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2008

MIMS EPrint: **2006.107**

ISSN 1749-9097

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March 5, 2008

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1 Introduction

If R is a commutative noetherian ring then its Zariski spectrum - points, topology and structure sheaf - has a definition, originating with Matlis and Gabriel, which is purely in terms of its category Mod-R of modules. That definition applies to other categories: applied to Mod-R with R noncommutative and right coherent it gives a spectrum based on the indecomposable injective R-modules; applied to the functor category $(R\text{-mod}, \mathbf{Ab})$, where R is any ring, it gives the space referred to in the title of the paper. This latter space is the dual of the Ziegler spectrum, hence may be regarded as a topology on the indecomposable pure-injective (= algebraically compact) R-modules. At a fair level of generality our definition gives "the spectrum of finite type localisations of a locally coherent category". We present the definitions, including some background on finite type localisations, show that this does nicely generalise the usual Zariski spectrum, and then compute the result in a variety of specific contexts.

This paper is a rewritten and updated version of (part of) [34], with material from [39] added [1]. The more recent [42] can be seen as a sequel to this paper; it deals with, among other things, functors between definable categories and the morphisms of ringed spaces induced by these.

For assumed general background on modules and abelian categories see [50], [31]. The references [33] and [20] give some perspective on where this line of investigation arose. There is a fairly comprehensive account, [41], of the area, due to be published.

2 Finite type localisation

Let $\mathcal{C}$ be a locally coherent abelian category. For example: the category Mod-R of right modules over a right coherent ring; the category $(\text{mod-}R, \mathbf{Ab})$ of additive functors from the category, mod-R, of finitely presented R-modules to the category, $\mathbf{Ab}$, of abelian groups, where R is any ring. More generally, if $\mathcal{A}$ is a skeletally small preadditive category then the functor category $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ is locally coherent. Here $\mathcal{A}\text{-Mod} = (\mathcal{A}, \mathbf{Ab})$ denotes the category of covariant additive functors from $\mathcal{A}$ to $\mathbf{Ab}$ and $\mathcal{A}$-mod denotes the category of finitely presented functors. We write Mod-$\mathcal{A}$ for $(\mathcal{A}^{\text{op}}, \mathbf{Ab})$.

So $\mathcal{C}$ is abelian and is generated by its skeletally small subcategory, $\mathcal{C}^{\text{fp}}$, of finitely presented objects, and these objects are all coherent. Recall that an object $C \in \mathcal{C}$ is said to be **finitely presented** if the representable functor $(C, -) : \mathcal{C} \to \mathbf{Ab}$ commutes with direct limits ("directed colimits" in the more logical terminology) and a finitely presented object is **coherent** if every

[1] This paper, the first version of which, [38], was written ten years ago, has undergone a number of revisions. I expect not to make any further changes beyond annotation (for example if there are corrections). Given this history and the fact that most of the content will appear in the book [41], it is not now my intention to submit this to a journal for publication. This, therefore, is the version for reference.

finitely generated subobject is finitely presented. It follows, [8, 2.4], that $\mathcal{C}$ is Grothendieck, hence every object C of $\mathcal{C}$ has an injective hull, denoted $E(C)$, and the full subcategory $\mathcal{C}^{\mathrm{fp}}$ is an abelian subcategory, meaning that it is abelian and also that the inclusion of $\mathcal{C}^{\mathrm{fp}}$ into $\mathcal{C}$ is exact.

Let $\mathrm{Inj}(\mathcal{C})$ denote the class of injective objects of $\mathcal{C}$. A **hereditary torsion theory** on $\mathcal{C}$ is specified by a subclass, $\mathcal{E} \subseteq \mathrm{Inj}(\mathcal{C})$ which is closed under direct products and direct summands. We will usually drop the term "hereditary" since we do not consider any other kind. The corresponding class of **torsion** objects is $\mathcal{T} = \{C \in \mathcal{C} : (C, \mathcal{E}) = 0\}$, where $(C, \mathcal{E}) = 0$ is shorthand for $(C, E) = 0$ for every $E \in \mathcal{E}$. The class, $\mathcal{F}$, of **torsionfree** objects consists of those which embed in some member of $\mathcal{E}$. Our notation for a typical torsion theory is τ and this serves also to denote the endofunctor on $\mathcal{C}$ which takes an object C to its largest torsion subobject, τC.

There is a well-developed theory of localisation at hereditary torsion theories in Grothendieck categories, which may be found in [10], [31], [50] and which we will assume to be known to the reader, giving only a brief reminder here.

There is a functor, $Q_\tau : \mathcal{C} \to \mathcal{C}_\tau$, **localisation at** τ, which is universal for, i.e. initial among, the exact functors $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is Grothendieck abelian, F is exact and commutes with direct limits and $F\mathcal{T}_\tau = 0$ where $\mathcal{T}_\tau = \{C : \tau C = C\}$ is the torsion class corresponding to τ. The category $\mathcal{C}_\tau$ is referred to as the **localisation**, or **quotient category**, of $\mathcal{C}$ at τ. The right adjoint of Q_τ embeds $\mathcal{C}_\tau$ as a full subcategory of $\mathcal{C}$, and although the embedding does not in general preserve direct limits, it does in the case of a finite type torsion theory (see below) on a locally coherent category. Often we write C_τ instead of $Q_\tau C$, where $C \in \mathcal{C}$, especially when regarding this as an object $\mathcal{C}$ *via* the embedding $\mathcal{C}_\tau \subseteq \mathcal{C}$. The process of localisation does have more explicit descriptions and includes classical and Ore localisation, at least in their effect on modules.

A torsion theory τ on $\mathcal{C}$ is said to be **of finite type** if the functor τ commutes with direct limits, equivalently if the corresponding torsionfree class, $\mathcal{F}_\tau$, is closed under direct limits. If $\mathcal{G} \subseteq \mathcal{C}^{\mathrm{fp}}$ is generating in the sense that every object of $\mathcal{C}$ is an epimorphic image of a direct sum of copies of members of $\mathcal{G}$ then τ being of finite type is equivalent to each filter $\mathcal{U}_\tau(G)$, $G \in \mathcal{G}$, having a cofinal set of finitely generated objects. Here $\mathcal{U}_\tau(G)$ denotes the set of τ-**dense** subobjects of G: those $G' \leq G$ such that $G/G' \in \mathcal{T}_\tau$. Clearly this is a filter and the condition equivalent to finite type is that for each $G \in \mathcal{G}$ and each $G' \in \mathcal{U}_\tau(G)$ there is some finitely generated $G'' \in \mathcal{U}_\tau(G)$ with $G'' \leq G'$. This is the usual form of the definition of finite type in the case where $\mathcal{C} = \text{Mod-}R$ and where $\mathcal{G} = \{R\}$.

We need the following results, for which see, for instance, [21], [25] [41, Chpt. 12].

