Didactic Transformation

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Didactic Transformation

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1. Didactic transformation: back to Auguste Comte

2. Case studies
   - Pragmatic reduction
   - Separation of concepts
   - Partial computerisation
   - Replacement of concepts
   - Ephemera

3. Questions

Didactic transformation

J.-P. Kahane (President, International Commission on Mathematical Instruction), quoted by Hyman Bass:

- In no other living science is the part of presentation, of the transformation of disciplinary knowledge to knowledge as it is to be taught (*transformation didactique*) so important at a research level.
- In no other discipline, however, is the distance between the taught and the new so large.
- In no other science has teaching and learning such social importance.
- In no other science is there such an old tradition of scientists commitment to educational questions.

Didactic transformation: Auguste Comte

Auguste Comte, *Catéchisme positiviste*, 1852:

> *A discourse, then, which is in the full sense didactic, ought to differ essentially from one simply logical, in which the thinker freely follow his own course, paying no attention to the natural conditions of all communication. [...]*
> *On the other hand, this transformation for the purposes of teaching is only practicable where the doctrines are sufficiently worked out for us to be able to distinctly compare the different methods of expanding them as a whole and to easily foresee the objections which they will naturally elicit.*

Quality of content

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- Quality of course content is the quality of didactic transformation of the content.
- Quality of didactic transformation is the quality and depth of mathematical work involved.

Looking across UK university courses:

- it strikes how much *mathematical* effort is invested into course development.

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- Weaker students $\Longrightarrow$ larger investment of effort

Mathematical component of transformation

Empirical evidence shows that

- Simple conversion of content in a “psychologically acceptable form” is frequently not enough

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Empirical evidence shows that

- Simple conversion of content in a “psychologically acceptable form” is frequently not enough
- Serious mathematical work is really needed.

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Pragmatic reduction

It is the simplest form of didactic transformation: use of a higher-level technique at an elementary level.

Q: Engineering students badly need Laplace transform

$$\mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt,$$

for use in control theory. Not enough time in lectures.

A: Give them more practice by solving even simplest differential equations by Laplace transform – instead of more common elementary methods.

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Actually, this approach can be traced back to Heaviside.

For example, linear equation with constant coefficients

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

becomes

$$Y(s)(s^2 + a_1 s + a_0) - sy(0) - y'(0) - a_1 y(0) = 0,$$

$$Y(s) = \frac{sy(0) + y'(0) + a_1 y(0)}{s^2 + a_1 s + a_0}.$$

- Roots of the characteristic polynomial inevitably reappear in computation of inverse transform:
  - interestingly, in the form of decomposition of rational functions into sums of simple fractions,
  - which triggers change of focus in teaching integration.

Separation of concepts

Q Physics students badly need tensor algebra. Not enough time in lectures.

A Teach first year linear algebra in tensor notation.

Linear algebra in tensor notation

- clear separation of spaces of row and column vectors

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- purchase of goods g_i at prices p^i:

$$\text{Cost} = \sum_i p_i g^i = \begin{pmatrix} p_1 \; p_2 \; p_3 \end{pmatrix} \begin{pmatrix} g^1 \\ g^2 \\ g^3 \end{pmatrix}$$

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- Legacy of manual typesetting of mathematical textbooks in 19th and 20th centuries?

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- Shifting of Laplace transform to the earlier chapters of calculus appears to be technical, but actually very easy easy conceptually.
- Teaching linear algebra in tensor notation is a much deeper conceptual transformation.
- The difference becomes obvious when a lecturer gives instructions to graduates students who run example classes.

Linear algebra via spreadsheets

A completely different approach to linear algebra:

- A course of linear algebra built around Gaussian elimination procedure, together with
- a systematic use of spreadsheets for elementary row operations.

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  Given sets $\{\bar{u}_i\}$ and $\{\bar{w}_j\}$ of vectors spanning vector subspaces U and W of $\mathbb{R}^n$, find a basis in $U \cap W$.

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- Unlike previous example, this approach ignores duality.

What hides behind the spreadsheet

Cell decomposition of the Grassmannian:

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Cell decomposition of the Grassmannian:

$$G/B^+ = \bigsqcup_{w \in W} B^- w B^+ / B^+.$$

- For a student — no need to know.
- For a teacher — quite useful:

  For example, describes the range of available problems.

Again, the conceptual component of didactic transformation is quite challenging.

