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# NLEVP: A Collection of Nonlinear Eigenvalue Problems

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**Abstract**

We describe a collection of nonlinear eigenvalue problems that we provide in the form of a MATLAB toolbox. The collection contains problems from models of real-life applications as well as ones constructed specifically to have particular properties. A brief description is given of each problem and the problems are classified according to their structural properties.

**Key words.** test problem, nonlinear eigenvalue problem, rational eigenvalue problem, polynomial eigenvalue problem, quadratic eigenvalue problem, hyperbolic, overdamped, palindromic, proportionally-damped

## 1 Introduction

In many areas of scientific computing collections of problems are available that play an important role in testing and benchmarking software. Among the uses of such collections are

- testing the correctness of a code against some measure of success, where the latter is typically an error or residual whose nature is suggested by the underlying problem;
- measuring the performance of a code—for example, speed, execution rate, or again an error or residual;
- measuring the robustness of a code, that is, the behaviour in extreme situations, such as for very badly scaled and/or ill conditioned data;
- comparing two or more different codes with respect to the factors above.

A collection ideally combines problems artificially constructed to reflect a wide range of possible properties with problems representative of real applications. Problems for which something is known about the solution are always particularly attractive.

Two areas that have historically been well endowed with collections of problems are optimization and linear algebra. In optimization we mention just the collections in the widely used Cute and Cuter testing environments [3], [20], though various other, sometimes more specialized, collections are available. In linear algebra the Matrix Market website [33] provides access to several collections of matrices, in a variety of formats. A recent collection of sparse matrices is the University of Florida Sparse Matrix Collection [12], which comprises over 1800 matrices from practical applications, including those from many earlier collections. Both the latter collections include the Harwell–Boeing collection [14] of sparse matrices and the NEP collection [1] of standard and generalized eigenvalue problems.

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The growing interest in nonlinear eigenvalue problems has created a need for a collection of problems in this area. The standard form of a nonlinear eigenvalue problem is $F(\lambda)x = 0$, where $F : \mathbb{C} \to \mathbb{C}^{m\times n}$ is a given matrix-valued function and $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$ are the sought eigenvalue and eigenvector, respectively. Rational and polynomial functions are of particular interest, the most practically important case being the quadratic $Q(\lambda) = \lambda^2 A + \lambda B + C$, which corresponds to the quadratic eigenvalue problem. For recent surveys on nonlinear eigenproblems see [34] and [41]. Associated with an $n\times n$ matrix quadratic $Q(\lambda)$ are the matrix equations $X^2A+XB+C = 0$ and $AX^2 + BX + C = 0$, where the unknown $X \in \mathbb{C}^{n\times n}$ is called a solvent [13], [18]. Thus a matrix polynomial $P(\lambda)$ defines both an eigenvalue problem and two matrix equations.

We have built a collection of nonlinear eigenvalue problems that is described in the remainder of this report. In order to provide focus and keep the collection to a manageable size we have chosen to exclude linear problems from the collection.

Our matrices come from a variety of sources. Some are from models of real-life applications, while others have been constructed specifically to have particular properties. Many of the matrices have been used in previous papers to test numerical algorithms.

Nonlinear eigenvalue problems are often highly structured and it is important to take account of the structure both in developing the theory and in designing numerical methods. We therefore provide a thorough classification of our matrices that records the most relevant structural properties.

We have chosen to implement the collection in MATLAB, as a toolbox, recognizing that it is straightforward to convert the matrices into a format that can be read by other languages by using either the built-in MATLAB I/O functions or those provided in Matrix Market. The collection is accessed via a single MATLAB function `nlevp`, which is modelled on MATLAB's `gallery` function. A criterion for inclusion of problems is that the underlying MATLAB code and data files are not too large, since we want to provide the toolbox as a single file that can be downloaded in a reasonable time.

This document describes Version 1.0 of the toolbox. The collection will grow and contributions are welcome (see Appendix C).

## 2 Installation and Usage

The collection is available from

`http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html`

It is provided as both a zip file and a tar file. To install the toolbox create in a suitable location the directory `nlevp` and make this the current directory. Download `nlevp.zip` or `nlevp.tar` into this directory. Then use appropriate "unzip" software (making sure to preserve the directory structure) or type `tar xvf nlevp.tar`. This creates the subdirectory `private`. For serious use it is best to put the `nlevp` directory on the MATLAB path, which can be done using the `addpath` command (ideally in `startup.m`).

To try the toolbox from within MATLAB, change to the `nlevp` directory if it is not already on the MATLAB path, and run the demonstration script by typing `nlevp_example`. Then execute the following commands:

```
help nlevp
nlevp query problems
nlevp query properties

nlevp help railtrack
nlevp query railtrack
coeffs = nlevp('railtrack')
spy(coeffs{2})

coeffs = nlevp('bicycle')
polyeig(coeffs{:})
```

The collection has been tested in MATLAB 7.5 (R2007b) and 7.6 (R2008a). It does not work with versions 6.5 (R13) and earlier of MATLAB, since it uses functionality introduced in MATLAB 7.0 (R14).

# 3 Identifiers

We give in Table 1 a list of identifiers for the types of problems available in the collection and in Table 2 a list of identifiers that specify the properties of problems in the collection. These properties can be used to extract specialized subsets of the collection for use in numerical experiments. In the next two subsections we briefly recall some relevant definitions and properties of nonlinear eigenproblems.

## 3.1 Nonlinear Eigenproblems

The **polynomial eigenvalue problem** (PEP) is to find scalars $\lambda$ and nonzero vectors $x$ and $y$ satisfying $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$, where

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i, \qquad A_i \in \mathbb{C}^{m\times n}, \quad A_k \neq 0 \tag{1}$$

is an $m \times n$ matrix polynomial of degree $k$. Here, $x$ and $y$ are right and left eigenvectors corresponding to the eigenvalue $\lambda$. A **quadratic eigenvalue problem** (QEP) is a PEP of degree $k = 2$. For a survey of QEPs see [41]. PEPs are defined by their coefficient matrices $A_0$, $A_1$, ..., $A_k$. Polynomial and quadratic eigenproblems are identified by `pep` and `qep`, respectively, in the collection (see Table 1), and any problem of type `qep` is automatically also of type `pep`.