Theorem 2.1. *Let τ be a finite type torsion theory on a locally coherent abelian category $\mathcal{C}$. Then the localised category $\mathcal{C}_\tau$ is locally coherent and $(\mathcal{C}_\tau)^{\mathrm{fp}} = (\mathcal{C}^{\mathrm{fp}})_\tau$.*

Proposition 2.2. *Each finite type torsion theory τ on a locally coherent abelian category $\mathcal{C}$ is determined by the set of τ-torsionfree indecomposable injectives. In particular $\mathcal{F}_\tau$ consists of all objects which embed in a direct product of copies of these indecomposables.*

Note also that if τ is of finite type then the torsion class $\mathcal{T}_\tau$ is determined by the finitely presented objects in it, in the sense that $\mathcal{T}_\tau$ is the closure under direct limits of $\mathcal{T}_\tau \cap \mathcal{C}^{\mathrm{fp}}$.

If $\mathcal{E}$ is any set of injective objects then set

$$\mathrm{cog}(\mathcal{E}) = \{D \in \mathcal{C} : D \text{ embeds in a product of copies of objects of } \mathcal{E}\}$$

to be the torsionfree class **cogenerated** by $\mathcal{E}$.

A set X of indecomposable injectives which is closed under taking indecomposable direct summands of direct products will cogenerate a torsion theory $\tau = \mathrm{cog}(X)$ but this torsion theory might not be of finite type. Set $\mathcal{S} = \mathcal{T}_\tau \cap \mathcal{C}^{\mathrm{fp}}$ and $\overline{X} = \{E \in \mathrm{Inj}(\mathcal{C}) : E \text{ indecomposable}, (\mathcal{S}, E) = 0\}$; then $\overline{X} \supset X$ will be the set of indecomposable torsionfree injectives for a torsion theory of finite type, with torsion class the closure of $\mathcal{S}$ under direct limits - a possibly proper subclass of $\mathcal{T}_\tau$. The operation $X \mapsto \overline{X}$ is exactly closure in the Ziegler topology, discussed later.

3 The spectrum

Let $\mathcal{C}$ be locally coherent. Denote by $\mathrm{inj}(\mathcal{C})$ the set of isomorphism types of indecomposable injective objects of $\mathcal{C}$. We define a topology on $\mathrm{inj}(\mathcal{C})$ by declaring the sets

$$[A] = \{E \in \mathrm{inj}(\mathcal{C}) : (A, E) = 0\}$$

to be open, where A ranges over $\mathcal{C}^{\mathrm{fp}}$. Since $[A] \cap [B] = [A \oplus B]$ these do form a basis of open sets for a topology, which we denote $\mathrm{Zar}(\mathcal{C})$ and call the **Gabriel-Zariski topology on** $\mathrm{inj}(\mathcal{C})$.

It will be shown that this does generalise the Zariski spectrum of a commutative noetherian ring, in two senses. First, see Section 4.4, in the sense that it is a category-theoretic definition of the usual Zariski spectrum, applied in a more general context. Second, in the sense that the usual Zariski spectrum of a commutative noetherian ring sits nicely within the Gabriel-Zariski topology for an associated functor category, the "rep-Zariski spectrum" of that ring, see 3.12 and 4.10.

Lemma 3.1. *Suppose that $\mathcal{C}$ is locally coherent and that $\mathcal{B} \subseteq \mathcal{C}^{\mathrm{fp}}$ is such that every object of $\mathcal{C}^{\mathrm{fp}}$ has a finite composition series with factors in $\mathcal{B}$. Then the open sets $[B]$ with $B \in \mathcal{B}$ form a basis of the Gabriel-Zariski topology on* $\mathrm{inj}(\mathcal{C})$.

Proof. Indeed, if $A = A_n \geq A_{n-1} \geq \cdots \geq A_1 \geq A_0 = 0$ with each $B_{i+1} = A_{i+1}/A_i \in \mathcal{B}$ then $[A] = [B_n] \cup [B_{n-1}] \cup \ldots [B_1]$: for any morphism from some B_i to an injective E extends to a morphism from A to E. □

For example, if $\mathcal{C} = \text{Mod-}R$ for some right coherent ring R, then the sets of the form $[R/I]$ with I a finitely generated right ideal form a basis.

Let us consider the special case where $\mathcal{C}$ is a (generalised) module category. Let $\mathcal{A}$ be a small preadditive category and assume that $\mathcal{A}$ is right coherent, which we may take to mean that Mod-$\mathcal{A}$ is locally coherent or any of the usual equivalent characterisations lifted from rings to this level of generality ([30]). Write $\text{inj}_{\mathcal{A}} = \text{inj}(\text{Mod-}\mathcal{A})$ for the set of isomorphism classes of indecomposable injective $\mathcal{A}$-modules and refer to this set as the **Gabriel-Zariski spectrum of** $\mathcal{A}$ when it is equipped with the topology described above, denoting it GZspec($\mathcal{A}$). The terminology reflects Gabriel's replacement [10] of prime ideals by the corresponding indecomposable injective modules.

Example 3.2. Let $\mathcal{A}$ be the path category of the quiver A_∞ shown.

$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$

It is easily checked that the indecomposable injective objects of $\mathcal{C} = \mathcal{A}\text{-Mod}_k$, the category of representations of A_∞ in the category of k-vectorspaces, where k is any field, are as follows: the E_n where, using the representation-theoretic description, $E_n(i) = k$ if $i \leq n$, $= 0$ if $i > n$, and each arrow, if non-zero, is an isomorphism; E_∞, which is 1-dimensional at each vertex and each arrow is replaced by an isomorphism. Note that E_n is the injective hull of the simple representation S_n which is 0 everywhere except at n where it is 1-dimensional. This gives us the points of inj($\mathcal{C}$).

Dually, the indecomposable projective objects are the P_n ($n \geq 1$), where P_n is 1-dimensional at each vertex $m \geq n$ and 0 elsewhere (note that $P_1 = E_\infty$). Clearly the indecomposable representation $M_{[n,m]}$ which is 1-dimensional at i for $n \leq i \leq m$ and 0 elsewhere is finitely presented.

For any representation M one has $(M, E_n) = 0$ iff M does not have S_n as a subquotient, that is, iff $M(n) = 0$. Therefore each indecomposable injective E_n is an isolated point, isolated by $[S_1] \cap \cdots \cap [S_{n-1}] \cap [P_{n+1}]$, and a basis of open neighbourhoods of E_∞ consists of the cofinite sets which contain that point. For clearly the latter are open and, from the description of the P_n, it is easily seen that every infinite-dimensional finitely presented object is eventually > 0-dimensional, so there are no other open neighbourhoods of E_∞.

Therefore $\text{GZspec}(\mathcal{A}^{\text{op}}) = \text{Zar}(\mathcal{A}\text{-Mod}_k)$ is the one-point compactification, by E_∞, of the discrete set $\{E_n : n \geq 1\}$.

Let $\text{Mod}_k\text{-}\mathcal{A}$ denote the category of representations of the opposite quiver, which we regard as the same quiver but with arrows reversed. Now one has a finite-dimensional indecomposable projective P'_n for each n and, for each n, an indecomposable injective E'_n. The dimension vector of P'_n is as for E_n above and that of E'_n is as for P_n above.

