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Reconstruction Algorithms for Permittivity and Conductivity Imaging

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ABSTRACT

Linear reconstruction algorithms are reviewed using assumed covariance matrices for the conductivity and data and the formulation of Tikhonov regularization using the singular value decomposition (SVD) with covariance norms. It is shown how iterative reconstruction algorithms, such as Landweber and conjugate gradient, can be used for regularization and analysed in terms of the SVD, and implemented directly for a one–step Newton's method. Where there are known inequality constraints, such as upper and lower bounds, these can be incorporated in iterative methods and have a stabilizing effect on reconstructions.

Keywords Reconstruction algorithms, electrical imaging, ECT, ERT, regularization

1 INTRODUCTION

From the point of view of reconstruction algorithms permittivity imaging (or Electrical Capacitance Tomography ECT), and conductivity imaging (Electrical Resistance Tomography ERT) are very similar, the difference being in the detail of the forward model. They are also special cases of imaging the complex conductivity (admittivity) which is at present more developed in medical imaging where it is called Electrical Impedance Tomography (EIT). In this paper some of the basics of regularized linear reconstruction algorithms, which should be more widely disseminated, are reviewed. The books by Bertero and Boccacci (1998), Tarantola (1987) and Hansen (1998) are recommended for more detailed information on the solution of linear ill–posed problems.

The electrical imaging techniques considered here require a system of conducting electrodes on the boundary of the region to be imaged. Impedance measurements are made between the electrodes. In the case of permittivity imaging the standard method is to apply a voltage on each electrode in turn and measure the current while the others are grounded, see Figure 1 and Byars (2001) for details.

In conductivity imaging one applies a known current between two or more electrodes and measures the resulting voltages. In some systems the same electrodes are used for current drive and voltage measurement.

2 FORWARD PROBLEM

Before attempting to reconstruct the unknown material property, one needs to solve the 'forward problem'. For accurate reconstruction one needs to be able to predict the measurements at the boundary from an arbitrary inhomogeneous material. Also the electric field in the interior will be needed for the solution of the inverse problem.

2.1 Permittivity Imaging

Electrical imaging systems typically employ an alternating current at a fixed frequency. This frequency is sufficiently small that the quasi–static approximation is valid and magnetic field can be neglected. This means that the electric field $-\nabla\phi$ for a (possibly complex) scalar potential ϕ.

For permittivity imaging in a body with permittivity ε, and negligible conductivity one has

$$\nabla \cdot \varepsilon \nabla \phi = 0 \tag{1}$$

and on each of the electrodes we have the boundary condition $\phi = V$ a constant. In the usual scheme one electrode E_i is held at a positive voltage and the others at zero. The electrodes are typically on the surface of an insulating cylinder. At points of the cylinder not on the electrodes the charge flux $\varepsilon\partial\phi/\partial\nu$ is zero, where ν is the outward unit normal. On each of the earthed electrodes E_i, the total charge is measured

$$Q_{ij} = \int_{Ei} \varepsilon \partial\phi / \partial\upsilon dS. \tag{2}$$

The mixed boundary value problem where the voltage is specified on the electrode (Dirichlet condition) and the Neumann condition is specified elsewhere gives a complete boundary value problem, which can be solved for ϕ. The measured Q_{ij} constitute the additional data used to determine the permittivity.

2.2 Conductivity Imaging

In conductivity imaging one has similarly

$$\nabla \cdot \sigma \nabla \phi = 0 \tag{3}$$

where σ is the conductivity. Away from the electrodes on the boundary of the object one has the Neumann condition $\sigma\partial\phi/\partial\nu=0$ as no current crosses the boundary there. Electrodes are conductors, and thus the potential on each electrode E_i is a constant V_i. Typically the electrochemical interaction between the electrode and the electrolyte produces a contact impedance layer. We will assume this is resistive and constant z_i for each electrode. It will depend on frequency. On electrode E_i one has then

$$\phi + z_i \sigma \partial\phi / \partial\upsilon = V_i \tag{4}$$

and the current is on each electrode is specified at I_i so that

$$\int_{Ei} \sigma \partial\phi / \partial\upsilon dS = I_i. \tag{5}$$

This is the so called complete electrode model which has been found to agree with measured data, see Somersalo *et al.* (1992), and gives a well–posed boundary value problem with the V_i unknown. For simplicity assume that a unit current is driven from electrode E_i to the others equally. The voltages V_{ij} then constitute our measured data used in the reconstruction. In practice other drive patterns are used but the data from these can be synthasized from this data.