The matrix function $R(\lambda) \in \mathbb{C}^{n\times n}$ whose elements are rational functions

$$r_{ij}(\lambda) = \frac{p_{ij}(\lambda)}{q_{ij}(\lambda)}, \qquad 1 \le i \le m, \quad 1 \le j \le n,$$

where $p_{ij}(\lambda)$ and $q_{ij}(\lambda)$ are scalar polynomials of the same variable and $q_{ij}(\lambda) \not\equiv 0$, defines a **rational eigenvalue problem** (REP) $R(\lambda)x = 0$ [27]. Unlike for PEPs it is difficult to impose a unique format for specifying REPs. For the collection we use the form

$$R(\lambda) = P(\lambda)Q(\lambda)^{-1},$$

where $P(\lambda)$ and $Q(\lambda)$ are matrix polynomials, or the less general form (often encountered in practice)

$$R(\lambda) = Ax + \lambda Bx + \sum_{j=1}^{p} \frac{\lambda}{\sigma_j - \lambda} C_j x, \tag{2}$$

where $A$, $B$, and the $C_j$ are $m \times n$ matrices, and the $\sigma_j$ are the poles. Which form is used is specified in the help for the M-file defining the problem. Rational eigenproblems are identified by `rep` in the collection.

As mentioned in the introduction, PEPs and REPs are special cases of **nonlinear eigenvalue problems** (NEPs)

$$F(\lambda)x = 0, \tag{3}$$

where $F : \mathbb{C} \to \mathbb{C}^{m\times n}$. Any problem that is not polynomial, quadratic, or rational is identified by `nep` in the collection (see Table 1).

## 3.2 Some Definitions and Properties

Nonlinear eigenproblems are said to be **regular** if $m = n$ and $\det(F(\lambda)) \not\equiv 0$, and **nonregular** otherwise. Recall that a regular PEP possesses $nk$ (not necessarily distinct) eigenvalues [18]. As the majority of problems in the collection are regular we identify only nonregular problems, for which the identifier is `nonregular`.

Table 1: Problems available in the collection and their identifiers.

| | |
|---|---|
| `qep` | quadratic eigenvalue problem |
| `pep` | polynomial eigenvalue problem |
| `rep` | rational eigenvalue problem |
| `nep` | other nonlinear eigenvalue problem |

Table 2: List of identifiers for the problem properties.

| | | |
|---|---|---|
| `nonregular` | `symmetric` | `hyperbolic` |
| `real` | `Hermitian` | `elliptic` |
| `nonsquare` | `T-even` | `overdamped` |
| `sparse` | `*-even` | `proportionally-damped` |
| `scalable` | `T-odd` | `gyroscopic` |
| `parameter-dependent` | `*-odd` | |
| `solution` | `T-palindromic` | |
| | `*-palindromic` | |
| | `T-anti-palindromic` | |
| | `*-anti-palindromic` | |

The identifiers `real`, `Hermitian`, and `symmetric` are defined in Table 3. For PEPs, the `real` identifier corresponds to $P$ having real coefficient matrices, while `hermitian` corresponds to Hermitian (but not all real) coefficient matrices. Similarly, `symmetric` indicates (complex) symmetric coefficient matrices, and the `real` identifier is added if the coefficient matrices are real symmetric. For problems that are parameter-dependent the identifiers `real` and `Hermitian` are used if the problem is real or Hermitian for real values of the parameter.

The **reversal** of the matrix polynomial (1) is defined by

$$\operatorname{rev}\bigl(P(\lambda)\bigr) = \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^{k-i} A_i.$$

Identifiers for odd-even and palindromic-like square matrix polynomials, together with the special symmetry properties of their spectra (see [31]) are given in Table 4.

**Gyroscopic** systems of the form $Q(\lambda) = \lambda^2 M + \lambda G + K$ with $M$, $K$ Hermitian, $M > 0$, and $G = -G^*$ skew-Hermitian are a subset of $*$-even ($T$-even when the coefficient matrices are real) QEPs and are identified with `gyroscopic`. Here for a Hermitian matrix $A$, we write $A > 0$ to denote that $A$ is positive definite and $A \geq 0$ to denote that $A$ is positive semidefinite. When

Table 3: Some identifiers and the corresponding spectral properties. For parameter-dependent problems, the problem is classified as `real` or `hermitian` if it is so for real values of the parameter.

| Identifier | Property of $F(\lambda) \in \mathbb{C}^{m\times n}$ | Spectral properties |
|---|---|---|
| `real` | $\overline{F(\lambda)} = F(\bar{\lambda})$ | eigenvalues real or come in pairs $(\lambda, \bar{\lambda})$ |
| `symmetric` | $m = n$, $\bigl(F(\lambda)\bigr)^T = F(\lambda)$ | none unless $F$ is real |
| `Hermitian` | $m = n$, $(F(\lambda))^* = F(\bar{\lambda})$ | eigenvalues real or come in pairs $(\lambda, \bar{\lambda})$ |

Table 4: Some identifiers and the corresponding spectral symmetry properties.

| Identifier | Property of $P(\lambda)$ | Eigenvalue pairing |
|---|---|---|
| `T-even` | $P^T(-\lambda) = P(\lambda)$ | $(\lambda, -\lambda)$ |
| `*-even` | $P^*(-\lambda) = P(\lambda)$ | $(\lambda, -\bar{\lambda})$ |
| `T-odd` | $P^T(-\lambda) = -P(\lambda)$ | $(\lambda, -\lambda)$ |
| `*-odd` | $P^*(-\lambda) = -P(\lambda)$ | $(\lambda, -\bar{\lambda})$ |
| `T-palindromic` | $\mathrm{rev}P^T(\lambda) = P(\lambda)$ | $(\lambda, 1/\lambda)$ |
| `*-palindromic` | $\mathrm{rev}P^*(\lambda) = P(\lambda)$ | $(\lambda, 1/\bar{\lambda})$ |
| `T-anti-palindromic` | $\mathrm{rev}P^T(\lambda) = -P(\lambda)$ | $(\lambda, 1/\lambda)$ |
| `*-anti-palindromic` | $\mathrm{rev}P^*(\lambda) = -P(\lambda)$ | $(\lambda, 1/\bar{\lambda})$ |

$K > 0$ the eigenvalues of $Q$ are purely imaginary and semi-simple [15], [29] and the quadratic $Q(\imath\lambda)$ is hyperbolic.