It is similarly checked that the open sets (apart from $\emptyset$) of $\mathrm{GZspec}(\mathcal{A}) = \mathrm{Zar}(\mathrm{Mod}_k\text{-}\mathcal{A})$ are the cofinite sets.

An important special case is where $\mathcal{A}$ itself has the form $(R\text{-mod})^{\mathrm{op}}$ for some ring (or small preadditive category) R, in which case we will refer to its Gabriel-Zariski spectrum, that is $\mathrm{inj}(R\text{-mod}, \mathbf{Ab})$ with the Gabriel-Zariski topology, as the (right) **rep-Zariski spectrum** of R and denote it Zar_R.

Obviously there are choices there: R-mod or mod-R; $(R\text{-mod})^{\mathrm{op}}$ or R-mod. Why we have chosen to define the right rep-Zariski spectrum of R as the Gabriel-Zariski spectrum of $(R\text{-mod}, \mathbf{Ab})$, rather than that of one of the other three possible categories is explained later.

But first we must define another space.

3.1 The Ziegler spectrum

The space subsequently named the Ziegler spectrum was defined in Ziegler's paper [52] on the model theory of modules. The points of this space for a ring R are the isomorphism classes of indecomposable pure-injective right R-modules; we define these now.

An embedding $A \longrightarrow B$ of right R-modules is said to be a **pure embedding** if for every left R-module L the induced map $A \otimes_R L \longrightarrow B \otimes_R L$ of abelian groups is monic; it is enough to test with L being finitely presented. A right R-module N is **pure-injective** if it is injective over pure embeddings in Mod-R, equivalently if every pure embedding with domain N is split. These are also known as the **algebraically compact** modules, reflecting a very different way of arriving at them.

Let us denote the set of isomorphism types of indecomposable pure-injective right R-modules by pinj_R. This set is topologised, and the result is called the (right) **Ziegler spectrum**, Zg_R, of R, by taking, for a basis of open sets, the sets of the form

$$(f) = \{N \in \mathrm{pinj}_R : (f, N) : (B, N) \to (A, N) \text{ is not onto } \}$$

where $f : A \to B$ ranges over morphisms in mod-R.

Theorem 3.3. *[52, 4.9] The sets of the above form constitute a basis for a topology and these basic open sets all are compact*[2].

All this makes sense if we start with any skeletally small preadditive category $\mathcal{A}$ in place of R and the result is denoted $\mathrm{Zg}_{\mathcal{A}}$. The only significant difference is that if $\mathcal{A}$ has infinitely many objects then it might be that the whole space is not compact: in the case of a ring R the open set defined by the forgetful functor $(R_R, -)$, that is, by the morphism $R_R \to 0$, is the whole space which is, therefore, compact.

[2]What some would call quasicompact since no separation property is intended.

More generally still, if $\mathcal{C}$ is locally coherent, or even just **locally finitely presented**, meaning that $\mathcal{C}^{\text{fp}}$ is skeletally small and generating, then we define the **Ziegler spectrum** of $\mathcal{C}$, $\text{Zg}(\mathcal{C})$, to be the set, $\text{pinj}(\mathcal{C})$ of isomorphism types of indecomposable pure-injectives of $\mathcal{C}$ equipped with the topology which has, for a basis of open sets, the (f) defined as above but using $\mathcal{C}^{\text{fp}}$ in place of mod-R. Missing from this is a definition of pure-injectivity in $\mathcal{C}$: we can get this by embedding $\mathcal{C}$ as a full subcategory of Mod-$(\mathcal{C}^{\text{fp}})$ *via* $C \mapsto (-, C) \restriction \mathcal{C}^{\text{fp}}$ and use the definition in the latter category. Indeed, all of the usual equivalent definitions make sense and are still equivalent in this generality.

Consider the functor Mod-$R \to (R\text{-mod}, \mathbf{Ab})$ which takes a module M_R to the functor $M \otimes_R -$ on finitely presented left R-modules. It is a result of Gruson and Jensen [17] that this is a full embedding and that N_R is a pure-injective module iff $N \otimes_R -$ is an injective functor. Moreover, every injective functor is isomorphic to one of this form. Therefore there is a bijection between pinj_R and $\text{inj}\big(R\text{-mod}, \mathbf{Ab}\big)$. Thus the Ziegler spectrum of R may be regarded as a topology on the set of indecomposable injective functors in $(R\text{-mod}, \mathbf{Ab})$. The basis given above then takes the following form:

$$(F) = \{N \in \text{pinj}_R : (F, N \otimes -) \neq 0\}$$

where F ranges over $(R\text{-mod}, \mathbf{Ab})^{\text{fp}}$. For, from $f : A \to B$ in mod-R we get an exact sequence $(B, -) \to (A, -) \to F_f \to 0$ and the cokernel F_f is a typical finitely presented functor in $(\text{mod-}R, \mathbf{Ab})$. Then one has the formula $\overrightarrow{F_f} N \simeq (dF_f, N \otimes -)$ from 3.6 below where the duality d and extension $\overrightarrow{F_f}$ are as there.

Compare this with the definition of the rep-Zariski spectrum of R, which has basic open sets those of the form

$$[F] = \{N \in \text{pinj}_R : (F, N \otimes -) = 0\}$$

where $F \in (R\text{-mod}, \mathbf{Ab})^{\text{fp}}$.

Thus the Ziegler spectrum of R and the rep-Zarski spectrum of R can be regarded as topologies on the same underlying set and they are "opposite", informally, and dual in the following precise sense.

Hochster [22] defined a duality for spectral spaces. The spaces we are dealing with are not spectral, but no matter, we use the same definition and declare the complements of compact Ziegler-open sets to form a basis of open sets for a new, "dual-Ziegler", topology. The resulting topology is precisely the rep-Zariski topology. More generally, we have the following. We include the proof for the case of a ring at 3.12; the proof in the general case is essentially the same.

Theorem 3.4. *Suppose that $\mathcal{C}$ is locally coherent. Then the Gabriel-Zariski topology on* $\text{inj}(\mathcal{C})$ *coincides with the dual-Ziegler topology restricted to* $\text{inj}(\mathcal{C}) \subseteq \text{pinj}(\mathcal{C})$.

Because $\mathcal{C}$ is locally coherent $\mathrm{inj}(\mathcal{C})$ is a closed subset with respect to the Ziegler topology on $\mathrm{pinj}(\mathcal{C})$ (essentially [9]) so it makes no difference whether we move to the dual-Ziegler topology on the latter then restrict to $\mathrm{inj}(\mathcal{C})$, or restrict the Ziegler topology to $\mathrm{inj}(\mathcal{C})$ then apply the Hochester-dual process.

Corollary 3.5. *Suppose that $\mathcal{A}$ is a small preadditive category. Then the dual-Ziegler topology on* $\mathrm{pinj}_{\mathcal{A}} \leftrightarrow \mathrm{inj}(\mathcal{A}\text{-mod}, \mathbf{Ab})$ *coincides with the rep-Zariski topology.*

Thus we see why, in order to define the rep-Zariski spectrum for *right* R-modules, we moved to the category of functors on finitely presented *left* R-modules. In fact there is a duality, stated next, between the finitely presented functors in $(R\text{-mod}, \mathbf{Ab})$ and those in $(\text{mod-}R, \mathbf{Ab})$ so, despite the switch from right to left, we do stay close to the category of right modules.