2.3 Solving the forward problem

In both cases it is important to realize that the forward models are three–dimensional problems, as one cannot constrain the electric field to a plane when the material is not homogeneous. In the cylindrical geometry, using long electrodes goes some way towards this, but a two–dimensional approximation is only valid if the material is translationally invariant.

In simple geometries, such as concentric cylinders of homogeneous materials, it is possible to solve the forward problem by separation of variables and series solution (see for example Somersalo 1991). In more general cases one needs to use a discretization of the interior, the finite element method being the preferred technique due to ease of applying boundary conditions.

3 LINEARIZATION

As the electric field depends on the material present, the measured data V or Q is a nonlinear function of σ or ε. Our first step is then to linearize the problem. For a change in $\delta\sigma$ or $\delta\varepsilon$ we have to first order

$$\delta V_{ij} = \frac{dV_{ij}}{d\sigma} \delta\sigma \tag{6}$$

and

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$$\delta Q_{ij} = \frac{dQ_{ij}}{d\varepsilon}\delta\varepsilon \tag{7}$$

where the Frechét derivatives (these are operators) are given by

$$\frac{dV_{ij}}{d\sigma}\delta\sigma = -\int \delta\sigma \nabla\phi_i \cdot \nabla\phi_j \; dxdydz \tag{8}$$

and

$$\frac{dQ_{ij}}{d\varepsilon}\delta\varepsilon = \int \delta\varepsilon \nabla\phi_i \cdot \nabla\phi_j \; dxdydz \tag{9}$$

(Derivations can be found in Breckon 1990). We have deliberately chosen a symmetrical drive and measurement scheme – in the case where measurements are not made in the same pattern as the drives we would need also the potentials for lead fields, which are the fields, which would result if the measurement electrodes were used as drives. Note that we can calculate potentials, and thus derivatives, for other patterns by superposition. For example adjacent pair drives and measurements would have potentials $\phi_i - \phi_{i-1}$

The permittivity or conductivity must be discreteized so as to have a finite number of linear equations to solve. For example one could choose to represent the permittivity or conductivity as constant on square pixels (voxels) ε or $\sigma = \sum_{k=1}^{K} m_k \chi_k$, where $\chi_k(\mathbf{x})$ is one if $\mathbf{x}$ is in the voxel and zero otherwise. (We will use m_k for the material parameters of the model we seek in both cases.) Then

$$J_{(ij)k} = \frac{\partial Q_{ij}}{\partial m_k} = \int \chi_k \nabla\phi_i \cdot \nabla\phi_j \; dxdydz \tag{10}$$

and similarly for conductivity. While this basis simplifies the computation (the integral reduces to the *k*–th voxel) there is considerable merit in choosing other types of basis such as splines or wavelets. We will regard $J_{(ij)k}$ as a matrix (the Jacobian matrix) rather than a matrix of matrices.

This Jacobian (or a normalized form of it) is often referred to as a sensitivity matrix, and in the process tomography literature a row of this matrix is termed a sensitivity map. If the derivative is taken assuming a homogeneous material and the geometry is simple, it may be possible to calculate the Jacobian using a series solution of the forward problem.

As electrical imaging problems are highly non–linear, it usually necessary to use nonlinear reconstruction methods such as regularized Gauss–Newton methods to get accurate absolute images. Each step in such a method will be a regularized linear step as described below.

