F}} \mathbb{F}\Sigma_k$ with $E\Sigma_k$. Write $\mathbb{F}\Sigma_k$ as a direct sum of principal indecomposables, $\mathbb{F}\Sigma_k = \bigoplus_{j=1}^r P_j$. Thus $E\Sigma_k = \bigoplus_{j=1}^r E \otimes_{\mathbb{F}} P_j$, and, by what was proved above, each $E \otimes_{\mathbb{F}} P_j$ is indecomposable. Since w is an idempotent, the right ideal $wE\Sigma_k$ is a direct summand of $E\Sigma_k$. Hence, by the Krull–Schmidt theorem, it is isomorphic, as a right $E\Sigma_k$-module, to the direct sum of some subset of the modules $E \otimes_{\mathbb{F}} P_j$. Hence there is a direct summand U of $\mathbb{F}\Sigma_k$ such that $wE\Sigma_k \cong E \otimes_{\mathbb{F}} U$. Since U is a direct summand, we may write $U = w_0\mathbb{F}\Sigma_k$ for some idempotent w_0 of $\mathbb{F}\Sigma_k$. It follows that $wE\Sigma_k \cong w_0E\Sigma_k$. □

Recall from §B.2 that if F is a field and n is a positive integer then $F(n)$ denotes an F-space of dimension n regarded as the natural $S_F(n,1)$-module.

Let $\mathbb{F}$ be a prime field and let k be a positive integer not divisible by $\operatorname{char}\mathbb{F}$. Let C be the (cyclotomic) field obtained from $\mathbb{F}$ by adjoining a primitive kth root of unity ϵ_k. For each divisor d of k write $\epsilon_d = \epsilon_k^{k/d}$, so that $|\epsilon_d| = d$, and let $e_{\epsilon_d} \in C\Sigma_k$ be defined as in (B.4.1) with $\xi = \epsilon_d$. By Lemmas B.4.2 and B.2.1 there exists an idempotent $u_{k,d}$ of $\mathbb{F}\Sigma_k$ such that, for any positive integer n, $e_{\epsilon_d}C(n)^{\otimes k}$ and $u_{k,d}C(n)^{\otimes k}$ are isomorphic $S_C(n,k)$-modules. Hence, by (B.4.2), if η is any element of $\langle \epsilon_k \rangle$ of order d, there is an isomorphism of $S_C(n,k)$-modules

$$e_\eta C(n)^{\otimes k} \cong u_{k,d}C(n)^{\otimes k}. \tag{B.4.5}$$

Let E be any extension field of $\mathbb{F}$ such that E contains a primitive kth root of unity ϵ, and write $C' = \mathbb{F}(\epsilon)$. Let χ be an isomorphism from C to C'. Of course, χ is the identity on $\mathbb{F}$. Applying χ to (B.4.5) and recalling that $u_{k,d} \in \mathbb{F}\Sigma_k$, we get an $S_{C'}(n,k)$-module isomorphism $e_\xi C'(n)^{\otimes k} \cong u_{k,d}C'(n)^{\otimes k}$, for all $\xi \in C'$ with $|\xi| = d$. Hence, by tensoring with E, we get an $S_E(n,k)$-module isomorphism

$$e_\xi E(n)^{\otimes k} \cong u_{k,d}E(n)^{\otimes k}, \tag{B.4.6}$$

for all $\xi \in E$ with $|\xi| = d$.

Now suppose that F is any extension field of $\mathbb{F}$ and let E be obtained from F by adjoining (if necessary) a primitive kth root of unity. Let G be a group and let V be an FG-module of finite dimension n. For each divisor d of k let $U_{k,d}$ be an FG-module satisfying (B.4.4). We regard V_E as the natural $S_E(n,1)$-module. Let

$\xi \in E$ with $|\xi| = d$. Then, by (B.4.6), there is an $S_E(n,k)$-module isomorphism

$$\alpha : e_\xi V_E^{\otimes k} \longrightarrow u_{k,d} V_E^{\otimes k}. \tag{B.4.7}$$

If we think of V_E as the natural $E\mathrm{GL}(n,E)$-module, the map α of (B.4.7) is an isomorphism of $E\mathrm{GL}(n,E)$-modules. Also, since V is an FG-module, V_E is an EG-module, and there is an associated homomorphism $\rho : G \to \mathrm{GL}(n,E)$. The action of G on $V_E^{\otimes k}$ is given by the composition of ρ and the action of $\mathrm{GL}(n,E)$ on $V_E^{\otimes k}$. It follows that α is an isomorphism of EG-modules.

By (B.4.3) and (B.4.4), $E \otimes U_{k,d}$ and $e_\xi V_E^{\otimes k}$ are isomorphic EG-modules. Thus, since (B.4.7) is an EG-module isomorphism, $E \otimes U_{k,d}$ and $u_{k,d} V_E^{\otimes k}$ are isomorphic EG-modules. Hence, by the Noether–Deuring theorem, $U_{k,d}$ and $u_{k,d} V^{\otimes k}$ are isomorphic FG-modules. Thus we have proved the following proposition.