Theorem 3.6. *([2], [18]) There is a duality* $d : (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}} \simeq ((R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}}$ *such that, if M is any right R-module and* $F \in (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ *then* $(dF, M \otimes -) \simeq \overrightarrow{F}M$*, where $\overrightarrow{F}$ denotes the unique extension of F to a functor from* Mod-R *to* $\mathbf{Ab}$ *which commutes with direct limits (sometimes we write just F for this extension).*

The definition of d on objects is $dF \cdot L = (F, - \otimes {}_R L)$. Regarding the extension $\overrightarrow{F}$; if $M = \varinjlim M_\lambda$ with the M_λ finitely presented then $\overrightarrow{F}M$ is (well-) defined to be $\varinjlim FM_\lambda$. If $(B, -) \longrightarrow (A, -) \longrightarrow F \longrightarrow 0$ with $A, B \in \text{mod-}R$ is a projective presentation of F then, interpreting the representable functors as functors on Mod-R, this can also be read as a projective presentation of this canonical extension of F.

It follows that an alternative form for the basic open sets of Zar_R is what we will still write as

$$[F] = \{N \in \mathrm{pinj}_R : \overrightarrow{F}N = 0\}$$

where now F ranges over the finitely presented functors in $(\text{mod-}R, \mathbf{Ab})$.

Also reflecting this duality there is, in general almost and in many cases actually, a homeomorphism between the right and left rep-Zariski spectra of a ring. What we mean by "almost a homeomorphism" is that the lattices (indeed complete Heyting algebras) of open sets are isomorphic. The next result, stated for a ring R, are also valid for a skeletally small preadditive category in place of R. It follows from the corresponding result, [20, Section 4], for the Ziegler spectrum. The notation ${}_R\mathrm{Zar}$ is used for $\mathrm{Zar}_{R^{\mathrm{op}}}$.

Theorem 3.7. *For any ring R there is a bijection between the open subsets of* Zar_R *and those of* ${}_R\mathrm{Zar}$ *which preserves containment, intersection and arbitrary union.*

We can describe this by saying that there is "a homeomorphism at the level of topology". More precisely, there is a homeomorphism of locales.

If R is countable, also under various conditions such as R having Krull-Gabriel dimension (see [20, 4.10]), this is induced by a homeomorphism $(\mathrm{Zar}_R/\sim) \simeq ({}_R\mathrm{Zar}/\sim)$ where $\sim$ denotes the equivalence relation on a topological space which identifies topologically indistinguishable points - points which belong to exactly the same open sets.

Corollary 3.8. *If R is countable and commutative, then duality induces a self-homeomorphism* $(\mathrm{Zar}_R/\sim) \simeq (\mathrm{Zar}_R/\sim)$.

This is the composition of the homeomorphism, $(\mathrm{Zar}_R/\sim) \simeq ({}_R\mathrm{Zar}/\sim)$, of 3.7 with that induced by right/left equivalence.

In the case of a commutative Dedekind domain R which, even if not countable, does have Krull-Gabriel dimension (equal to $= 2$ if R is not a field), the above self-homeomorphism of Zar_R, which is already equal to $\mathrm{Zar}_R/\sim$, fixes all finite length points, interchanges, for each prime P, the P-adic and P-Prüfer points, and fixes the generic point, that is, the quotient field of R. For these modules see Section 5.1.

Note that, just from the definition of the topologies, $N' \in \text{Zar-cl}(N)$ iff $N \in \text{Zg-cl}(N')$. We also mention that, although the rep-Zariski topology can be defined in terms of the Ziegler topology, the reverse is not true. This was shown in [6, 3.1] where one sees a homeomorphism of Zar_R for a certain finite-dimensional algebra R which is not a homeomorphism with respect to the Ziegler topology.

We reiterate the point that the definition of the Gabriel-Zariski topology can be applied at different levels. Given a ring R, it can be applied with $\mathcal{C} = \text{Mod-}R$, yielding a topology on the set inj_R of (isomorphism types of) indecomposable injective right R-modules. Or it can be applied to the functor category $(R\text{-mod}, \mathbf{Ab})$ and, as we have seen, this topology on the set of indecomposable injective functors may be viewed as a topology on the set of indecomposable pure-injective right R-modules.

In principle one may go arbitrarily further: the next stage would be to have a topology on the set of indecomposable pure-injective functors, and so on. But here at most three levels will concern us: a ring R, or small preadditive category; its module category Mod-R; 'the functor category' $(R\text{-mod}, \mathbf{Ab})$.

3.2 Embedding the Gabriel-Zariski spectrum

If R is any ring then, since any injective module certainly is pure-injective, $\mathrm{inj}_R \subseteq \mathrm{pinj}_R$. If R is right coherent then there is the Gabriel-Zariski topology on inj_R. There is also the rep-Zariski topology on pinj_R, hence an induced topology on inj_R. We prove that this is just the Gabriel-Zariski topology.

We will use the following, where $\mathrm{ann}_M(I) = \{a \in M : aI = 0\}$ for $I \subseteq R$.

Lemma 3.9. *Let $I \leq J$ be right ideals of a ring R and let E be an injective right R-module. Then* $\mathrm{ann}_E I/\mathrm{ann}_E J \simeq \mathrm{Hom}(J/I, E)$.

Proof. Just apply $(-, E)$ to the short exact sequence $0 \longrightarrow J/I \longrightarrow R/I \longrightarrow R/J \longrightarrow 0$ and note that $\mathrm{Hom}(R/I, E) \simeq \mathrm{ann}_E I$, similarly for J, and $\mathrm{Ext}^1(R/J, E) = 0$. □

If $F \leq ({}_RR^m, -)$ is finitely generated then, under the duality d of 3.6, this gives an epimorphism $(R^m_R, -) = d({}_RR^m, -) \to dF \to 0$. Denote by DF the, finitely presented, kernel of this epimorphism. Since, applying duality in the other direction, $F \simeq D^2F$, F also has the form DG for some finitely generated $G \leq (R^m_R, -)$.

Proposition 3.10. *[44, 1.1] If E is any injective right R-module and F is any finitely presented functor from R-mod to $\mathbf{Ab}$ which is a subfunctor of a power of the forgetful functor then $\overrightarrow{DF}(E) = \mathrm{ann}_E\big(F({}_RR)\big)$.*

If $F \leq ({}_RR^m, -)$, so $F({}_RR) \leq R^m$, then by $\mathrm{ann}_E(F({}_RR))$ we mean $\{(a_1, \ldots, a_m) \in E^m : \sum_{i=1}^m a_i r_i = 0$ for all $(r_1, \ldots, r_m) \in F({}_RR)\}$.

