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VORTEX DYNAMICS ON A CYLINDER

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ABSTRACT. Point vortices on a cylinder (periodic strip) are studied geometrically. The Hamiltonian formalism is developed, a non-existence theorem for relative equilibria is proved, equilibria are classified when all vorticities have the same sign, and several results on relative periodic orbits are established, including as corollaries classical results on vortex streets and leapfrogging.

1. INTRODUCTION

Spatially periodic rows of point vortices in a 2-dimensional ideal fluid have long attracted the attention of fluid dynamicists, one of the earliest and the most popular instances being Kármán's vortex street [6], [16, photos 94–98]. The general problem is as follows: analyse the motion of an infinite configuration consisting of vortices $z_1,\ldots,z_N \in \mathbb{C}$ with vorticities $\Gamma_1,\ldots,\Gamma_N \in \mathbb{R}$ together with their translates $\{z_k + 2\pi rm \mid k = 1,\ldots,N, m \in \mathbb{Z}\}$, where $2\pi r > 0$ is the spatial period of translation. Traditionally the problem is analysed on the plane $\mathbb{C}$, but in this paper we place the vortices on a cylinder $\mathbb{C}/2\pi r\mathbb{Z}$ (fig. 1). Though the two pictures—periodic planar and cylindrical—are for

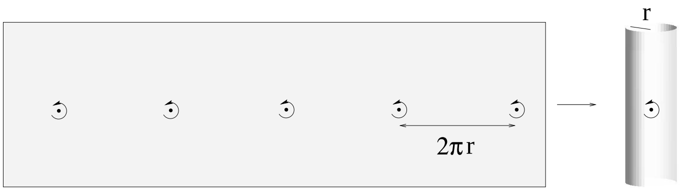

FIGURE 1

most purposes equivalent, as we shall see there are advantages, both conceptual and computational, to working on a cylinder rather than on the plane. The proviso 'for most purposes' is necessary because the cylindrical picture posits that everything in the dynamics be $2\pi r$-periodic, whereas in the planar picture one could allow, for example, non-periodic perturbations to the periodic row. Physically, however, perturbations are usually due to some small change in the mechanism generating the vortex row, and the simplest type of change generates spatially periodic perturbations. Symmetry-breaking perturbations, which do occur in real fluids and are very interesting, arise at the next level of complexity. So it is natural to look at the cylindrical picture first.

We shall be interested in how vortices move relative to one another, more precisely in their dynamics modulo the translational action of the symmetry group $\mathbb{C}/2\pi\mathbb{Z}$. The basic objects of interest are relative equilibria and relative periodic orbits. A *relative equilibrium* is a motion of vortices that lies entirely in a group orbit (i.e. it looks stationary up to translation), and a *relative periodic orbit* is a motion that revisits the same group orbit after some time (i.e. it looks periodic

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in time up to translation). Equilibria and periodic orbits in the ordinary sense are special examples of relative equilibria and relative periodic orbits. When we wish to exclude ordinary equilibria or periodic orbits, we speak of relative equilibria or relative periodic orbits *with nonzero drift*.

As on the plane, dynamics of point vortices on a cylinder lends itself to a Hamiltonian formalism. The model presented here is then a finite-dimensional Hamiltonian approximation to the vortex dynamics of the Euler equation. This approximation is mathematically very rich and in the context of the plane can claim a pedigreed history [7, chap. VII], [17]. Conversely, the motion of point vortices is amenable to desingularization to a solution of the Euler equation.

For vortices on the plane or on a sphere, an extensive theory of relative equilibria is available (especially when the vorticities are identical or opposite) [1, 9, 8]. In contrast, apart from a study on 3 vortices on the periodic strip [2], and a study of rings of point vortices on surfaces of revolution [4], no literature seems to exist on relative equilibria and relative periodic orbits of N vortices on a cylinder. In this paper we develop the Hamiltonian formalism for vortex dynamics on a cylinder (section 2), prove that if the vorticities do not sum to zero a cylinder supports no relative equilibrium with nonzero drift (section 3), classify equilibria when all vorticities have the same sign (section 3), show that 3 vortices form a relative periodic orbit for 'small' initial conditions or for vorticities dependent over $\mathbb{Q}$ with zero sum, and establish several results on a class of relative periodic orbits called leapfrogging [16, photo 79] (section 4), which may be regarded as splitting of Kármán's vortex street.

Although Noether's theorem tells us that associated to any 1-parameter group of symmetries three is a corresponding first integral, there is a topological hypothesis (that certain closed 1-forms are exact) which is not fulfilled by the cylinder, and while the subgroup of horizontal translations $\mathbb{R}/2\pi\mathbb{Z} \subset \mathbb{C}/2\pi\mathbb{Z}$ does have a conserved quantity associated to it, the subgroup $\mathrm{i}\mathbb{R}$ of vertical translations does not. However, since any closed 1-form is *locally* exact, this subgroup does have *locally* well-defined first integrals, and one of the novelties of the present work is to exploit these local first integrals (Theorems 2, 3, 4).

Many of the results have analogues in the theory of vortices on a torus, i.e. for spatially biperiodic arrays of vortices.

2. Hamiltonian formalism of vortices on a cylinder

Throughout the paper *cylinder* means the surface $\mathbb{C}/2\pi r\mathbb{Z} \simeq (\mathbb{R}/2\pi r) \times \mathbb{R}$, where $r > 0$ is some fixed constant, the *radius* of the cylinder. The coordinate $z = x + iy$ on $\mathbb{C}/2\pi r\mathbb{Z}$ is to be read modulo $2\pi r$, i.e. $x \equiv x + 2\pi rn$ for all $n \in \mathbb{Z}$; the x-axis (which is a circle) is *horizontal*, the y-axis *vertical*. The phase space for the motion of vortices $z_1, \ldots, z_N$ with vorticities $\Gamma_1, \ldots, \Gamma_N$ is the product of N copies of the cylinder with diagonals removed (to exclude collisions). The Hamiltonian is a weighted combination $H(z_1, \ldots, z_N) = \sum_{k<l} \Gamma_k \Gamma_l \psi(z_k, z_l)$ of Green's function ψ for the Laplacian on the cylinder: $\nabla^2\psi(z, z_0) = -\delta_{z_0}(z)$ (see e.g. [15, section 2]). Hamilton's equations are

$$\frac{dz_k}{dt} = \frac{2}{i} \frac{\partial H}{\partial(\Gamma_k \bar{z}_k)}, \qquad (k = 1, \ldots, N).$$

The quickest way to derive the Hamiltonian on a cylinder is to periodize Green's function on the plane $\psi(z_k, z_l) = -\frac{1}{2\pi} \log|z_k - z_l|$ by taking into account contributions from $2\pi\mathbb{Z}$-translates.