4 REGULARIZED LINEAR SOLUTIONS

Moving to a general setting of a linear problem, we have a vector $\mathbf{m}$, which represents the unknown parameters of our material (the image) and data $\mathbf{D}$, which are voltage or charge measurements. In or linearized problem

$$\delta\mathbf{D} = \mathbf{J}\delta\mathbf{m} \tag{11}$$

This could be interpreted as seeking either a difference image from the difference between two sets of measured data (typically one would be a data set from a homogeneous object), or it could be a step in a non–linear iterative algorithm where $\delta\mathbf{D}$ is the difference between measured and calculated data and $\delta\mathbf{m}$ is an update to our model. For this section we will drop the δ and consider solving

$$\mathbf{D} = \mathbf{J}\mathbf{m} \tag{12}$$

Our first problem is that the matrix $\mathbf{J}$ may not be square, as we may have chosen more or fewer voxels than we have independent measurements. A natural solution to this is to choose the Moore–

Penrose generalized inverse $\mathbf{J}^{?} = \mathbf{J}^T\left(\mathbf{J}\mathbf{J}^T\right)^{-1}$ in the underdetermined case and $\mathbf{J}^{?} = \left(\mathbf{J}^T\mathbf{J}\right)^{-1}\mathbf{J}^T$ in the over determined case. In both cases $\mathbf{m}_{MP} = \mathbf{J}^{?}\mathbf{D}$ is the least squares solution to (12).

Unfortunately our inverse problem is extremely ill–posed and the inversion of $\mathbf{J}^T\mathbf{J}$ or $\mathbf{J}\mathbf{J}^T$ will be swamped by numerical error. This is a basic physical problem with electrical imaging; arbitrary large amplitude interior variations of the property being imaged can produce negligible changes in boundary measurement. As in any ill–posed problem we must change the problem. Rather than find the least squares solution to (12) we must add some additional assumptions, prior information, about $\mathbf{m}$, and seek a solution consistent with this prior information. The simplest additional assumption is that the norm of $\mathbf{m}$, $\|\mathbf{m}\| = \sqrt{\sum m_k^2}$, is not too big. More precisely we seek an $\mathbf{m}$ to minimize

$$\|\mathbf{D} - \mathbf{J}\mathbf{m}\|^2 + \alpha^2 \|\mathbf{m}\|^2 \tag{13}$$

where the regularization parameter α controls the trade–off between fitting our data and consistency with our prior information. There both systematic and empirical methods of choosing the regularization parameter, Morozov's discrepancy principle, generalized cross validation and the L–curve criterion being some of the more popular, see Hansen (1998), chapter 7. The solution to this, also called the Tikhonov regularized solution, is

$$\mathbf{m}_{\text{Tik}} = \left(\mathbf{J}^T\mathbf{J} + \alpha^2\mathbf{I}\right)^{-1}\mathbf{J}^T\mathbf{D} = \mathbf{J}^T\left(\mathbf{J}\mathbf{J}^T + \alpha^2\mathbf{I}\right)^{-1}\mathbf{D}. \tag{14}$$

Here $\mathbf{I}$ is the identity matrix of the appropriate size (the same size as an image and a data set respectively). In X–ray computed tomography $\mathbf{J}$ is integration along rays, and multiplying a data vector by $\mathbf{J}^T$ is called *back projection*.

The positive definite matrices $\left(\mathbf{J}^T\mathbf{J} + \alpha^2\mathbf{I}\right)^{-1}$ and $\left(\mathbf{J}\mathbf{J}^T + \alpha^2\mathbf{I}\right)^{-1}$ can be interpreted as filters in the image and data spaces respectively. An interpretation of (14) is that the first form is back projection followed by spatial filtering, and the second is filtering of the data followed by back–projection.

5 SINGULAR VALUE DECOMPOSITION

The singular value decomposition (SVD) of a matrix provides a tool to study the ill–conditioning of the matrix as well as to calculate a regularized inverse. Singular values and singular vectors generalize the idea of eigenvalues and eigenvectors. We have a set of orthonormal vectors in the image space $\mathbf{u}_i$ and in the data space $\mathbf{v}_i$ such that $\mathbf{J}\mathbf{u}_i = \lambda_i\mathbf{v}_i$ for scalars λ_i, $\lambda_1 \geq \lambda_2 \geq L \geq 0$ (actually σ is the traditional letter for a singular value but that is taken for conductivity). The interpretation is that a component of an image corresponding to $\mathbf{u}_i$ is attenuated by λ_i. If we measure at a fixed precision δ we will not be able to detect changes in the image in components for which $\lambda_i < \delta$. In electrical imaging problems the singular values decay exponentially, Breckon (1988) Compared with conventional imaging modalities such as X–ray computerized tomography the ill–posed nature of electrical imaging is severe.