A Hermitian matrix polynomial $P(\lambda)$ is **hyperbolic** if there exists $\mu \in \mathbb{R} \cup \infty$ such that $P(\mu)$ is positive definite and for every nonzero $x \in \mathbb{C}^n$ the scalar equation $x^*P(\lambda)x = 0$ has $m$ distinct zeros in $\mathbb{R} \cup \{\infty\}$. All the eigenvalues of such a $P$ are real, semisimple, and grouped in $k$ intervals, each of them containing $n$ eigenvalues [32], [21]. These polynomials are identified in the collection by `hyperbolic`. **Overdamped** systems $Q(\lambda) = \lambda^2 M + \lambda C + K$ are particular hyperbolic QEPs for which $M > 0$, $C > 0$, and $K \geq 0$; they have the identifier `overdamped`. Finally, a QEP is said to be **proportionally damped** when $M$, $C$, and $K$ are simultaneously diagonalizable; such a QEP is identified by `proportionally-damped`.

Hermitian matrix polynomials $P(\lambda)$ with even degree $k$ that are **elliptic**, i.e., $P(\lambda) > 0$ for all $\lambda \in \mathbb{R}$ [32, §34], are identified by `elliptic`. Elliptic matrix polynomials have nonreal eigenvalues.

The identifier `sparse` is used if the defining matrices are stored in MATLAB's sparse format. Problems that depend on one or more parameters are identified with `parameter-dependent`. A separate identifier, `scalable`, is used to denote that the problem dimension (or a function of it) is a parameter; for such problems a default value of the parameter is provided, typically being a value used in previously published experiments.

For some problems a *supposed* solution (eigenvalues and/or eigenvectors) is returned via the last output parameter, being either an exactly known solution or an approximate or computed solution. These problems are identified with `solution`. The documentation for the matrix will provide more information on the nature of the supposed solution.

# 4 Collection of Problems

This section contains a brief description of all the problems in the collection. The identifiers for the problem properties are listed inside curly brackets after the name of each problem.

**Acoustic wave 1D** {`pep,qep,real,symmetric,parameter-dependent,scalable`}. This problem is a QEP $Q(\lambda) = \lambda^2 M + \lambda C + K$ that arises from the finite element discretization of the wave equation for the acoustic pressure in a bounded domain, where the boundary conditions are partly pressure release and partly impedance [11].

On the 1D domain $[0, 1]$ the $n \times n$ matrices are defined by

$$M = \frac{1}{n}(I_n - \frac{1}{2}e_n e_n^T), \quad C = \frac{1}{\zeta}\, e_n e_n^T, \quad K = n \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix},$$

where $e_n$ is the last column of the $n \times n$ identity matrix and the parameter $\zeta$ is the (possibly complex) impedance.

**Acoustic wave 2D** {`pep,qep,real,symmetric,parameter-dependent,scalable`}. A 2D version of Acoustic wave 1D. On the unit square $[0,1] \times [0,1]$ with mesh size $h$ the $n \times n$ coefficient matrices of $Q(\lambda)$ with $n = \frac{1}{h}(\frac{1}{h} - 1)$ are given by

$$M = h^2 I_{m-1} \otimes (I_m - \frac{1}{2} e_m e_m^T), \quad C = \frac{h}{\zeta} I_{m-1} \otimes (e_m e_m^T), \quad K = I_{m-1} \otimes D_m + T_{m-1} \otimes (-I_m + \frac{1}{2} e_m e_m^T),$$

where $\otimes$ denotes the Kronecker product, $m = 1/h$, $\zeta$ is the (possibly complex) impedance, and

$$D_m = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 4 & -1 \\ & & -1 & 2 \end{bmatrix}_{m \times m}, \qquad T_{m-1} = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix}_{(m-1)\times(m-1)}.$$

**Bicycle** {`pep,qep,real,parameter-dependent`}. This is a $2 \times 2$ quadratic polynomial arising in the study of bicycle self-stability [36]. The linearized equations of motion for the Whipple bicycle model can be written as

$$M\ddot{q} + C\dot{q} + Kq = f,$$

where $M$ is a symmetric mass matrix, the nonsymmetric damping matrix $C = vC_1$ is linear in the forward speed $v$, and the stiffness matrix $K = gK_0 + v^2 K_2$ is the sum of two parts: a velocity independent symmetric part $gK_0$ proportional to the gravitational acceleration $g$ and a nonsymmetric part $v^2 K_2$ quadratic in the forward speed.