Formally the periodized Hamiltonian becomes

$$-\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{k<l}\Gamma_k\Gamma_l\log|z_k-z_l-2\pi rn|,$$

which, as it stands, diverges. But since additive constants in H do not affect the dynamics, we can subtract off a constant divergent series to force the remaining functional part to converge. Jettisoning $-\frac{1}{2\pi}\sum_n\sum_{k<l}\Gamma_k\Gamma_l\log|2\pi rn|$ and pairing terms in n and $-n$,

$$H=-\frac{1}{2\pi}\sum_{k<l}\Gamma_k\Gamma_l\log\left|(z_k-z_l)\prod_{n\geqslant 1}\left(1-\left(\frac{z_k-z_l}{2\pi rn}\right)^2\right)\right|=-\frac{1}{2\pi}\sum_{k<l}\Gamma_k\Gamma_l\log\left|\sin\frac{z_k-z_l}{2r}\right|. \tag{2.1}$$

The equations of motion on a cylinder are therefore

$$\frac{dz_k}{dt}=\frac{i}{4\pi r}\sum_{l,l\neq k}\Gamma_l\operatorname{cotan}\frac{\overline{z_k}-\overline{z_l}}{2r},\qquad (k=1,\ldots,N). \tag{2.2}$$

For reference, we list expressions in real coordinates:

$$H=-\frac{1}{4\pi}\sum_{k<l}\Gamma_k\Gamma_l\log\left\{\sin^2\left(\frac{x_k-x_l}{2r}\right)+\sinh^2\left(\frac{y_k-y_l}{2r}\right)\right\}, \tag{2.3}$$

$$\begin{cases}\dfrac{dx_k}{dt}=-\dfrac{1}{8\pi r}\displaystyle\sum_{l,l\neq k}\Gamma_l\frac{\sinh\dfrac{y_k-y_l}{r}}{\sin^2\left(\dfrac{x_k-x_l}{2r}\right)+\sinh^2\left(\dfrac{y_k-y_l}{2r}\right)}\\[2ex] \dfrac{dy_k}{dt}=\dfrac{1}{8\pi r}\displaystyle\sum_{l,l\neq k}\Gamma_l\frac{\sin\dfrac{x_k-x_l}{r}}{\sin^2\left(\dfrac{x_k-x_l}{2r}\right)+\sinh^2\left(\dfrac{y_k-y_l}{2r}\right)}\end{cases},\qquad (k=1,\ldots,N). \tag{2.4}$$

One noteworthy feature of (2.4) is that as $y_k-y_l\to\infty$ (infinite vertical separation), the velocity induced by z_l on the vortex z_k does not decay to 0, but tends to $\Gamma_l/4\pi r$, as is obvious upon calculating in the planar theory the circulation around a tall window of width $2\pi r$ enclosing z_l. Another way to interpret the feature is to note that in the planar theory, up to rescaling, stretching vertical separation amounts to narrowing the spatial period $2\pi r\to 0$; the latter limit produces a *vortex sheet* (or more aptly *vortex line* in this 2-dimensional theory), which induces a velocity field constant above (and the opposite constant below) the sheet independently of the distance to the sheet. This is exactly as in 2-dimensional electromagnetism or gravity where the force induced by a homogeneous charge or mass distribution along an infinite line is independent of the distance to the line.

Physically, periodizing the plane with period $2\pi r$ and considering N vortices on the resulting cylinder is the same as periodizing with period $2\pi rn$ and considering nN vortices on the resulting wider cylinder. The equivalence between these periodizations is trivial yet sometimes useful:

Proposition. *Let $z_1, \dots, z_N$ be vortices with vorticities $\Gamma_1, \dots, \Gamma_N$ on a cylinder of radius r. Next let $z_1, \dots, z_N, z_1 + 2\pi r, \dots, z_N + 2\pi r, \dots, z_1 + 2\pi rn, \dots, z_N + 2\pi rn$ be their 'n-fold copies' with corresponding vorticities on a cylinder of radius rn, where n is any strictly positive integer. Then the dynamics on the cylinder of radius rn covers the dynamics on the cylinder of radius r.*

In particular, given a relative equilibrium or a relative periodic orbit, we can reel off infinite families of relative equilibria or relative periodic orbits at no extra cost by replicating the configuration sideways on a wider cylinder.

Remark 1. A torus has the form $\mathbb{C}/(\pi\mathbb{Z} + \tau\pi\mathbb{Z})$, where the parameter $\tau \in \mathbb{C}$, $\mathrm{Im}\tau > 0$ controls the conformal class. The Hamiltonian is

$$H = -\frac{1}{2\pi} \sum_{k<l} \Gamma_k \Gamma_l \left\{ \log |\vartheta_1(z_k - z_l|\tau)| - \frac{(\mathrm{Im}(z_k - z_l))^2}{\pi \mathrm{Im}\tau} \right\},$$

where ϑ_1 is the 1st Jacobian theta function [12], [14], [15].

A cylinder has a translational symmetry of $\mathbb{C}/2\pi r\mathbb{Z}$ acting on itself, hence acting diagonally on the phase space. The plane has a supplementary rotational symmetry $z \mapsto e^{i\theta} z, \theta \in \mathbb{R}$; this is lost on the cylinder. Via Noether's theorem the translational symmetry of $\mathbb{C}/2\pi r\mathbb{Z}$ should give rise to a first integral, a *momentum map* $(z_1, \dots, z_N) \mapsto \sum_k \Gamma_k z_k$, but there is a rub: because z's are defined only modulo $2\pi r$ this 'momentum map' is not well-defined as a map to the dual of the Lie algebra of the symmetry group $\mathbb{C}/2\pi\mathbb{Z}$. Nor is it advisable to treat this 'momentum map' as a multi-valued function, for generically $\Gamma_1, \dots, \Gamma_N$ are independent over $\mathbb{Q}$ and so the ambiguity $\{2\pi r \sum_k \Gamma_k n_k \,|\, n_1, \dots, n_N \in \mathbb{Z}\}$ in the value of the 'map' is dense in $\mathbb{R}$. Nevertheless, the momentum map is *locally* (i.e. on each chart) well-defined. From now on, whenever we write $\sum_k \Gamma_k z_k$, some suitable ad hoc chart will be understood.