Using the singular vectors as columns of orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ we have $\mathbf{J} = \mathbf{V}\Lambda\mathbf{U}^T$ where Λ is $\text{diag}(\lambda_i)$ padded with zeros to make it the same shape as $\mathbf{J}$. We also have $\mathbf{J}^{?} = \mathbf{U}\ddot{\mathbf{E}}\mathbf{V}^T$ (where Λ^{-1} is $\text{diag}(\lambda_i^{-1})$ suitably padded with zeros). We can define a more general approximate inverse by $\mathbf{U}f(\ddot{\mathbf{E}})\mathbf{V}^T$, where for example

$$f(\lambda) = f_\alpha^{\text{Tik}}(\lambda) = \frac{\lambda}{\lambda^2 + \alpha^2} \tag{15}$$

gives Tikhonov regularization.

If one is using regularized linear reconstruction algorithms, with a precomputed Jacobian, one may as well precompute the singular vectors and values of $\mathbf{J}$. Although this is expensive it need be done only once for each geometry and assumed background. One can then use this singular value decomposition to calculate the regularized image, for any value if the regularization parameter, with a cost of only two matrix times vector operations.

6 ITERATIVE LINEAR ALGORITHMS

Landweber's[1] method is an iterative algorithm for solving $\mathbf{Jm} = \mathbf{D}$ which for a well posed problem converges to the least squares solution $\mathbf{J}^{?}\mathbf{m}$. The n–th iteration giving the next estimate of $\mathbf{m}$ is

$$\mathbf{m}_n = \mathbf{m}_{n-1} + \tau \mathbf{J}^T(\mathbf{D} - \mathbf{Jm}_{n-1}) \qquad (16)$$

where τ is a relaxation parameter satisfying $0 < \tau < 2/\lambda_1^2$. For an ill–posed problem one can use an iterative algorithm as a regularization method. According to Morozov's discrepancy principle one stops the iteration as soon as the error $\| \mathbf{D} - \mathbf{Jm}_n \|$ falls bellow the estimated error in the measured data. Notice that if $\mathbf{m}_0 = \mathbf{0}$ is the initial guess then the first approximation is proportional to $\mathbf{J}^{?}\mathbf{D}$.

The n–th Landweber iteration can be computed directly using the SVD filter

$$f(\lambda) = f_{n,\tau}^{\text{Land}}(\lambda) = \frac{1-(1-\tau\lambda^2)^n}{\lambda} \qquad (17)$$

Which means one can cheaply calculate the result of doing any number of Landweber iterations, the SVD of $\mathbf{J}$ having been computed in advance. See Byars (2001), and Figures 1. and 2. for experimental results. Landweber's iteration is very slowly converging, which could be seen as an advantage where data are very noisy – only a small number of iterations should be employed. A hybrid method, also used by Byars (2001), is to replace the 'back projection' term $\mathbf{J}^T$ in (16) by the Tikhonov regularized inverse $\mathbf{J}^T\left(\mathbf{JJ}^T + \alpha^2\mathbf{I}\right)^{-1}$. In this case the SVD 'filter function' is given by

$$f(\lambda) = f_{n,\tau,\alpha}^{\text{Hybrid}}(\lambda) = \frac{1-(1-\tau\lambda^2/(\lambda^2+\alpha^2))^n}{\lambda}, \qquad (18)$$

see Section 6.2 of Hansen (1998).

One interesting variation on Landweber's or indeed any iterative method is to solve a constrained problem (such as upper and lower bound constraints on the material property) enforcing the constraint (projecting on to the feasible set) at each iteration. Knowing prior upper and lower bounds on the image stabilizes the electrical imaging inverse problem; in particular the inverse problem has logarithmic continuity under these conditions, Alessandrini (1991). What this means is that one can guarantee a reduction in the error ε in the image by an improvement in the noise on the data δ, but one has diminishing returns: specifically $d\delta/d\varepsilon$ tends to infinity as ε tends to zero.

Projected Landweber method (Goldstein (1964), Bertero (1998)) has been employed with some success in industrial applications of ECT, Byars (2001), where it is common to have a two phase mixture with known permittivities. It is not possible to accelerate this method so easily using a precomputed factorisation such as the SVD, and it seems a higher computational cost is inevitable.