The statement at [44, 1.1] is, like various of our other references, said in terms of pp formulas. We give some explanation of that terminology in Section 7.

Theorem 3.11. *([53, 1.3], [48, Prop. 7]) A ring R is right coherent iff every right ideal of the form $F({}_RR)$ with F a finitely presented subfunctor of $({}_RR, -) \in (R\text{-mod}, \mathbf{Ab})$ is finitely generated.*

Now we prove 3.4 in this case.

Proposition 3.12. *Let R be right coherent. Then the Gabriel-Zariski topology on inj_R coincides with the topology induced from the rep-Zariski topology on pinj_R. That is, we may regard $\mathrm{GZspec}(R)$ as a subspace of Zar_R.*

Proof. One direction needs no assumption on R: given $A \in \text{mod-}R$, the basic Gabriel-Zariski-open set $[A] = \{E \in \mathrm{inj}_R : (A, E) = 0\}$ is just the intersection of the basic rep-Zariski-open set $[(A, -)]$ with $\mathrm{inj}_R \subseteq \mathrm{pinj}_R$.

For the other direction, given $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ we have, see just after 3.6, the basic rep-Zariski-open set $[F] = \{N \in \mathrm{pinj}_R : (F, N \otimes -) = 0\} = \{N \in \mathrm{pinj}_R : \overrightarrow{dF}(N) = 0\}$ by 3.6. Since every finitely presented functor is the quotient of two finitely generated subfunctors of some power of the forgetful functor we have $dF \simeq DG/DH$ for some finitely generated $G \leq H \leq ({}_RR^n, -)$ for some n. It follows from 3.10 that for $E \in \mathrm{inj}_R$, $\overrightarrow{dF}(E) = \mathrm{ann}_E I/\mathrm{ann}_E J$ for some right ideals I, J which, by 3.11 are finitely generated. Hence, by 3.9, $\{E \in \mathrm{inj}_R : \overrightarrow{dF}(E) = 0\} = [J/I]$. Since J/I is finitely presented this is a basic Gabriel-Zariski open set, as required. □

In general inj_R is neither an open nor closed subset of Zar_R; compare with the comment regarding the Ziegler topology after 3.4. It is enough to take $R = \mathbb{Z}$. To see that $\mathrm{inj}_{\mathbb{Z}}$ is not closed, just note that $\overline{\mathbb{Z}_p} \in \text{Zar-cl}(\mathbb{Q})$ according to the list after 5.2. To see that it is not open, again just check the list after 5.2, and note that it is not the case that every injective point has an open neighbourhood completely contained in $\mathrm{inj}_{\mathbb{Z}}$. The same will be true for any PI Dedekind domain with infinitely many primes.

4 Associated structure

4.1 The presheaf structure

Let $\mathcal{C}$ be a locally coherent abelian category. We define a presheaf of localisations over $\mathrm{Zar}(\mathcal{C})$. In fact, as for the usual definition of the structure sheaf of an affine algebraic variety, we define a presheaf just on a basis: that is enough data for the sheafification process. A difference is that, even on the basis, what we define here is a presheaf rather than a sheaf: one may have a basic open set $U = V \cup W$ where V, W are basic open and disjoint yet with the set of sections over U not equal to the product of that over V with that over W (see after 5.4).

Let $A \in \mathcal{C}^{\mathrm{fp}}$, so $[A] = \{E \in \mathrm{inj}(\mathcal{C}) : (A, E) = 0\}$ is basic open in $\mathrm{Zar}(\mathcal{C})$. Let $\tau_{[A]} = \mathrm{cog}([A])$ denote the torsion theory on $\mathcal{C}$ cogenerated by this set. This torsion theory is, since the torsion class is generated by the finitely presented object A, of finite type. If $B \in \mathcal{C}^{\mathrm{fp}}$ with $[B] \subseteq [A]$ then $\mathcal{T}_{[B]} \supseteq \mathcal{T}_{[A]}$ so the localisation $\mathcal{C} \to \mathcal{C}_{[B]}$ factors through $\mathcal{C} \to \mathcal{C}_{[A]}$. Thus we obtain the presheaf (on a basis) of finite-type localisations of $\mathcal{C}$.

This is a presheaf of large categories. By 2.1 there is induced a presheaf of localisations of $\mathcal{C}^{\mathrm{fp}}$ - a somewhat smaller object. Note that this is just the presheaf of quotients of $\mathcal{C}^{\mathrm{fp}}$ by its Serre subcategories.

Because functors between categories are often defined only up to natural equivalence there is the issue of whether we do have precise commutativity of restriction. That is, if $[A] \supseteq [B] \supseteq [C]$ then is localisation from $\mathcal{C}_{[A]}$ to $\mathcal{C}_{[B]}$ followed by localisation from $\mathcal{C}_{[B]}$ to $\mathcal{C}_{[C]}$ equal to, or only naturally equivalent to, localisation from $\mathcal{C}_{[A]}$ to $\mathcal{C}_{[C]}$? At least for current purposes, we can have exact commutativity, hence avoid introducing more sophisticated concepts, provided we use the definition of the localised category as being the original category on objects but with modified morphism sets as at [10, p. 365].

We will refer to this structure, regarded as a presheaf of localisations of $\mathcal{C}^{\mathrm{fp}}$, as the **finite type presheaf** on $\mathcal{C}$, denoting it by $\mathrm{FT}(\mathcal{C})$ and its sheafification by $\mathrm{LFT}(\mathcal{C})$. If $\mathcal{C}$ has the form Mod-$\mathcal{A}$ for some skeletally small preadditive category $\mathcal{A}$ then we will write $(\mathrm{L})\mathrm{FT}_{\mathcal{A}}$ for the subpresheaf which has for its values, not the finite-type localisations of $\mathcal{C}^{\mathrm{fp}}$ but the image, induced by the Yoneda-embedding of $\mathcal{A}$ in Mod-$\mathcal{A}$, of $\mathcal{A}$ in these. In particular if $\mathcal{A}$ is a ring R then the larger presheaf has, for its values, various localisations of mod-R and the smaller has rings, various localisations of R, for its values. We will see, 4.7, that if R is commutative noetherian then $\mathrm{FT}_R = \mathcal{O}_{\mathrm{Spec}(R)}$, the usual structure sheaf.

In the case where $\mathcal{C} = (R\text{-mod}, \mathbf{Ab})$, so the space is the rep-Zariski spectrum, Zar_R, of a ring R, there is another interpretation of this presheaf, in terms of definable scalars with respect to definable subcategories.

4.2 Definable subcategories and rings of definable scalars

Let $\mathcal{A}$ be a skeletally small preadditive category. A subcategory of Mod-$\mathcal{A}$ is said to be **definable** if it is closed in Mod-$\mathcal{A}$ under products, pure submodules and direct limits. The terminology "definable" reflects the model-theoretic origin of interest in these.