When $\sum_k \Gamma_k \neq 0$, the *center of vorticity* $\sum_k \Gamma_k z_k / \sum_k \Gamma_k$ is a more intuitive first integral [7, art. 154]. The next result provides a substitute for center of vorticity when $\sum_k \Gamma_k = 0$.

Theorem 1. *Let $\{z\}$ be vortices on the plane or on a cylinder whose vorticities sum to zero: $\sum \Gamma = 0$. Suppose the vortices are partitioned into two groups $\{z'\}$, $\{z''\}$ and within each group $\sum \Gamma' \neq 0$, $\sum \Gamma'' \neq 0$, so that the center of vorticity for each group is well-defined. Then the vector connecting the two centers of vorticity is a local first integral (fig. 2).*

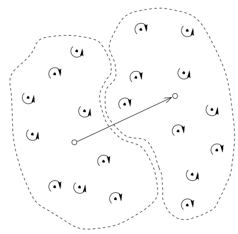

FIGURE 2

Proof. Since $\sum \Gamma' + \sum \Gamma'' = 0$, the vector in question is

$$\frac{\sum \Gamma' z'}{\sum \Gamma'} - \frac{\sum \Gamma'' z''}{\sum \Gamma''} = \frac{\sum \Gamma' z'}{\sum \Gamma'} + \frac{\sum \Gamma'' z''}{\sum \Gamma'} = \frac{\sum \Gamma z}{\sum \Gamma'},$$

and $\sum \Gamma z$ is a local first integral. □

Theorem 1 is serviceable in many problems. The simplest illustration is the motion of a *vortex pair* z_1, z_2 with vorticities $\Gamma, -\Gamma$ [16, photos 77, 78]. Treating z_1 as one group and z_2 as the other group, we check against Theorem 1 that $z_2 - z_1$ is constant during the motion. In fact, according to (2.4) the vortex pair on a cylinder forms a relative equilibrium moving with slope

$$-\sin\frac{x_2-x_1}{r}\Big/\sinh\frac{y_2-y_1}{r}.$$

When $x_2 - x_1 = 0$ or πr the pair moves horizontally: the corresponding configurations on the plane are the unstaggered or fully staggered cases of Kármán's vortex street, see also [4]. When z_1, z_2 are in general position, the corresponding vortex street on the plane translates at an angle to the horizontal, a case studied in [11]. The 'plane limit' $r \to \infty$ yields the angle of progression of a vortex pair on the plane $-(x_2 - x_1)/(y_2 - y_1)$. For a beautiful study of the stability of variants of vortex streets, see [5].

3. Relative equilibria

The first fact about relative equilibria of vortices on a cylinder is that there are not many of them.

Theorem 2. *Let $z_1, \dots, z_N$ be vortices with vorticities $\Gamma_1, \dots, \Gamma_N$ on a cylinder $\mathbb{C}/2\pi r\mathbb{Z}$. Suppose $\sum_k \Gamma_k \neq 0$. Then all relative equilibria are in fact equilibria. Moreover, if all Γ's have the same sign, then for each cyclic ordering there exists a unique (up to translation by $\mathbb{C}/2\pi r\mathbb{Z}$) equilibrium, and all the vortices are aligned on a single horizontal circle.*

Proof. If $z_1, \dots, z_N$ form a relative equilibrium, then all z's move with some common drift velocity v. The local first integral should not vary:

$$0 = \frac{d}{dt}\sum_k \Gamma_k z_k = v\sum_k \Gamma_k,$$

so $\sum_k \Gamma_k = 0$ or else $v = 0$.

If the vortices are not aligned on a single horizontal circle, pick a 'top vortex' (one with maximal y-coordinate) and a 'bottom vortex' (one with minimal y-coordinate). If all Γ's have the same sign, then by (2.4) the velocities of the top and bottom vortices must have x-components with opposite signs, so this position cannot constitute an equilibrium.

Now suppose all the vorticities are of the same sign. Fix a cyclic ordering of the vortices, and place the vortices in order on a single horizontal circle. The Hamiltonian is given by

$$H = -\frac{1}{4\pi}\sum_{k<l}\Gamma_k\Gamma_l \log\sin^2\left(\frac{x_k - x_l}{2r}\right).$$

One readily checks that this is a convex function of $x_1, \dots, x_N$: one first checks that wherever they are defined the second derivatives satisfy $\partial^2 H/\partial x_k \partial x_l < 0$ for $k \neq l$ and $\partial^2 H/\partial x_k^2 > 0$ and $\sum_l \partial^2 H/\partial x_k \partial x_l = 0$ for each k ; it then follows from a variant of Gershgorin's theorem (Lemma 1 below) that 0 is a simple eigenvalue of the Hessian of H and all other eigenvalues are strictly positive. Consequently on each connected component of the domain of definition there is a unique minimum and no other critical point, and different connected components correspond to different cyclic orderings. This is the same argument as for [9, Theorem 4.8]. □

Lemma 1. *Let $A = (a_{kl})$ be a symmetric $N \times N$ matrix satisfying $a_{kl} < 0$ for $k \neq l$, and $a_{kk} > 0$, $\sum_{l=1}^{N} a_{kl} = 0$ for each k. Then 0 is a simple eigenvalue of A and all other eigenvalues are strictly positive.*

Proof. Let $u = (u_1, \ldots, u_n)^T$ be an eigenvector of A with eigenvalue λ, normalized so that there is an index k for which $u_k = 1$ and $|u_l| \leqslant 1$ for all l. The kth row of the equation $Au = \lambda u$ is $a_{kk} + \sum_{l, l \neq k} a_{kl} u_l = \lambda$, which in view of the hypotheses on a_{kl} may be written $\sum_l |a_{kl}|(1 - u_l) = \lambda$. But $1 - u_l \geqslant 0$ and $|a_{kl}| > 0$ for each l; it follows that $\lambda \geqslant 0$ and $\lambda = 0$ if and only if all $u_l = 1$. On the other hand, $(1, \ldots, 1)^T$ is obviously an eigenvector with eigenvalue 0. □

If the vortices are placed on a single horizontal circle so that successive vorticities have alternating signs, then we also get the existence of an equilibrium, though the uniqueness problem is open as the function is no longer convex. In full generality, if the signs are neither the same nor alternating, the argument for existence fails as $H \to +\infty$ for some collisions and $\to -\infty$ for others.