**Bilby** {`pep,qep,real,parameter-dependent`}. This $5 \times 5$ quadratic matrix polynomial arises in a model from [2] for the population of the greater bilby (*Macrotis lagotis*), an endangered Australian marsupial. Define the $5 \times 5$ matrix

$$M(g,x) = \begin{bmatrix} gx_1 & (1-g)x_1 & 0 & 0 & 0 \\ gx_2 & & (1-g)x_2 & 0 & 0 \\ gx_3 & & 0 & (1-g)x_3 & 0 \\ gx_4 & & 0 & & (1-g)x_4 \\ gx_5 & & 0 & & (1-g)x_5 \end{bmatrix}.$$

The model is a quasi-birth-death process some of whose key properties are captured by the elementwise minimal solution of the equation

$$R = \beta(A_0 + RA_1 + R^2 A_2), \qquad A_0 = M(g,b), \quad A_1 = M(g, e-b-d), \quad A_2 = M(g,d),$$

where $b$ and $d$ are vectors of probabilities and $e$ is the vector of ones. The corresponding quadratic matrix polynomial is $Q(\lambda) = \lambda^2 A + \lambda B + C$, where

$$A = \beta A_2^T, \quad B = \beta A_1^T - I, \quad C = \beta A_0^T.$$

We take $g = 0.2$, $b = [1,\ 0.4,\ 0.25,\ 0.1,\ 0]^T$, and $d = [0,\ 0.5,\ 0.55,\ 0.8,\ 1]^T$, as in [2].

**Butterfly** {`pep,real,parameter-dependent,T-even,scalable`}. This is a quartic matrix polynomial $P(\lambda) = \lambda^4 A_4 + \lambda^2 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$ of dimension $m^2$ with T-even structure, depending on a $10 \times 1$ parameter vector $c$ [35]. Its spectrum has a butterfly shape. The coefficient matrices are Kronecker products, with $A_4$ and $A_2$ real and symmetric and $A_3$ and $A_1$ real and skew-symmetric, assuming $c$ is real. The default is $m = 8$.

**CD player** {`pep,qep,real`}. This is a $60 \times 60$ quadratic polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M = I_{60}$ arising in the study of a CD player control task [9], [10]. The mechanism that is modeled consists of a swing arm on which a lens is mounted by means of two horizontal leaf springs. This is a small representation of a larger original rigid body model (which is also quadratic).

**Closed-loop** {`pep,qep,real,parameter-dependent`}. This is a quadratic polynomial

$$Q(\lambda) = \lambda^2 I + \lambda \begin{bmatrix} 0 & 1+\alpha \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

associated with a closed-loop control system with feedback gains 1 and $1+\alpha$, $\alpha \geq 0$. The eigenvalues of $Q(\lambda)$ lie inside the unit disc if and only if $0 \leq \alpha < 0.875$ [40].

**Concrete** {`pep,qep,symmetric,parameter-dependent,sparse`}. This is a quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + (1+i\mu)K$ arising in a model of a concrete structure supporting a machine assembly [16]. The matrices have dimension 2472. $M$ is real diagonal and low rank. $C$, the viscous damping matrix, is pure imaginary and diagonal. The factor $1+i\mu$ adds uniform hysteretic damping. The default is $\mu = 0.04$.

**Damped beam** {`pep,qep,real,symmetric,scalable`}. This QEP arises in the vibration analysis of a beam simply supported at both ends and damped in the middle [22]. The corresponding quadratic $Q(\lambda) = \lambda^2 M + \lambda C + K$ has real symmetric coefficient matrices with $M > 0$, $K > 0$, and $C = ce_n e_n^T \geq 0$, where $c$ is a damping parameter. Half of the eigenvalues of the problem are pure imaginary and are eigenvalues of the undamped problem ($C = 0$).

**Dirac** {`pep,qep,real,symmetric,parameter-dependent,scalable`}. The spectrum of this matrix polynomial is the second order spectrum of the radial Dirac operator with an electric Coulombic potential of strength $\alpha$,

$$D = \begin{bmatrix} 1+\dfrac{\alpha}{r} & -\dfrac{d}{dr}+\dfrac{\kappa}{r} \\ \dfrac{d}{dr}+\dfrac{\kappa}{r} & -1+\dfrac{\alpha}{r} \end{bmatrix}.$$

For $-\sqrt{3}/2 < \alpha < 0$ and $\kappa \in \mathbb{Z}$, $D$ acts on $L^2((0,\infty),\mathbb{C}^2)$ and it corresponds to a spherically symmetric decomposition of the space into partial wave subspaces [38]. The units are chosen so that $c = m = 1$. The problem discretization is relative to subspaces generated by the Hermite functions of odd order. The size of the matrix coefficients of the QEP is $n+m$: $n$ Hermite functions in the first component of the $L^2$ space and $m$ in the second component [5].

For $\kappa = -1$, $\alpha = -1/2$ and $n$ large enough, there is a conjugate pair of isolated points of the second order spectrum near the ground eigenvalue $E_0 \approx 0.866025$. The essential spectrum, $(-\infty,-1] \cup [1,\infty)$, as well as other eigenvalues, also seem to be captured for large $n$.

**Gun** {`nep,sparse`}. This nonlinear eigenvalue problem models a radio-frequency gun cavity. The eigenvalue problem is of the form

$$T(\lambda)x = \Big[K - \lambda M + i(\lambda - \sigma_1^2)^{\frac{1}{2}} W_1 + i(\lambda - \sigma_2^2)^{\frac{1}{2}} W_2\Big] x = 0,$$

where $M, K, W_1, W_2$ are real symmetric matrices of size $9956 \times 9956$. $K$ is positive semidefinite and $M$ is positive definite. In this example $\sigma_1 = 0$ and $\sigma_2 = 108.8774$. The eigenvalues of interest are the $\lambda$ for which $\lambda^{1/2}$ is close to 146.71 [30, p. 59].

**Hospital** {`pep,qep,real`}. This is a $24 \times 24$ quadratic polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M = I_{24}$, arising in the study of the Los Angeles University Hospital building [9], [10]. There are 8 floors, each with 3 degrees of freedom.

**Loaded string** {`rep,real,symmetric,parameter-dependent,scalable`}. This rational eigenvalue problem arises in the finite element discretization of a boundary problem describing the eigenvibration of a string with a load of mass $m$ attached by an elastic spring of stiffness $k$. It has the form

$$R(\lambda)x = \left(A - \lambda B + \frac{\lambda}{\lambda - \sigma} C\right) x = 0,$$

where the pole $\sigma = k/m$, and $A > 0$ and $B > 0$ are $n \times n$ matrices defined by

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}, \quad B = \frac{h}{6}\begin{bmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & 4 & 1 \\ & & 1 & 2 \end{bmatrix},$$

and $C = ke_n e_n^T$ with $h = 1/n$ [37].