If $F \in (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ then those $M \in \text{Mod-}\mathcal{A}$ such that $(F, M \otimes -) = 0$ are easily seen to form a definable subcategory. More generally, for any $\mathcal{H} \subseteq (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ those M such that $(H, M \otimes -) = 0$ for every $H \in \mathcal{H}$ form a definable subcategory of Mod-$\mathcal{A}$. In fact, this relation gives a bijection between Serre subcategories of $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ and definable subcategories of Mod-$\mathcal{A}$. Equally there is a bijection with closed subsets, X, of $\mathrm{Zg}_{\mathcal{A}}$. Namely to X associate the subcategory of Mod-$\mathcal{A}$ consisting of all those right $\mathcal{A}$-modules M such that $M \otimes -$ is cogenerated by the set of $N \otimes -$ with $N \in X$. To get from a definable subcategory of Mod-$\mathcal{A}$ to a closed subset of $\mathrm{Zg}_{\mathcal{A}}$, just intersect with $\mathrm{Zg}_{\mathcal{A}}$ (4.2).

Theorem 4.1. *For any small preadditive category $\mathcal{A}$ there are natural bijections between:*
the definable subcategories of Mod-$\mathcal{A}$*;*
Serre subcategories of $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$*;*
closed subsets of $\mathrm{Zg}_{\mathcal{A}}$.

By 3.6 these are also in natural bijection with the Serre subcategories of $(\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$ and hence also with definable subcategories of $\mathcal{A}$-Mod and closed subsets of ${}_{\mathcal{A}}\mathrm{Zg}$.

Theorem 4.2. *[52, 4.10] A subset of* $\mathrm{Zg}_{\mathcal{A}}$ *is closed in the Ziegler topology iff it has the form $\mathcal{X} \cap \mathrm{Zg}_{\mathcal{A}}$ for some definable subcategory $\mathcal{X}$ of* Mod-$\mathcal{A}$.
In this case $\mathcal{X}$ is the definable subcategory generated by the modules in X, that is, the closure of X under products, direct limits and pure submodules.

To every module $M_{\mathcal{A}}$ is associated a closed subset, $\mathrm{supp}(M)$, of $\mathrm{Zg}_{\mathcal{A}}$, called the **support** of M and defined by $\mathrm{supp}(M) = \{N \in \mathrm{Zg}_{\mathcal{A}} : \mathrm{Hom}(F, N \otimes -) = 0 \text{ for all } F \in (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \text{ such that } \mathrm{Hom}(F, M \otimes -) = 0\}$. Equivalently, $N \in \mathrm{pinj}_{\mathcal{A}}$ is in $\mathrm{supp}(M)$ iff N is in the definable subcategory of Mod-$\mathcal{A}$ generated by M.

Let us now suppose, mainly for ease of statement, that $\mathcal{A}$ is a ring R. So we have the distinguished object $({}_RR, -)$, the forgetful functor, in $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. If τ is any finite type torsion theory on $(R\text{-mod}, \mathbf{Ab})$ then we may consider the endomorphism ring, $\mathrm{End}\big(({}_RR, -)_\tau\big)$, of the localisation of this functor at τ.

Note that, since $\mathrm{End}({}_RR,-) \simeq R^{\mathrm{op}}$ (if the action is written on the left), there is a natural ring morphism $R^{\mathrm{op}} \to \mathrm{End}\big(({}_RR,-)_\tau\big)$ induced by the localisation functor. If τ corresponds to the definable subcategory $\mathcal{X}$ of Mod-R then write $R_{\mathcal{X}}$ for the opposite of $\mathrm{End}\big(({}_RR,-)_\tau\big)$ and call it the **ring of definable scalars** of $\mathcal{X}$. Equivalently, in view of 3.6, and perhaps more naturally, we can define $R_{\mathcal{X}}$ to be the endomorphism ring of the localisation of $(R_R,-)$ at the finite type torsion theory on (mod-R, **Ab**) dual to τ. If M is a right R-module which generates the definable subcategory $\mathcal{X}$, equivalently if $\mathrm{supp}(M) = X$, then we write R_M and call this the **ring of definable scalars** of M.

The terminology, which comes from model theory, is explained as follows. To each definable subcategory $\mathcal{X}$ of Mod-R there corresponds the ring, under pointwise addition and composition, of all those additive relations which are definable in the natural first-order language of R-modules and which well-define a function on each member of $\mathcal{X}$ (equivalently, on each member of X or, indeed, on any generating set). Of course the action of each element $r \in R$, the multiplication-by-r map, is definable in this sense so there is a natural morphism of rings $R \to R_{\mathcal{X}}$. This ring is exactly the ring of definable scalars of $\mathcal{X}$.

This does also make sense with a small preadditive category $\mathcal{A}$ in place of the ring R, *via* the Yoneda embedding of $\mathcal{A}$ into its module category and the further Yoneda embedding into the functor category.

Actually, the whole localised category of finitely presented functors has a model-theoretic interpretation as the category of "pp-imaginaries" of $\mathcal{X}$, see [41].

Thus the ring of definable scalars associated to a definable subcategory/an R-module is a kind of localisation of R, but one representation level up: it is a localisation of the corresponding functor, rather than of the ring itself. The resulting "localisations" do include classical localisations of R, see 4.10.

We may also obtain this ring as the biendomorphism ring of a suitably large module.

Theorem 4.3. *([34, 4.3], see [5]) If X is a closed subset of Zg_R then the corresponding ring, R_X, of definable scalars is the biendomorphism ring of any pure-injective module N which satisfies the conditions:* $\mathrm{supp}(N) = X$*; $(N \otimes -)$ cogenerates the corresponding finite type torsion theory on* (R-mod, **Ab**)*; N is cyclic over its endomorphism ring.*

If M is any module then every definable scalar of M is a biendomorphism. If M is **of finite endolength**, that is, of finite length over $\mathrm{End}(M)$, then the rings are equal.

Proposition 4.4. *([34, A1.5], [5, 3.6]) If M is a module of finite endolength then its ring of definable scalars coincides with its biendomorphism ring.*

By way of contrast, the biendomorphism ring of the Prüfer group $\mathbb{Z}_{p^\infty}$ is the ring, $\overline{\mathbb{Z}_{(p)}}$, of p-adic integers whereas the ring of definable scalars of $\mathbb{Z}_{p^\infty}$ is just the localisation $\mathbb{Z}_{(p)}$. The same goes for $\overline{\mathbb{Z}_{(p)}}$ regarded as a $\mathbb{Z}_{(p)}$-module, though, if $\overline{\mathbb{Z}_{(p)}}$ is regarded as a $\overline{\mathbb{Z}_{(p)}}$-module, then the two rings will coincide (more actions are (finitely) definable over $\overline{\mathbb{Z}_{(p)}}$ than over $\mathbb{Z}_{(p)}$).