Remark 2. For $N = 2$, if $\Gamma_1 + \Gamma_2 \neq 0$, we have generically a periodic orbit and exceptionally an equilibrium of antipodal vortices $z, z + \pi r$ or a separatrix connecting equilibria. For $N > 2$, if $\sum_k \Gamma_k \neq 0$ but Γ's have mixed signs, equilibria are less severely constrained. For example, for $N = 3$, let z_1, z_2 be vortices with vorticities $\Gamma_1, \Gamma_2 > 0$. To secure an equilibrium, the third vortex z_3 with vorticity $\Gamma_3 < 0$ must be placed at one of the 2 stagnation points of the velocity field induced by z_1, z_2, given in view of (2.2) as roots of

$$\Gamma_1 \operatorname{cotan} \frac{z - z_1}{2} + \Gamma_2 \operatorname{cotan} \frac{z - z_2}{2} = 0.$$

Having chosen z_3 as one of the roots and thereby immobilized z_3, adjust Γ_3 so as to immobilize z_1 :

$$\Gamma_2 \operatorname{cotan} \frac{z_1 - z_2}{2} + \Gamma_3 \operatorname{cotan} \frac{z_1 - z_3}{2} = 0.$$

Then z_2 too is automatically immobilized:

$$\Gamma_3 \operatorname{cotan} \frac{z_2 - z_3}{2} + \Gamma_1 \operatorname{cotan} \frac{z_2 - z_1}{2} = 0.$$

The upshot is that given any z_1, z_2 with vorticities of the same sign, we have 2 positions to place z_3 with the right vorticity of the opposite sign to secure an equilibrium. For example, vortices z_1, z_2 both of vorticity Γ such that $z_2 - z_1 = 2ib$ are immobilized by the adjunction of a vortex $(z_1 + z_2)/2$ of vorticity

$$\Gamma \left(\frac{1}{2} \operatorname{sech}^2 \frac{b}{2r} - 1 \right).$$

This is always less than $-\Gamma/2$ and in the plane limit $r \to \infty$ tends to the corresponding value in the planar theory $-\Gamma/2$. On the other hand, in the 'vortex sheet limit' $b \to \infty$ this tends to $-\Gamma$, also as it should. Similarly, vortices z_1, z_2 of vorticity Γ such that $z_2 - z_1 = 2a$ are immobilized by the adjunction of a vortex $(z_1 + z_2)/2$ of vorticity

$$\Gamma \left(\frac{1}{2} \sec^2 \frac{a}{2r} - 1 \right).$$

In the planar limit this tends again to $-\Gamma/2$. On the other hand, it is 0 when $a = \pi r/2$: z_1, z_2 are antipodal on the cylinder and are stationary already by themselves. When $a \to \pi r$, z_1, z_2 nearly meet at the back and a stronger and stronger vortex is required at the front to prevent them from moving.

Remark 3. Now suppose $\sum_k \Gamma_k = 0$. It was pointed out at the end of section 2 that a vortex pair $N = 2$ is always a relative equilibrium. For $N = 3$, Aref and Stremler [2] made a detailed study of relative equilibria; the patterns of some trajectories are surprisingly complicated. For $N > 3$ and N even, we have for any $a, b > 0$ a family of relative equilibria consisting of $n = N/2$ vortices with vorticity Γ at

$$ib, ib + \frac{2\pi r}{n}, \ldots, ib + (n-1)\frac{2\pi r}{n}, \tag{3.1}$$

and n vortices with vorticity $-\Gamma$ at

$$a - ib, a - ib + \frac{2\pi r}{n}, \ldots, a - ib + (n-1)\frac{2\pi r}{n}. \tag{3.2}$$

This is merely a crowded vortex street with spatial period $2\pi r/n$, or equivalently a single vortex pair on a thinner cylinder of radius r/n (see stability calculations in [3]). No essentially different family of relative equilibria seems to be known for $N > 3$.

Incidentally, even the trivial equivalence between 1 vortex on a cylinder of radius r and n horizontally equidistributed vortices on a cylinder of radius nr leads to amusing identities [1] : for example, equating the induced velocity fields and rescaling the variables in (2.2),

$$\frac{1}{n}\sum_{l=1}^{n} \operatorname{cotan}\frac{z + \pi l}{n} = \operatorname{cotan} z, \quad \forall z \in \mathbb{C}.$$

Remark 4. On the plane equilibria do not exist either when all Γ's are of the same sign (even the possibility of a horizontal circle is lost), and the non-existence of translational relative equilibria with nonzero drift when $\sum_k \Gamma_k \neq 0$ holds also on the plane and on a torus; the proof carries over verbatim from the cylindrical theorem. A torus, however, accommodates more varied families of equilibria: for example, $n_1 n_2$ vortices with identical vorticity Γ placed on a sub-lattice $(\pi/n_1)\mathbb{Z} + (\tau\pi/n_2)\mathbb{Z}$ form an equilibrium [15]. Many further patterns of equilibria may be designed on a torus with identical or alternating vortices.

4. Relative periodic orbits

Once a relative equilibrium of vortices is known, a frequently successful recipe for creating relative periodic orbits consists in *splitting* the vortices. Assume the vortices $z_1, \ldots, z_N$ with vorticities $\Gamma_1, \ldots, \Gamma_N$ form a relative equilibrium. Let us split each z_k into a cluster, near the original position of z_k, of n_k vortices $z_{k,1}, \ldots, z_{k,n_k}$ whose vorticities are of the same sign and sum to Γ_k. We expect the child vortices $z_{k,1}, \ldots, z_{k,n_k}$ to orbit around one another and remain a cluster, while seen from far away they still look like the original parent vortex z_k with vorticity Γ_k. It is reasonable to conjecture that for suitable intial configurations the child vortices form a relative periodic orbit, and for perhaps generic splittings they form a relative *quasi-periodic* orbit.