Projected iterative reconstruction algorithms are common in conventional tomography, and there are some convergence proofs, Viergever (1988). Projected iterative methods of course are only one way of solving a quadratic minimization problem with inequality constraints and this is an active area of research in optimisation theory.

7 MORE GENERAL REGULARIZATION

[1] This algorithm is regularly reinvented and has many different names.

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Conventional Tikhonov regularization allows us to impose only a crude 2−norm bound as prior information. We define a more general norm for a positive definite matrix P by $\| \mathbf{x} \|_P^2 = \mathbf{x}^T\mathbf{P}\mathbf{x}$. In generalized Tikhonov regularization we seek an $\mathbf{m}$ to minimize the functional

$$\| \mathbf{Jm} - \mathbf{D} \|_P^2 + \alpha^2 \| \mathbf{m} - \mathbf{m}_0 \|_Q^2 \tag{19}$$

where $\mathbf{m}_0$ is the most likely background level. The norm on the data space determined by $\mathbf{P}$ reflects our confidence in the measurements −we not strive too hard to fit an unreliable datum. The norm on the image side reflects our prior knowledge of the material, or at least how it differs from the prior. For example to enforce a smoothing constraint we use a differential operator. Borsic (2001) uses non−local operators, whereas Polydorides (2001) and Vauhkonen (2001) use differential operators. The solution of the minimization problem is

$$\mathbf{m} = \left(\mathbf{J}^\mathbf{T}\mathbf{PJ} + \alpha^2\mathbf{Q}\right)^{-1} (\mathbf{J}^\mathbf{T}\mathbf{PD} - \alpha^2\mathbf{Qm}_0) \tag{20}$$

and from Tarantola (1987) we have the equivalent forms

$$\mathbf{m} = \mathbf{m}_0 + \left(\mathbf{J}^\mathbf{T}\mathbf{PJ} + \alpha^2\mathbf{Q}\right)^{-1} \mathbf{J}^\mathbf{T}\mathbf{P}(\mathbf{D} - \mathbf{Jm}_0) \tag{21}$$

$$= \mathbf{m}_0 + \mathbf{Q}^{-1}\mathbf{J}^T \left(\mathbf{JQ}^{-1}\mathbf{J}^T + \alpha^2\mathbf{P}^{-1}\right)^{-1} (\mathbf{D} - \mathbf{Jm}_0) \tag{22}$$

Note that (22) uses the invertibility of $\mathbf{Q}$ which is otherwise not required provided the kernel of $\mathbf{Q}$ does not contain any non−zero vectors which are in the (effective numerical) kernel of $\mathbf{J}$. For example one might use a differential operator for $\mathbf{Q}$ which is zero on constants. In this case $\| \cdot \|_\mathbf{Q}$ is only a *seminorm*. (The case of non−invertible **Q** can be handled using the Generalized Singular Value Decomposition, Hansen (1998) section 2.1). SVD filter methods can be used in this more general context by first transforming to the 'decorelated' variables $\mathbf{D}' = \mathbf{P}^{-1/2}\mathbf{D}$ and $\mathbf{m}' = \mathbf{Q}^{-1/2}(\mathbf{m} - \mathbf{m}_0)$. We take the SVD of $\mathbf{J}' = \mathbf{P}^{-1/2}\mathbf{JQ}^{1/2}$ and the Tikhonov filter applied to this will result in generalized Tikhonov regularization.

Tarantola (1987) gives a statistical interpretation of this regularization procedure. If the errors on the data are multivariate Gaussian with mean $\mathbf{D}$ and covariance $\mathbf{P}^{-1}$ and the probability distribution representing the distribution of the a priori information about the conductivity is multivariate Gaussian with mean $\mathbf{m}_0$ and covariance $\alpha^{-2}\mathbf{Q}^{-1}$ then (20) represents the maximum a posteriori estimate.

In inverse problems it is important to incorporate as much prior information as possible, by representing known structures in the forward model as well as using a prior probability distribution in regularization. It is also important to characterize the systematic and random errors in the instrumentation, as well as the forward model. It is worth noting that the above formulation can only be used if the errors in the data are approximately Gaussian.

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