Proposition B.4.3. *Let $\mathbb{F}$ be a prime field, k a positive integer not divisible by* $\mathrm{char}\,\mathbb{F}$*, and d a divisor of k. Then there is an idempotent $u_{k,d}$ of $\mathbb{F}\Sigma_k$ such that if G is any group, F is any extension field of $\mathbb{F}$, and V is any finite-dimensional FG-module, then the FG-module $U_{k,d}$ of (B.4.4) satisfies $U_{k,d} \cong u_{k,d} V^{\otimes k}$.*

From now on we assume that p is a prime and F is an extension field of $\mathbb{F}_p$, the field of p elements. We take k to be a positive integer not divisible by p. For each divisor d of k we use the idempotent $u_{k,d}$ of $\mathbb{F}_p\Sigma_k$ given by Proposition B.4.3.

Let m be a non-negative integer and let W be an F-space of dimension $p^m k$. Let E be an infinite extension field of F and write $W_E = E \otimes_F W$, where we regard W_E as the natural $E\mathrm{GL}(p^m k, E)$-module.

We apply Theorem B.4.1 but with E replacing F, $G = \mathrm{GL}(p^m k, E)$ and $V = W_E$. Recall that $B_{p^m k}$ may be written up to isomorphism as $b_{p^m k} W_E^{\otimes p^m k}$ and, by Proposition B.4.3, for each divisor d of k, $U_{k,d}$ may be written up to isomorphism as $u_{k,d} W_E^{\otimes k}$. Hence Theorem B.4.1 gives an isomorphism of $E\mathrm{GL}(p^m k, E)$-modules

$$b_{p^m k} W_E^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} u_{k,\lambda(1)} W_E^{\otimes k} \otimes \cdots \otimes u_{k,\lambda(p^m)} W_E^{\otimes k}.$$

Therefore, by (B.3.16),

$$b_{p^m k} W_E^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} (u_{k,\lambda(1)} \# \cdots \# u_{k,\lambda(p^m)}) W_E^{\otimes p^m k}. \tag{B.4.8}$$

Since E is infinite, (B.4.8) is an isomorphism of $S_E(p^m k, p^m k)$-modules. Hence, by the Noether–Deuring theorem, there is an $S_F(p^m k, p^m k)$-module isomorphism

$$b_{p^m k} W^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} (u_{k,\lambda(1)} \# \cdots \# u_{k,\lambda(p^m)}) W^{\otimes p^m k}. \tag{B.4.9}$$

Let V be an FG-module of arbitrary, possibly infinite, dimension. Then, by (B.4.9) and Corollary B.2.3, there is an isomorphism of FG-modules

$$b_{p^m k} V^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} (u_{k,\lambda(1)} \# \cdots \# u_{k,\lambda(p^m)}) V^{\otimes p^m k}.$$

In other words, by (B.3.16),

$$b_{p^m k} V^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} u_{k,\lambda(1)} V^{\otimes k} \otimes \cdots \otimes u_{k,\lambda(p^m)} V^{\otimes k}.$$

Thus we have proved the following result.

Theorem B.4.4. *Theorem B.4.1 holds for an FG-module V of arbitrary dimension, where we take $B_{p^m k} = b_{p^m k} V^{\otimes p^m k}$ and $U_{k,d} = u_{k,d} V^{\otimes k}$ for every divisor d of k.*

We conclude with an observation on the idempotent $u_{k,k}$.

Proposition B.4.5. *Under the hypotheses of Theorem B.4.4, $u_{k,k} V^{\otimes k} \cong L^k(V)$.*

Proof. We use the idempotent ω_k of $\mathbb{F}_p \Sigma_k$ defined in §B.3. Let E be an infinite extension field of F and let W be an F-space of dimension k. Write $W_E = E \otimes_F W$ and regard W_E as the natural $E\mathrm{GL}(k, E)$-module. By [B2, Lemma 2.3] combined with Proposition B.4.3, $u_{k,k} W_E^{\otimes k}$ and $L^k(W_E)$ are isomorphic as $E\mathrm{GL}(k, E)$-modules. Thus, by (B.3.12), $u_{k,k} W_E^{\otimes k} \cong \omega_k W_E^{\otimes k}$. Since E is infinite, this is an isomorphism of $S_E(k, k)$-modules. Hence, by the Noether–Deuring theorem, $u_{k,k} W^{\otimes k}$ and $\omega_k W^{\otimes k}$ are isomorphic as $S_F(k, k)$-modules. By Corollary B.2.3, if V is an FG-module of arbitrary dimension, $u_{k,k} V^{\otimes k}$ and $\omega_k V^{\otimes k}$ are isomorphic as FG-modules. Thus, by (B.3.12), $u_{k,k} V^{\otimes k} \cong L^k(V)$. □

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