A vortex pair on a cylinder, which corresponds in the planar picture to Kármán's vortex street, is a relative equilibrium. In this section we shall create various relative periodic orbits by splitting a vortex pair; as a special case we recover the phenomenon classically known in the planar picture as leapfrogging. In Theorem 3 we split one of the vortices, while in Theorem 4 we split both.

The split is measured by a complex variable $\zeta = \xi + i\eta$ (or rather by 2ζ), and we are principally interested in small values of $|\zeta|$. In all the formulae the radius of the cylinder is normalized to $r = 1$; denormalization is a matter of dimensional analysis. Later in the section additional classes of relative periodic orbits are described.

Take a vortex pair at $c, -c$, where $c = a + ib \in \mathbb{C}$. We split it into 3 or 4 vortices as in fig. 3: the left diagram illustrates Theorem 3; the middle one Theorem 4, case $-b(1+\Gamma/\Gamma')/2 < \eta < b(1+\Gamma'/\Gamma)/2$; the right one case $b(1+\Gamma'/\Gamma)/2 < \eta$. Theorem 4, case $\eta < -b(1+\Gamma/\Gamma')/2$ is like the right diagram reflected laterally with Γ, Γ' interchanged.

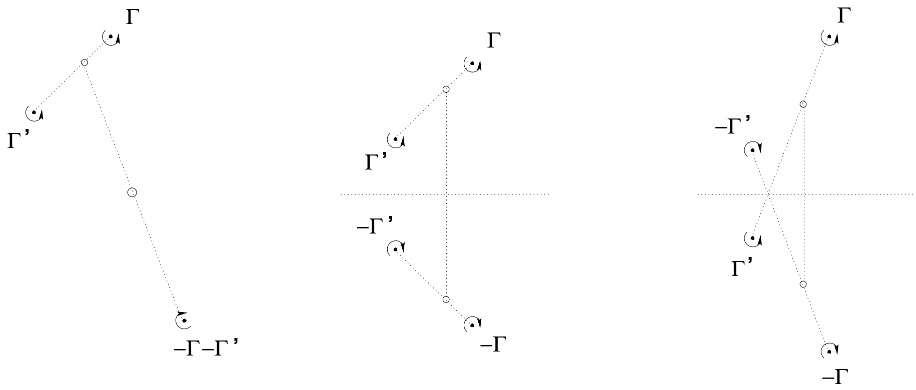

FIGURE 3

Theorem 3. *Let $c \in \mathbb{C} \setminus \{0\}$. On a cylinder, consider the configuration of 3 vortices with vorticities $\Gamma, \Gamma', -\Gamma-\Gamma'$ (Γ and Γ' being of the same sign) at*

$$c + \frac{2\Gamma'}{\Gamma+\Gamma'}\zeta, \quad c - \frac{2\Gamma}{\Gamma+\Gamma'}\zeta, \quad -c.$$

There exists an open punctured neighborhood of $\zeta = 0$ such that for every initial condition $\zeta(0) \neq 0$ in this neighborhood, these vortices form a relative periodic orbit. If $\Gamma/\Gamma' \in \mathbb{Q}$, then for a generic choice of $\zeta(0)$ (no restriction on its size) these vortices form a relative periodic orbit, and for isolated choices of $\zeta(0)$ they form a relative equilibrium or a separatrix connecting relative equilibria.

Combined with Proposition of section 2, Theorem 3 gives relative equilibria and relative periodic orbits of $N = 3n$ vortices for all $n \geqslant 1$. The result for $N = 3$ when $\Gamma/\Gamma' \in \mathbb{Q}$ is in [2], but we give a somewhat different proof. The relative periodicity for small $\zeta(0)$ is new.

The proof invokes the following elementary lemma.

Lemma 2. *Let H be a function with only nondegenerate critical points on a compact surface with p punctures such that $|H| \to \infty$ near each puncture. Then the generic level sets of H are disjoint unions of loops. If $p > 2$, then besides loops there exist isolated saddles and separatrices connecting the saddles.*

Proof. By rescaling the values of H and compactifying the punctures, we reduce to the situation where H is defined on a compact surface, takes values in $[-1, 1]$ and attains ± 1 at the points where the punctures used to be. The first part of the conclusion is immediate from Sard's theorem and the implicit function theorem. Moreover, from Morse theory

$$p - \#\text{saddles} \leqslant \#\text{max} + \#\text{min} - \#\text{saddles} = \text{Euler characteristic } \leqslant 2,$$

whence the second part of the conclusion. □

The idea now for the proof of Theorem 3 is to use symmetries and Theorem 1 to rewrite the Hamiltonian as a function on a punctured 2-dimensional sphere, satisfying the condition of divergence near the punctures. Applying Lemma 2 and recalling that a phase point in a Hamiltonian system moves along a level set of the Hamiltonian, we shall be home.

Proof of Theorem 3. The center of vorticity of the group Γ, Γ' is at c, that of the singleton group $-\Gamma - \Gamma'$ at $-c$. By Theorem 1, the vector connecting these centers is a local first integral. Hence passing to the quotient by translations, these centers may be assumed immobile. Within the group Γ, Γ', the position of one vortex determines the position of the other (it is at a definite ratio of distances across their center). Hence the trajectory of the vortex with vorticity Γ determines the trajectories of all 3 vortices up to translation, and the hamiltonian H may be regarded as a function of $\zeta = \xi + i\eta$ alone *as long as the trajectory of ζ lies on a single chart* . If the vortices Γ, Γ' are very close, they orbit like a binary star around their immobile center c within the chart, so that sooner or later $\arg\zeta$ increases by 2π. Since $H(\zeta) \to +\infty$ as $\zeta \to 0$, for large enough $E \in \mathbb{R}$ the connected component of $\{\zeta \in \mathbb{C}\diagdown 0 \mid |H(\zeta)| > E\}$ surrounding the singularity $\zeta = 0$ is topologically a punctured open disk, free of critical points of H. (The infimum of such E is the largest of the saddle values of H.) The level sets of H on this neighborhood are topologically circles, and so every ζ starting from $\zeta(0) \neq 0$ in this neighborhood returns to $\zeta(0)$, guaranteeing relative periodicity.