**Mobile manipulator** {`pep,qep,real`}. This is a $5 \times 5$ QEP arising from the modelling as a time-invariant descriptor control system of a two-dimensional three-link mobile manipulator [8, Ex. 14], [7]. The system in its second-order form is

$$\begin{aligned} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) &= Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

where the coefficient matrices are $5 \times 5$ and of the form

$$M = \begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_0 & -F_0^T \\ F_0 & 0 \end{bmatrix},$$

with

$$M_0 = \begin{bmatrix} 18.7532 & -7.94493 & 7.94494 \\ -7.94493 & 31.8182 & -26.8182 \\ 7.94494 & -26.8182 & 26.8182 \end{bmatrix}, \quad D_0 = \begin{bmatrix} -1.52143 & -1.55168 & 1.55168 \\ 3.22064 & 3.28467 & -3.28467 \\ -3.22064 & -3.28467 & 3.28467 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 67.4894 & 69.2393 & -69.2393 \\ 69.8124 & 1.68624 & -1.68617 \\ -69.8123 & -1.68617 & -68.2707 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The quadratic $Q(\lambda) = \lambda^2 M + \lambda D + K$ is close to being nonregular [8], [23].

**Orr-Sommerfeld** {`pep,parameter-dependent,scalable`}. This example is a quartic polynomial eigenvalue problem arising in the spatial stability analysis of the Orr-Sommerfeld equation [40].

**Power plant** {`pep,qep,symmetric,parameter-dependent`}. This is a QEP $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$ describing the dynamic behaviour of a nuclear power plant simplified into an eight-degrees-of-freedom system [26], [41]. The mass matrix $M$ and damping matrix $D$ are real symmetric and the stiffness matrix has the form $K = (1 + \imath\mu)K_0$, where $K_0$ is real symmetric (hence $K = K^T$ is complex symmetric). The parameter $\mu$ describes the hysteretic damping of the problem. The matrices are badly scaled.

**Railtrack** {`pep,qep,t-palindromic,sparse`}. This is a T-palindromic QEP of size 1005: $Q(\lambda) = \lambda^2 A^T + \lambda B + A$ with $B = B^T$. It stems from a model of the vibration of rail tracks under the excitation of high speed trains [24], [25], [31]. This problem has the property that the matrix $A$ is of the form

$$A = \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} \in \mathbb{C}^{1005 \times 1005},$$

where $A_{21} \in \mathbb{C}^{201 \times 67}$, that is, $A$ has low rank ($\mathrm{rank}(A) = 67$). Hence this eigenvalue problem has many eigenvalues at zero and infinity.

**Schrödinger** {`pep,qep,real,symmetric,sparse`}. The spectrum of this matrix polynomial is the second order spectrum, relative to a subspace $\mathcal{L} \subset H^2(\mathbb{R})$, of the Schrödinger operator $Hf(x) = f''(x) + (\cos(x) - e^{-x^2})f(x)$ acting on $L^2(\mathbb{R})$ [6]. The subspace $\mathcal{L}$ has been generated using fourth order Hermite elements on a uniform mesh on the interval $[-49, 49]$, subject to clamped boundary conditions. The corresponding QEP is given by $K - 2\lambda C + \lambda^2 B$ where

$$K_{jk} = \langle Hb_j, Hb_k \rangle, \qquad C_{jk} = \langle Hb_j, b_k \rangle \qquad \text{and} \qquad B_{jk} = \langle b_j, b_k \rangle.$$

Here $\{b_k\}$ is a basis of $\mathcal{L}$. The matrices are of size 1998.

The essential spectrum of $H$ consists of a set of bands separated by gaps. The end points of these bands are the Mathieu characteristic values. The presence of the short-range potential gives rise to isolated eigenvalues of finite multiplicity. The portion of the second order spectrum that lies in the box $[-1/2, 2] \times [-10^{-1}, 10^{-1}]$ is very close to the spectrum of $H$.

**Sign1** {`pep,qep,hermitian,parameter-dependent,scalable`}. The spectrum of this quadratic matrix polynomial is the second order spectrum of the linear operator $Mf(x) = \mathrm{sign}(x)f(x) + a\widehat{f}(0)$ acting on $L^2(-\pi, \pi)$ with respect to the Fourier basis $\mathcal{B}_n = \{e^{-inx}, \ldots, 1, \ldots, e^{inx}\}$, where $\widehat{f}(0) = (1/2\pi)\int_{-\pi}^{\pi} f(x)\,dx$ [4]. The corresponding QEP is given by $K_n - 2\lambda C_n + \lambda^2 I_n$ where

$$K_n = \Pi_n M^2 \Pi_n, \qquad C_n = \Pi_n M \Pi_n$$

and $I_n$ is the identity matrix of size $2n + 1$. Here $\Pi_n$ is the orthogonal projector onto $\mathrm{Span}(\mathcal{B}_n)$.

As $n$ increases, the limit set of the second order spectrum is the unit circle, together with two real points: $\lambda_\pm$. The intersection of this limit set with the real line is the spectrum of $M$. The points $\lambda_\pm$ comprise the discrete spectrum of $M$.

**Sign2** {`pep,qep,hermitian,parameter-dependent,scalable`}. This problem is analogous to problem Sign1, the only difference being that the operator is $Mf(x) = (2\sin(x) + \mathrm{sign}(x))f(x) + a\widehat{f}(0)$.

Near the real line, the second order spectrum accumulates at $[-3, -1] \cup [1, 3] \cup \{\lambda_\pm\}$ as $n$ increases. The two accumulation points $\lambda_\pm \approx \{-0.7674, 3.5796\}$ are the discrete spectrum of $M$.