Theorem 4.5. *([35], [34, A4.4] for the last statement) If $f : R \longrightarrow S$ is an epimorphism of rings then* Mod-S*, regarded as a subcategory of* Mod-R *via f, is definable. Also* Zar_S *may be identified with the subset* Mod-$S \cap \mathrm{Zar}_R$ *of* Zar_R*, and the ring of definable scalars of this Ziegler-closed closed subset is precisely S, regarded as an R-algebra via f. Moreover, if S is regarded as an R-module via f then the ring of definable scalars of S_R is exactly S.*

4.3 The sheaf of locally definable scalars

Recall that each basic open set $X = [F]$ of Zar_R is Ziegler-closed. So we have a presheaf on this basis which assigns to $[F]$ the corresponding ring of definable scalars, $R_{[F]}$. We call this the **presheaf of definable scalars** of R and denote it Def_R. It follows direct from the definitions that it is a separated presheaf, hence embeds in its sheafification, which we call the **sheaf of locally definable scalars** and denote LDef_R. Observe that this is just a small part of the presheaf, FT(mod-$\mathcal{A}$, $\mathbf{Ab}$), of small categories, namely the thread given by the localisations of the forgetful functor.

As remarked already, even in 'nice' cases, Def_R will be a presheaf, rather than a sheaf, though, in many examples, significant parts of it will have the gluing property.

4.4 The Zariski spectrum through representations

Recall the definition of the **Zariski spectrum**, $\mathrm{Spec}(R)$, of a commutative ring R. The points are the prime ideals of R and a basis of open sets for the topology is given by the sets $D(r) = \{P \in \mathrm{Spec}(R) : r \notin P\}$ for $r \in R$.

Assuming now that R is commutative noetherian, we recall how the spectrum may be defined ([10]) purely in terms of the category, Mod-R, of R-modules.

Each point $P \in \mathrm{Spec}(R)$ is replaced by the injective hull, $E_P = E(R/P)$, of the corresponding quotient module R/P. Because P is prime E_P is indecomposable and every indecomposable injective is isomorphic to one of this form. So, for R commutative noetherian, we have a natural bijection $\mathrm{Spec}(R) \leftrightarrow \mathrm{inj}_R$.

As for the topology, under this bijection $D(r)$ corresponds to

$$[R/rR] = \{E \in \mathrm{inj}_R : \mathrm{Hom}(R/rR, E) = 0\}.$$

For, if $r \in R \setminus P$ and if $f : R/rR \longrightarrow E_P$ then $\mathrm{ann}_R(f(1 + rR)) \geq rR$ and so, since P is the *unique* maximal annihilator, $\mathrm{ass}(E_P)$, of non-zero elements of E_P (see the proof of 6.2), it must be that $f(1 + rR) = 0$ hence $f = 0$. For the converse, if $r \in P$ then the canonical surjection from R/rR to R/P followed by inclusion is a non-zero morphism from R/rR to E_P. It follows by 3.1 that this is just the Gabriel-Zariski topology which we defined earlier.

Theorem 4.6. *Let R be a commutative noetherian ring. Then* $\mathrm{Zar}(\text{Mod-}R)$ *is naturally homeomorphic to* $\mathrm{Spec}(R)$. *The bijection is given by $P \mapsto E(R/P)$ and $E \mapsto \mathrm{ass}(E)$.*

The above statement rather begs the question of whether it applies to arbitrary commutative rings and we do consider these in Section 6, showing that one obtains almost the same result if R is commutative and coherent.

This result also holds for FBN rings. The main difference between this and the commutative case is that if P is a prime ideal then $E(R/P)$ need not be indecomposable. It will, however, be a direct sum of finitely many copies of a unique indecomposable injective and this is enough.

Next we check that our presheaf of definable scalars, if restricted to inj_R when R is commutative noetherian, is isomorphic as a ringed space to the usual structure sheaf $\mathcal{O}_X$, $X = \mathrm{Spec}(R)$.

That structure sheaf associates to a basic open set $D(r)$ (r not nilpotent) of $\mathrm{Spec}(R)$ the ring $\mathcal{O}_X D(r) = R[r^{-1}]$ - the localisation of R obtained by inverting $r \in R$ - and the restriction maps are just the canonical localisation maps. This presheaf defined on a basis extends to a sheaf on the space, see e.g. [51, 4.2.6], [19, Section II.2], indeed a sheaf of local rings, with the stalk at a prime P being the localisation, $R_{(P)}$, of R at P.

Recall that, given any presheaf F on a topological space T and given any point $t \in T$ the **stalk** of F at t is defined to be $F_t = \varinjlim\{F(U) : t \in U \text{ and } U \text{ is open}\}$. If $\mathcal{U}_0$ is a basis for the topology then clearly $F_t = \varinjlim\{F(U) : t \in U \in \mathcal{U}_0\}$ so it is enough to know, or to have defined, the presheaf on a basis of open sets.

Given $r \in R$, consider the torsion theory on Mod-R cogenerated by the set $[R/rR]$ of indecomposable injectives. Since every hereditary torsion theory over a (right) noetherian ring is of finite type this is the torsion theory previously denoted by $\tau_{[R/rR]}$ and it is easy to check that the localisation of R at this torsion theory is precisely $R[r^{-1}]$, as required.

Proposition 4.7. *If R is commutative noetherian then the torsion-theoretic presheaf,* FT_R *defined earlier coincides with the usual structure (pre)sheaf* $\mathcal{O}_{\mathrm{Spec}(R)}$. *More precisely, the bijection $P \mapsto E_P$ induces an isomorphism of ringed spaces* $\big(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}\big) \simeq \big(\mathrm{GZSpec}(R), \mathrm{FT}_R\big)$.

Proposition 4.8. *([34, C1.1], see [37, 3.1]) Let $\mathcal{A}$ be a small preadditive category and let $E \in \mathrm{inj}_{\mathcal{A}}$. Then the stalk of the presheaf,* $\mathrm{FT}_{\mathcal{A}}$ *at E is the localisation of $\mathcal{A}$ at the torsion theory of finite type corresponding to E. Here we identify $\mathcal{A}$ with its image in* Mod-$\mathcal{A}$ *under the Yoneda embedding $A \mapsto (-, A)$.*

In particular, taking $\mathcal{A} = R$-mod where R is a ring, if $N \in \mathrm{Zar}_R$ then the stalk of the presheaf, Def_R, *of definable scalars at N is the ring of definable scalars at N:* $(\mathrm{Def}_R)_N = R_N$.

It is the case, see [52, 5.4] or [33, 2.$\mathbb{Z}$.8], that if N is any indecomposable pure-injective then the multiplications by elements of the centre, $C(R)$, of R, which are not automorphisms of N form a prime ideal of $C(R)$. So there is a prime ideal, $P = P(N)$, of $C(R)$ such that N is a module over the localisation of R obtained by inverting the elements of $C(R) \setminus P$. The same applies to the stalk R_N of the presheaf of definable scalars at N, so there is the following result.

Theorem 4.9. *([34, D1.1], see [37, 6.1]) The centre of the presheaf of definable scalars is a presheaf of commutative local rings.*

The following result is [34, A1.7] and is also given a proof avoiding model theory (but still using pp conditions) in [41]

Theorem 4.10. *Let R be any ring and let τ be a torsion theory of finite type on* Mod-R. *Set $\mathcal{E} = \mathcal{F}_\tau \cap \mathrm{inj}_R \subseteq \mathrm{Zar}_R$ to be the corresponding set of indecomposable torsionfree injectives. Then the torsion-theoretic localisation, R_τ, of R at τ coincides, as an R-algebra, with the ring of definable scalars, $R_{\mathcal{E}}$, of $\mathcal{E}$.*

This, together with the comments at the beginning of the section, shows that if R is commutative noetherian then the embedding of inj_R into pinj_R discussed in Section 3.2 extends to an embedding of the usual structure sheaf into the presheaf of definable scalars.