We must deal with the scenario where the trajectory of ζ does not lie on a single chart. Since $\Gamma/\Gamma' \in \mathbb{Q}$, the lowest common multiple L of $2, 1+\Gamma/\Gamma', 1+\Gamma'/\Gamma$ makes sense. To define ζ on the whole cylinder, we must swell the cylinder to $\mathbb{C}/L\pi\mathbb{Z}$. The swollen cylinder $\mathbb{C}/L\pi\mathbb{Z}$ covers the original cylinder $\mathbb{C}/2\pi\mathbb{Z}$ and H as a function of ζ lifts to a function on $\mathbb{C}/L\pi\mathbb{Z}\diagdown\{\text{singularities}\}$. The singularities represent the collisions between

$$\Gamma \sim \Gamma' \text{ (front and back)}, \quad \Gamma \sim -\Gamma - \Gamma', \quad \Gamma' \sim -\Gamma - \Gamma'$$

where $|H| \to \infty$; off the singularities, by (2.1),

$$e^{2\pi H/\Gamma\Gamma'} = \frac{\left|\sin\left(c + \dfrac{\zeta}{1+\Gamma/\Gamma'}\right)\right|^{1+\Gamma/\Gamma'} \left|\sin\left(c - \dfrac{\zeta}{1+\Gamma'/\Gamma}\right)\right|^{1+\Gamma'/\Gamma}}{|\sin\zeta|}. \tag{4.1}$$

Toward the 'ends' $\eta \to \pm\infty$, $|H| \to \infty$ as well. Topologically $\mathbb{C}/L\pi\mathbb{Z}\diagdown\{\text{singularities}\}$ is a sphere with at least 4 punctures. (4.1) shows that the critical points of H are all nondegenerate and $|H| \to \infty$ near each puncture. By Lemma 2, the generic level sets of H are loops, representing (putting horizontal translation back in) relative periodic orbits, and there exist values of ζ representing relative equilibria as well as separatrices (relative heteroclinic orbits) connecting relative equilibria. □

Remark 5. In Theorem 3, relative periodicity when $\Gamma/\Gamma' \notin \mathbb{Q}$ is spoilt only for $\zeta(0)$ too large. For such $\zeta(0)$, the orbit is relative quasi-periodic. Of course, even when $\Gamma/\Gamma' \notin \mathbb{Q}$ there are questions that can be settled within a chart. Thus, for 3 vortices with arbitrary vorticities that sum to zero, topological reasons imply the existence of a configuration that forms a relative equilibrium.

Theorem 4. *Let* $b \in \mathbb{R}\setminus\{0\}$. *On a cylinder, consider the configuration of* 4 *vortices with vorticities* $\Gamma, \Gamma', -\Gamma', -\Gamma$ *(*Γ *and* Γ' *being of the same sign) at*

$$ib + \frac{2\Gamma'}{\Gamma+\Gamma'}\zeta, \quad ib - \frac{2\Gamma}{\Gamma+\Gamma'}\zeta, \quad -ib - \frac{2\Gamma}{\Gamma+\Gamma'}\overline{\zeta}, \quad -ib + \frac{2\Gamma'}{\Gamma+\Gamma'}\overline{\zeta};$$

Let $\Gamma/\Gamma' \neq 1$. *Then for a generic choice of the initial condition* $\zeta(0)$ *these vortices form a relative periodic orbit, and for isolated choices of* $\zeta(0)$ *they form a relative equilibrium or a separatrix connecting relative equilibria. If* $\Gamma/\Gamma' = 1$, *the same conclusion holds for* $\zeta(0)$ *such that* $|\operatorname{Im}\zeta(0)| < b$ *or* $\pi H(\zeta(0))/\Gamma^2 < \log\sinh b$.

Combined with Proposition of section 2, Theorem 4 gives relative equilibria and relative periodic orbits of $N = 4n$ vortices for all $n \geqslant 1$.

Proof. As in the proof of Theorem 3, the positions of all 4 vortices are determined by those of the ones with vorticities Γ and $-\Gamma$. Thanks to a supplementary reflexive symmetry $z \mapsto \bar{z}$, the position of Γ determines that of $-\Gamma$. This time, after passing to the quotient by translations, H is a genuine function on the cylinder $\mathbb{C}/\pi\mathbb{Z}$ of $\zeta = \xi + i\eta$, $-\pi/2 < \xi \leqslant \pi/2$, with the singularities removed. Off the singularities, by (2.1),

$$e^{2\pi H/\Gamma\Gamma'} = \left| \frac{\sin\left(ib + \dfrac{\Gamma'\zeta + \Gamma\overline{\zeta}}{\Gamma+\Gamma'}\right)}{\sin\zeta} \right|^2 \left|\sin\left(ib + \frac{\zeta - \overline{\zeta}}{1+\Gamma/\Gamma'}\right)\right|^{\Gamma/\Gamma'} \left|\sin\left(ib - \frac{\zeta - \overline{\zeta}}{1+\Gamma'/\Gamma}\right)\right|^{\Gamma'/\Gamma}. \tag{4.2}$$

In particular, when $\Gamma/\Gamma' = 1$,

$$e^{2\pi H/\Gamma^2} = \frac{\sin^2\xi + \sinh^2 b}{\sin^2\xi + \sinh^2\eta}\,|\sinh(b+\eta)\sinh(b-\eta)|. \tag{4.3}$$