**Sleeper** {`pep,qep,real,symmetric,scalable,proportionally-damped,solution`}. This QEP describes the oscillations of a rail track resting on sleepers [28]. The QEP has the form

$$Q(\lambda) = \lambda^2 I + \lambda(I + A^2) + A^2 + A + I,$$

where $A$ is the circulant matrix with first row $[-2,\ 1,\ 0,\ \ldots,\ 0,\ 1]$. The eigenvalues of $A$ and corresponding eigenvectors are explicitly given as

$$\mu_k = -4\sin^2\left(\frac{(k-1)\pi}{n}\right), \quad x_k(j) = \frac{1}{\sqrt{n}}\exp\left(\frac{-2i\pi(j-1)(k-1)}{n}\right), \qquad k = 1\colon n.$$

The eigenvalues of $Q$ can be determined from the scalar equations

$$\lambda^2 + \lambda(1 + \mu_k^2) + (1 + \mu_k + \mu_k^2) = 0.$$

Due to the symmetry, manifested in $\sin(\pi - \theta) = \sin(\theta)$, there are several multiple eigenvalues.

**Spring dashpot** {`pep,qep,real,parameter-dependent,scalable`}. Gotts [19] describes a QEP arising from a finite element model of a linear spring in parallel with Maxwell elements (a Maxwell element is a spring in series with a dashpot). The quadratic polynomial is $Q(\lambda) = \lambda^2 M + \lambda D + K$, where the mass matrix $M$ is rank deficient and symmetric, the damping matrix $D$ is rank deficient and block diagonal, and the stiffness matrix $K$ is symmetric and has arrowhead structure. This example reflects the structure only, since the matrices themselves are not from a finite element model but randomly generated to have the desired properties of symmetry etc. The matrices have the form

$$M = \operatorname{diag}(\rho\widetilde{M}_{11}, 0), \qquad D = \operatorname{diag}(0, \eta_1\widetilde{K}_{11}, \dots, \eta_m\widetilde{K}_{m+1,m+1}),$$

$$K = \begin{bmatrix} \alpha_\rho\widetilde{K}_{11} & -\xi_1\widetilde{K}_{12}, & \dots & -\xi_m\widetilde{K}_{1,m+1} \\ -\xi_1\widetilde{K}_{12} & e_1\widetilde{K}_{22} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -\xi_m\widetilde{K}_{1,m+1} & 0 & 0 & e_m\widetilde{K}_{m+1,m+1} \end{bmatrix},$$

where $\widetilde{M}_{ij}$ and $\widetilde{K}_{ij}$ are element mass and stiffness matrices, $\xi_i$ and $e_i$ measure the spring stiffnesses, and $\rho$ is the material density.

**String** {`pep,qep,real,symmetric,proportionally-damped,parameter-dependent,scalable`}. This example is an $n \times n$ QEP arising from a linearly damped vibrating string [39]. The $n \times n$ matrices $K$, $D$, and $M$ are defined by

$$M = \mu I, \quad D = \tau T, \quad K = \kappa T, \quad T = \begin{bmatrix} 3 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 3 \end{bmatrix},$$

where $\mu$, $\tau$, and $\kappa$ are real nonnegative parameters.

**Wing** {`pep,qep,real`}. This example is a $3 \times 3$ quadratic $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ from [17, Sec. 10.11], with numerical values modified as in [29, Sec. 5.3]. The eigenproblem for $Q(\lambda)$ arose from the analysis of the oscillations of a wing in an airstream. The matrices are

$$A_2 = \begin{bmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix}.$$

**Wiresaw1** {`pep,qep,real,t-even,gyroscopic,parameter-dependent,scalable`}. This gyroscopic QEP arises in the vibration analysis of a wiresaw [42]. It takes the form $Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0$, where the $n \times n$ coefficient matrices are defined by

$$M = I_n/2, \quad K = \operatorname*{diag}_{1 \le j \le n}\left(j^2\pi^2(1 - v^2)/2\right),$$

and

$$C = -C^T = (c_{jk}), \quad \text{with} \quad c_{jk} = \begin{cases} \dfrac{4jk}{j^2 - k^2}v, & \text{if } j + k \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Here, $v$ is a real nonnegative parameter corresponding to the speed of the wire.

Note that for $0 < v < 1$, $K$ is positive definite and the quadratic

$$G(\lambda) := -Q(-\imath\lambda) = \lambda^2 M + \lambda(\imath C) - K$$

is hyperbolic (but not overdamped).

**Wiresaw2** {`pep,qep,real,parameter-dependent,scalable`}. When the effect of viscous damping is added to the problem in **Wiresaw1**, the corresponding QEP has the form [42]

$$\widetilde{Q}(\lambda) = \lambda^2 M + \lambda(C + \eta I) + K + \eta C,$$

where $M$, $C$, and $K$ are the same as in **Wiresaw1** and the damping parameter $\eta$ is real and nonnegative.

## 5 Contributors

We are grateful to the following people for contributing to the collection.

- Zhaojun Bai, University of California, Davis.
- Younes Chahlaoui, The University of Manchester.
- Lyonell Boulton, Heriot-Watt University, Edinburgh.
- Chris Munro, The University of Manchester.
- Valeria Simoncini, University of Bologna.
- Nils Wagner, University of Stuttgart.