Replacement of concepts: gauge integral

A new idea in teaching analysis:

use of gauge integral as a replacement for either

- *the Riemann integral, or*
- *the Lebesgue integral.*

Old: Riemann integral

S is the Riemann integral of $f : [a, b] \to \mathbb{R}$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever

$$a = t_0 \le s_1 \le t_1 \le s_2 \le \cdots \le t_{n-1} \le s_n = t_n = b$$

and $t_i - t_{i-1} < \delta$ for all i,

then

$$\left| S - \sum_{i=1}^{n} f(s_i)(t_i - t_{i-1}) \right| < \epsilon.$$

New: gauge integral

S is the gauge integral of $f : [a, b] \to \mathbb{R}$

if for every $\epsilon > 0$ there exists $\delta : [a, b] \to (0, +\infty)$ such that whenever

$$a = t_0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq t_{n-1} \leq s_n = t_n = b$$

and $t_i - t_{i-1} < \delta(s_i)$ for all i,

then

$$\left| S - \sum_{i=1}^{n} f(s_i)(t_i - t_{i-1}) \right| < \epsilon.$$

Which one to go, Lebesgue or Riemann?

- Gauge integral is a formal generalisation of Riemann integral.
- But can replace Lebesgue integral: If $S \subset [0, 1]$,

$$\mu(S) = \int_0^1 1_S dx$$

exists if S is Lebesgue measurable, and gives the measure of S.

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- A challenging methodological problem, both at technical and conceptual levels.
- Mathematics involved borders on open problems.

Didactic transformation in an occasional chat

- A problem in a (good) junior school:

  Which integers n, $1 < n < 100$*, can be written as sum of two squares?*

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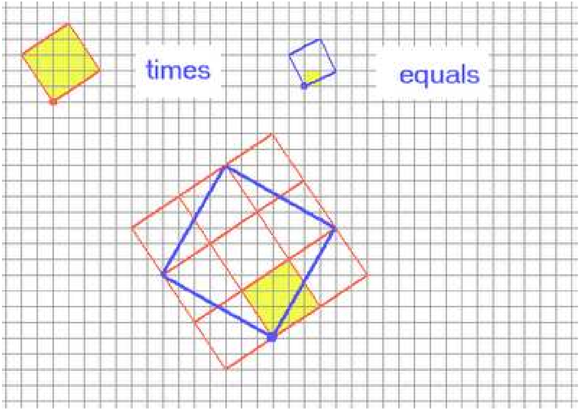
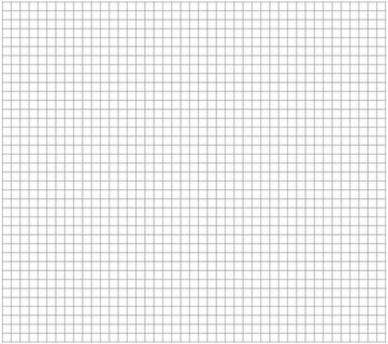
- A boy experimentally discovers that product of bisquare numbers is bisquare.
- Explanation: the Brahmagupta-Fibonacci identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

  or the norm of complex numbers.



An improvised geometric explanation.



Humble graph paper.

History of graph paper

- First commercially published “coordinate paper”: 1795.
- 1890–1910: E. H. Moore served on U.S. education panels and fought for teaching students to graph curves using paper with “squared lines.”
- Powerful but underused tool of algebra and number theory:
  - a lattice in a Lie group: $\mathbb{Z} \times \mathbb{Z} < \mathbb{R} \times \mathbb{R}$
  - integral domain $\mathbb{Z}[\sqrt{-1}]$.
  - . . . and more

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  - integral domain $\mathbb{Z}[\sqrt{-1}]$.
  - ...and more
  - Hence extensive use in “olympiad” mathematics.

Questions

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- Is there a usable methodology for comparing efficiency of different mathematical treatments of the same mathematical discipline?
- How would an “evidence based” comparative study of pedagogical suitability of gauge integral vs. Riemann integral look like?
- How can findings of educational studies direct and guide didactic transformation?

Policy implications

- Can the concept “didactic transformation” be reformulated in more accessible language and turned into a catchword explaining to policymakers the special needs of mathematical education?

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- Can it be used to fight the encroachment of “generic” staff development in universities?
- and BTW what are these “special needs”?