The isolated singularities represent simultaneous collisions between

$$\Gamma \sim \Gamma' \text{ and } -\Gamma' \sim -\Gamma$$

where $H \to +\infty$, and, if $\Gamma/\Gamma' \neq 1$, between

$$\Gamma \sim -\Gamma' \text{ and } \Gamma' \sim -\Gamma$$

where $H \to -\infty$. Toward the ends, $H \to +\infty$. There are also circles of singularities $\eta = -b(1+\Gamma/\Gamma')/2$, $b(1+\Gamma'/\Gamma)/2$ representing collisions between

$$\Gamma \sim -\Gamma, \quad \Gamma' \sim -\Gamma'$$

where $H \to -\infty$. Let us saw the cylinder $\mathbb{C}/\pi\mathbb{Z}$ of ζ into 3 trunks:

$$\begin{aligned} C_+ &= \{\zeta \,|\, b(1+\Gamma'/\Gamma)/2 < \eta\}, \\ C_0 &= \{\zeta \,|\, -b(1+\Gamma/\Gamma')/2 < \eta < b(1+\Gamma'/\Gamma)/2\}, \\ C_- &= \{\zeta \,|\, \eta < -b(1+\Gamma/\Gamma')/2\}. \end{aligned}$$

Topologically C_+, C_0, C_- are spheres with punctures. C_0 contains $\zeta = 0$, the simultaneous collisions between $\Gamma \sim \Gamma', -\Gamma' \sim -\Gamma$, so C_0 has at least 3 punctures and $|H| \to \infty$ near each puncture. Lemma 2 applies to C_0 and implies the existence of relative periodic orbits and relative equilibria.

For the moment, suppose $\Gamma/\Gamma' \neq 1$. ζ representing the simultaneous collisions between $\Gamma \sim -\Gamma', \Gamma' \sim -\Gamma$ is in C_+ or C_- accordingly as $\Gamma/\Gamma' > 1$ or < 1. If $\Gamma/\Gamma' > 1$, this puts on C_+ at least 3 punctures near each of which $|H| \to \infty$, so Lemma 2 applies and implies the existence of relative periodic orbits and relative equilibria; whereas C_- acquires only 2 punctures, so we can conclude the existence only of relative periodic orbits. If $\Gamma/\Gamma' < 1$, the rôles of C_+, C_- are reversed.

Note that as H is symmetric under the lateral reflection along $\xi = 0$ and along $\xi = \pi/2$, every point on either line where $\partial H/\partial\eta$ vanishes is critical. Let $\Gamma/\Gamma' > 1$ and work on C_+. The strip $0 < \xi < \pi/2$ is free of critical points, for here by (4.3) H is strictly monotone in ξ along any line $\eta =$ constant. Along $\xi = 0$, $H \to -\infty$ as $\eta \to b(1+\Gamma'/\Gamma)/2$ or $b(\Gamma+\Gamma')/(\Gamma-\Gamma')$, between which $\partial H/\partial\eta$ must vanish, signalling a saddle at say ζ_1. Along $\xi = \pi/2$, $H \to -\infty$ or $+\infty$ as $\eta \to b(1+\Gamma'/\Gamma)/2$ or $+\infty$. These bits of information, together with the fact that all critical points of H are nondegenerate, imply that $\partial H/\partial\eta$ vanishes twice along $\xi = \pi/2$, signalling a maximum at say ζ_2 and a saddle (which shall be left nameless). As a bonus we learn that 2 relative equilibria are represented in C_+, whereas a count of 3 singularities just predicts at least 1 relative equilibrium. The analysis works mutatis mutandis on C_- if $\Gamma/\Gamma' < 1$.

Finally, suppose $\Gamma/\Gamma' = 1$. Then the simultaneous collisions $\Gamma \sim -\Gamma', \Gamma' \sim -\Gamma$ as well as ζ_1, ζ_2 escape to the ends $\eta \to \pm\infty$, and toward the ends $2\pi H/\Gamma^2$ asymptotes to $\log(\sin^2\xi + \sinh^2 b)$, which remains bounded. Hence all the critical points in C_+, C_- disappear. Relative periodic orbits are represented by compact level sets of H, i.e. those that fill the region $e^{\pi H/\Gamma^2} < \sinh b$ of C_+, C_-; there is no relative equilibrium on these trunks. □

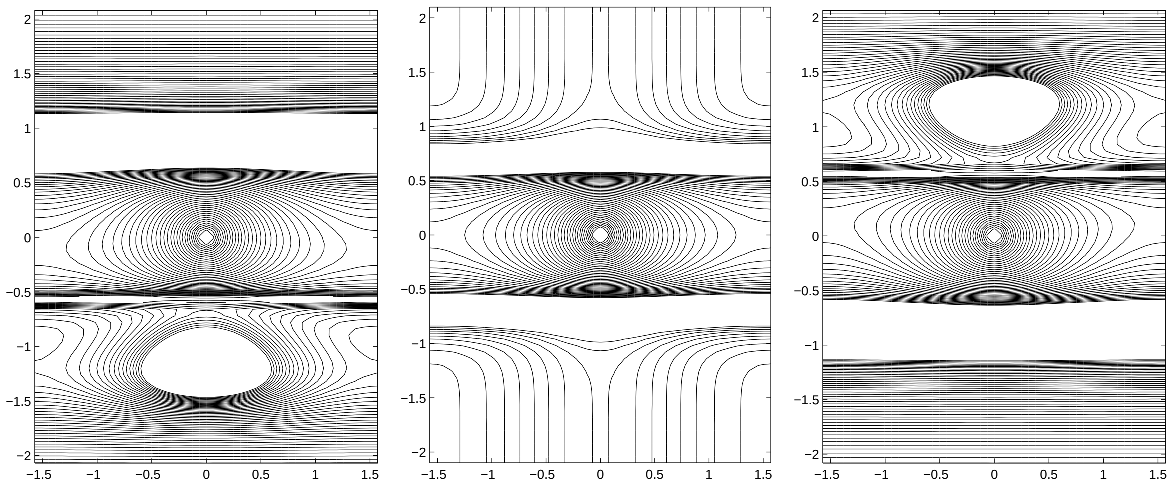

FIGURE 4

The plots of fig. 4 depict the level sets of H as a function of ζ for $\Gamma/\Gamma' < 1, = 1, > 1$ respectively; they were drawn at $b = 1$. By (4.2), the levels for $\Gamma/\Gamma' < 1$ and > 1 are mirror images of each other via $\zeta \mapsto \overline{\zeta}$. The blank holes and bands indicate where H diverges to $-\infty$ too steeply, while the diamond in the middle of each plot surrounds a peak $H \to +\infty$.