## A The MATLAB Function `nlevp`

```
function varargout = nlevp(name,varargin)
%NLEVP   Collection of nonlinear eigenvalue problems.
%  [OUT1,OUT2,...] = NLEVP(NAME, ARG1, ARG2,...)
%    generates the matrices defining the problem specified by NAME (a string).
%    ARG1, ARG2,... are input arguments and OUT1, OUT2, ...
%    are the output arguments.  See the list below for the available problems.
%
%  PROBLEMS = NLEVP('query','problems') or NLEVP QUERY PROBLEMS
%    returns a cell array containing the names of all problems
%    in the collection.
%  NLEVP('help','name') or NLEVP HELP NAME
%    gives additional information on problem NAME, including number and
%    meaning of input/output arguments.
%  NLEVP('query','name') or NLEVP QUERY NAME
%    returns a cell array containing the properties of the problem NAME.
%  PROPERTIES = NLEVP('query','properties') or NLEVP QUERY PROPERTIES
%    returns the properties used to classify problems in the collection.
%  NLEVP('query',PROPERTY1,PROPERTY2,...) or NLEVP QUERY PROPERTY1 ...
%    lists the names of all problems having the specified properties.
%
%  Available problems:
%
%  acoustic_wave_1d   Acoustic wave problem in 1 dimension.
%  acoustic_wave_2d   Acoustic wave problem in 2 dimensions.
%  bicycle            2-by-2 QEP from the Whipple bicycle model.
%  bilby              5-by-5 QEP from Bilby population model.
%  butterfly          Quartic matrix polynomial with T-even structure.
%  cd_player          QEP from model of CD player.
%  closed_loop        2-by-2 QEP associated with closed-loop control system.
%  concrete           Sparse QEP from model of a concrete structure.
%  damped_beam        QEP from simply supported beam damped in the middle.
%  dirac              QEP from Dirac operator.
%  gun                NEP from model of a radio-frequency gun cavity.
%  hospital           QEP from model of Los Angeles Hospital building.
%  loaded_string      REP from finite element model of a loaded vibrating
%                     string.
%  mobile_manipulator QEP from model of 2-dimensional 3-link mobile manipulator.
%  orr_sommerfeld     Quartic PEP arising from Orr-Sommerfeld equation.
%  power_plant        8-by-8 QEP from simplified nuclear power plant problem.
%  railtrack          QEP from study of vibration of rail tracks.
```

```
%  schrodinger        QEP from Schrodinger operator.
%  sign1              QEP from rank-1 perturbation of sign operator.
%  sign2              QEP from rank-1 perturbation of 2*sin(x) + sign(x)
%                     operator.
%  sleeper            QEP modelling a railtrack resting on sleepers.
%  spring_dashpot     QEP from model of spring/dashpot configuration.
%  string             QEP from finite element model of vibrating string.
%  wing               QEP from analysis of oscillations of a wing in an
%                     airstream.
%  wiresaw1           Gyroscopic system from the vibration analysis of wiresaw.
%  wiresaw2           QEP from vibration analysis of wiresaw with viscous
%                     damping effect.
%
%  Examples:
%  coeffs = nlevp('railtrack')
%    generates the matrices defining the railtrack problem.
%  nlevp('help','railtrack')
%    prints the help text of the railtrack problem.
%  nlevp('query','railtrack')
%    prints the properties of the railtrack problem.
%
%  For example code to solve all polynomial eigenvalue problems (PEPs)
%  in this collection of dimension < 500 with MATLAB's POLYEIG
%  see NLEVP_EXAMPLE.M.

%  Reference:
%  T. Betcke, N. J. Higham, V. Mehrmann, C. Schroeder, and F. Tisseur.
%  NLEVP: A Collection of Nonlinear Eigenvalue Problems,
%  MIMS EPrint 2008.40, Manchester Institute for Mathematical Sciences,
%  The University of Manchester, UK, 2008.

% Check inputs
if nargin < 1, error('Not enough input arguments'); end
if ~ischar(name), error('NAME must be a string'); end

if strcmpi(name,'query') && nargin == 1
   error('Not enough input arguments')
end

switch lower(name)
    case 'help'
        if nargin < 2, error('Not enough input parameters'); end
        name = varargin{1};
        if ~ischar(name), error('NAME must be a string'); end
        eval(['help ', name]);
        return
    otherwise
        [varargout{1:nargout}] = feval(name,varargin{:});
end
```

# B The MATLAB Function nlevp_example

The function `nlevp_example.m` illustrates the use of `nlevp` and running it provides a quick test that the toolbox is correctly installed.

```
%NLEVP_EXAMPLE  Run POLYEIG on PEP problems from NLEVP.
```

```
probs = nlevp('query','pep');
fprintf('            Problem   Dim  Max and min magnitude of eigenvalues\n')
fprintf('            -------   ---  -----------------------------------\n')
nprobs = length(probs);
m = ceil(nprobs/4);
j = 1;
for i=1:nprobs
   coeffs = nlevp(probs{i});
   n = length(coeffs{1});
   if n < 500
      % POLYEIG will convert sparse input matrices to full.
      e = polyeig(coeffs{:});
      fprintf('%20s   %3.0f  %9.2e, %9.2e\n', ...
               probs{i}, n, max(abs(e)), min(abs(e)))
      subplot(m,4,j)
      plot(e,'.')
      title(probs{i},'Interpreter','none')
      j = j+1;
   else
      fprintf('%20s   %3.0f is a PEP but is too large for this test.\n', ...
               probs{i}, n)
   end
end
```

This function produces the output

```
             Problem   Dim  Max and min magnitude of eigenvalues
             -------   ---  -----------------------------------
    acoustic_wave_1d    10  1.98e+001, 2.88e+000
    acoustic_wave_2d    30  1.64e+001, 4.29e+000
             bicycle     2  1.41e+001, 3.23e-001
               bilby     5        Inf, 8.76e-018
           butterfly    64  2.01e+000, 3.59e-001
           cd_player    60  1.87e+006, 2.23e-004
         closed_loop     2  1.07e+000, 3.31e-001
            concrete   2472 is a PEP but is too large for this test.
         damped_beam   200  3.69e+006, 7.26e+001
               dirac    80  1.14e+001, 8.68e-001
            hospital    24  8.97e+001, 5.24e+000
  mobile_manipulator     5        Inf, 2.30e-001
      orr_sommerfeld    64  3.39e+000, 1.73e-004
         power_plant     8  3.69e+002, 1.77e+001
           railtrack   1005 is a PEP but is too large for this test.
         schrodinger   1998 is a PEP but is too large for this test.
               sign1    81  1.00e+000, 1.00e+000
               sign2    81  3.00e+000, 1.00e+000
             sleeper    10  1.62e+001, 6.88e-001
      spring_dashpot    10        Inf, 1.08e-003
              string     5  4.68e+001, 5.05e-001
                wing     3  8.49e+000, 1.99e+000
            wiresaw1    10  3.14e+001, 3.14e+000
            wiresaw2    10  3.14e+001, 3.14e+000
```

and the eigenvalue plots in Figure 1.