Take the $N = 4$ case as in Theorem 4 and initially align the 4 vortices vertically: $\xi(0) = 0$. If $\eta(0)$ is sufficiently small, the vortices of the group Γ, Γ' orbit like a binary counter-clockwise, the vortices of the group $-\Gamma', -\Gamma$ orbit like a binary clockwise, while the 2 groups progress together like a vortex pair. The superposition produces *leapfrogging*, a relative periodic orbit whose plane limit $r \to \infty$ is observed as the motion of a cross-section of consecutive vortex rings as they overtake each other. By adjusting the parameters Γ/Γ', b, $\zeta(0)$, we can render leapfrogging on a cylinder not only relative periodic but periodic. Alternatively, if $\eta(0)$ is sufficiently close to $b(1+\Gamma'/\Gamma)/2$ or to $-b(1+\Gamma/\Gamma')/2$, the vortices $\Gamma', -\Gamma'$ or $\Gamma, -\Gamma$ form a pair and rush off without leapfrogging. In the planar theory, in the case $\Gamma/\Gamma' = 1$, [10] calculated the critical value of $\eta(0)$ that separates the leapfrogging and non-leapfrogging régimes. In our setup this value may be obtained at once as follows.

In the situation of Theorem 4, denote by $\rho(b, \Gamma/\Gamma')$ the distance from the origin $\zeta = 0$ to the nearest separatrix. Then $\eta(0) = \rho(b, \Gamma/\Gamma) = \rho(b, 1)$. Denote by $\zeta_{\mathrm{re}} = \xi_{\mathrm{re}} + i\eta_{\mathrm{re}}$ a value of ζ at a saddle of $H(\zeta)$, representing a relative equilibrium. Inside the separatrices connecting the saddles we have leapfrogging; outside, not. $\rho = \rho(b, 1)$ is the ordinate at which a separatrix cuts the η-axis. Since the value of H is the same along the separatrices as on the saddles, $H(0, \rho) = H(\xi_{\mathrm{re}}, \eta_{\mathrm{re}})$. It is clear that a relative equilibrium occurs when 2 vortex pairs are antipodal: $\xi_{\mathrm{re}} = \pm\pi/2, \eta_{\mathrm{re}} = 0$. This fixes ρ in the cylindrical theory: $\sqrt{2}\tanh\rho = \tanh b$. Restoring r and taking the plane limit $r \to \infty$, we get in the planar theory $\rho = b/\sqrt{2}$, agreeing with [10, section 3], which arrived at $(b+\rho)/(b-\rho) = 3 + 2\sqrt{2}$.

When $\Gamma/\Gamma' \neq 1$, ζ_{re} and $\rho(b, \Gamma/\Gamma')$ are difficult to pin down in closed form. At any rate $\xi_{\mathrm{re}} = \pm\pi/2$; η_{re} is the unique root of

$$
\begin{aligned}
(\Gamma+\Gamma')\tanh\eta \quad &+ \quad (\Gamma-\Gamma')\tanh\left(b - \frac{\Gamma-\Gamma'}{\Gamma+\Gamma'}\eta\right) \\
&- \quad \Gamma\coth\left(b + \frac{2\eta}{1+\Gamma/\Gamma'}\right) + \Gamma'\coth\left(b - \frac{2\eta}{1+\Gamma'/\Gamma}\right) = 0
\end{aligned}
$$

which in view of (2.4) is the condition that the vertically aligned pairs $\Gamma, -\Gamma$ and $\Gamma', -\Gamma'$, antipodal to each other, move with the same velocity. If $\Gamma/\Gamma' = 1 + \varepsilon$, then up to 2nd order in ε,

$$
\eta_{\mathrm{re}} \simeq \tanh b\,\mathrm{sech}^2 b\left(\frac{\varepsilon}{2} - \left(1 + \frac{\mathrm{sech}^4 b}{2}\right)\frac{\varepsilon^2}{4}\right), \qquad \rho(b, 1+\varepsilon) = \rho(b, 1) - \frac{\tanh b\,\mathrm{sech}^2 b}{1+\cosh^2 b}\frac{\varepsilon^2}{4\sqrt{2}}.
$$

Remark 6. By an argument parallel to that of Theorem 4 we see that 4 vortices with vorticities $\Gamma, \Gamma', -\Gamma', -\Gamma$ at

$$
a + \frac{2\Gamma'}{\Gamma+\Gamma'}\zeta, \quad a - \frac{2\Gamma}{\Gamma+\Gamma'}\zeta, \quad -a + \frac{2\Gamma}{\Gamma+\Gamma'}\overline{\zeta}, \quad -a - \frac{2\Gamma'}{\Gamma+\Gamma'}\overline{\zeta}
$$

leapfrog as well (fig. 5, left diagram).
Unlike the $N = 4$ case of Theorem 4, however, the configuration on the right does not leapfrog.

Remark 7. Leapfrogging vortices and their generalizations analysed above owe their relative periodicity to the type of symmetry compatible with the local first integral of Theorem 1. Other

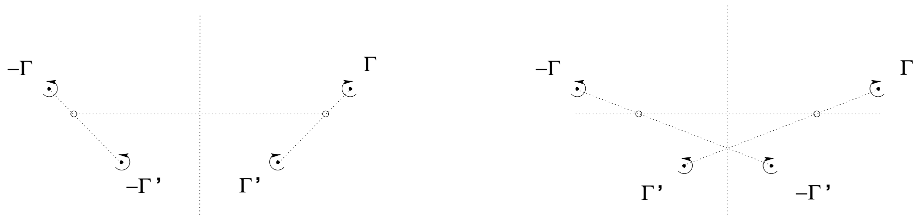

FIGURE 5

types of symmetry permit other types of relative periodic orbits. Thus, $2n$ vortices with identical vorticity Γ at (3.1), (3.2) form a relative periodic orbit [13, section 3.2].

Remark 8. Vortex streets and leapfrogging vortices can be adapted to a torus, where they form relative periodic orbits. A torus accommodates many further types of relative periodic orbits. For example on $\mathbb{C}/(\pi\mathbb{Z}+i\pi\mathbb{Z})$, by splitting each point of a sub-lattice into a rectangular quadruplet of vortices with vorticities $\Gamma, -\Gamma, \Gamma, -\Gamma$, we create a periodic orbit, the 'dancing vortices' of [15].

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