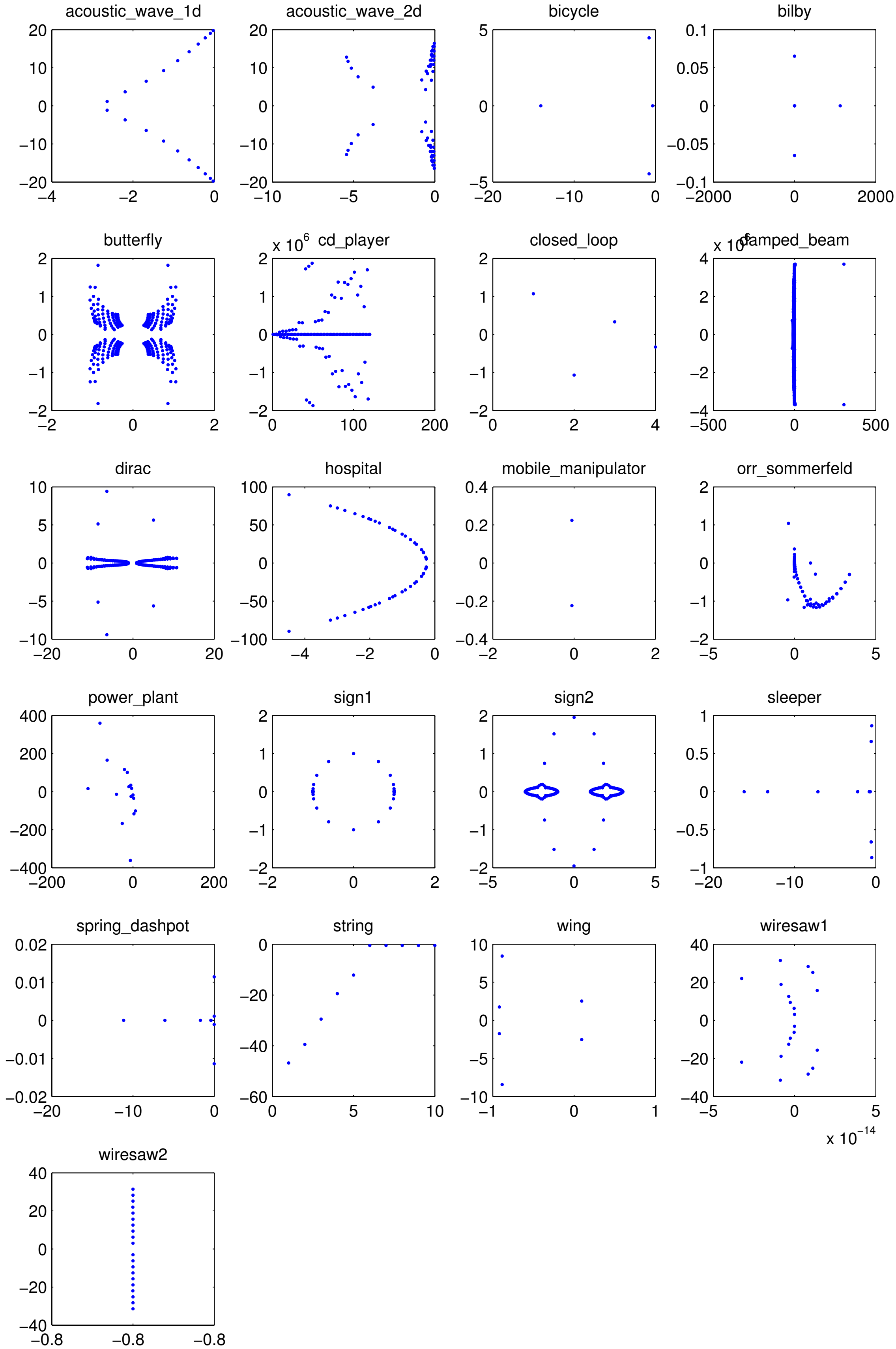


Figure 1: Eigenvalue plots produced by `nlevp_example.m`.

# C Contributing to the Collection

Contributions of suggested new problems for the collection are welcome. They should be sent to `ftisseur@ma.man.ac.uk`. The following rules should be followed when providing new problems.

Write a latex file called `problemname.tex`, where `problemname` is the proposed name of your example, describing the problem. The `tex` file should consist of a problem environment, with first line stating the relevant identifiers for the problem from Tables 1 and 2:

```
\begin{problem}{matrix_name}{identifier1,identifier2,...}
This is a xxx-problem of dimension nnn.
It arises in ...
\end{problem}
```

Provide any of your citations in a `bib` file; one `bib` file suffices even if multiple `tex` files are provided.

Write an M-file generating the coefficients of the example called `matrix_name.m`. Document the M-file in the leading comment lines with the most important information from the `tex` file. If the problem is parameter dependent set default values for any parameters not specified when the function is called. If you need extra data files, their names should begin with `matrix_name`, e.g., `matrix_name.mat`.

To specify a polynomial problem the first output of the M-file should be a cell array containing the coefficient matrices starting with the constant term. Thus if the first output is called `coeffs` and you want to define a PEP (1), then `coeffs{1}=`$A_0$, `coeffs{2}=`$A_1$, ..., `coeffs{k+1}=`$A_k$.

If a supposed solution is provided it should be returned in a structure `sol` with the following format:

sol.eval: an $m \times 1$ vector, where $m$ eigenvalues are provided,

sol.evec: an $m \times n$ matrix, where column `j` is the eigenvector corresponding to `sol.eval(j)`.

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