4.5 Lattices

We may order torsion theories by $\tau \leq \mu$ iff $\mathcal{T}_\tau \subseteq \mathcal{T}_\mu$. Let $\mathrm{FTT}(\mathcal{C})$ denote the set of finite type torsion theories on $\mathcal{C}$ with this ordering.

Remark 4.11. If $\mathcal{C}$ is locally coherent then $\mathrm{FTT}(\mathcal{C})$ is a complete lattice.

The intersection of a set of Serre subcategories is Serre so this is clear since, for any finite type torsion theory τ, the class $\mathcal{T}_\tau$ is determined by the Serre subcategory $\mathcal{T}_\tau \cap \mathcal{C}^{\mathrm{fp}}$ of $\mathcal{C}^{\mathrm{fp}}$. In terms of cogenerating sets of injectives; given a set $\{\tau_i\}_i$ of finite type torsion theories, let $\mathcal{E}_i$ denote the corresponding sets of torsionfree indecomposable injectives; then the torsion theory cogenerated by $\bigcap_i \mathcal{E}_\lambda$ is the supremum of the τ_i. The infimum of this set is the torsion theory cogenerated by the closure of $\bigcup_i \mathcal{E}_i$ under taking indecomposable summands of direct products. It is easy to find examples which show that this closure operation on sets of indecomposable injectives may be non-trivial. For instance take $\tau_i = \mathrm{cog}(N_i \otimes -)$ where the N_i form an infinite set of non-isomorphic indecomposable finite-dimensional modules over a finite-dimensional algebra R. Here the $N_i \otimes -$ are regarded as, indecomposable, injective, objects of the functor category $\mathcal{C} = (R\text{-Mod}, \mathbf{Ab})$. The torsion theory that they generate must contain at least one indecomposable injective of the form $N \otimes -$ where N is an infinite-dimensional indecomposable module. This follows since this is an infinite set of isolated points of the Ziegler spectrum, which is compact (see [33]).

If $A \in \mathcal{C}^{\mathrm{fp}}$ then it is clear that A belongs to the torsion class generated by a union $\bigcup_i \mathcal{T}_i$ iff it belongs to the torsion class generated by finitely many of the $\mathcal{T}_i$. Therefore the torsion theory $\mathrm{gen}(A)$, which has torsion class the closure under direct limits of the Serre subcategory generated by A, is a compact element of the lattice $\mathrm{FTT}(\mathcal{C})$. If $\mathcal{T}$ is a finite type torsion class $\mathcal{T} = \bigvee_{A\in\mathcal{T}} \mathrm{gen}(A)$ so the converse follows. That is, the ($\bigvee$-)compact elements of $\mathrm{FTT}(\mathcal{C})$ are the $\mathrm{gen}(A)$ with $A \in \mathcal{C}^{\mathrm{fp}}$. If we equip $\mathrm{FTT}(\mathcal{C})$ with the topology which has, as basic open sets, the intervals $[\tau, 1]$, where τ is compact and where 1 is the torsion theory with torsion class $\mathcal{C}$, then the resulting locale is isomorphic to that of the Gabriel-Zariski topology defined above. Note ([15]) that FTT is, with meet for the multiplication, an ideal lattice in the sense of [4].

5 Examples

We describe the rep-Zariski spectrum and the presheaf of definable scalars for various rings.

5.1 The rep-Zariski spectrum of a PI Dedekind domain

The class of Dedekind domains includes both the ring of integers $\mathbb{Z}$ and the archetypal tame algebra $k[X]$ where k is a field. What we say here extends with essentially no extra work to those non-commutative Dedekind domains which satisfy a polynomial identity, so we work in that generality.

First we need the points of the space: the indecomposable pure-injective modules. For $R = \mathbb{Z}$ this goes back to Kaplansky [23] and the general case is much the same (see, [52], [27], [36]).

As said already, if N is an indecomposable pure-injective R-module then the elements of the centre, $C(R)$, of R which do not act as automorphisms of N form a prime ideal $P = P(N)$, so N is a module over the corresponding localisation of R. This allows 5.2 below to be proved by reducing to the 'local' case, since there is, for such a ring, see [29, 13.7.9], a bijection between the (prime) ideals of the centre and the (prime) ideals of the ring.

Remark 5.1. Let R be a PI Dedekind domain, with centre $C(R)$. The bijection $P \mapsto P \cap C(R)$ between $\mathrm{Spec}(R)$ and $\mathrm{Spec}(C(R))$ induces a homeomorphism $\mathrm{Zar}_R \simeq \mathrm{Zar}_{C(R)}$ by the comment after 4.6 and since PI implies FBN.

Theorem 5.2. *([27], [36, 1.6]) Let R be a PI Dedekind domain. The points of Zar_R are the following where in each case P ranges over the non-zero, thus maximal, primes of R:*
- *the indecomposable modules, R/P^n, of finite length, for $n \geq 1$;*
- *the completion, $\overline{R}_P = \varprojlim_n R/P^n$, of R in the P-adic topology; we call these* **adic** *modules;*
- *the* **Prüfer** *modules $R_{P^\infty} = E(R/P)$;*
- *the quotient division ring, $Q = Q(R)$, of R.*

Since any Dedekind prime ring is Morita equivalent to a Dedekind domain ([29, 5.2.12]) this and what we say below apply equally well to such rings, since everything involved is Morita-invariant.

Now we describe the topology on Zar_R by giving a basis of open neighbourhoods at each point. Of course, to do this we need to know something about the finitely presented functors. This information is in the literature, though in most sources it is expressed in terms of 'pp-formulas' (we explain what these are in Section 7).

Over these rings all finitely presented functors in $(\text{mod-}R, \mathbf{Ab})$, and their extensions $\overrightarrow{F}$ as in 3.6, are, in a sense which may be made precise (e.g. see [33, 2.$\mathbb{Z}$.1] or [45, Section 2.2]), built up from annihilator and divisibility conditions. It is easy to check that, for each element $r \in R$, both the functors given on objects by $M \mapsto Mr$ and $M \mapsto \mathrm{ann}_M(r)$ are finitely presented. Note that these are subfunctors of the forgetful functor. If L is a finitely generated (by $s_1, \dots, s_l$) left ideal then the functor $M \mapsto ML$ is finitely presented, being the sum of the functors $M \mapsto Ms_j$. If I is a finitely generated (by $r_1, \dots, r_m$) right ideal then the functor $M \mapsto \mathrm{ann}_M(I)$ also is finitely presented, being the intersection of the functors $M \mapsto \mathrm{ann}_M(r_i)$; since every finitely presented functor is coherent, such an intersection will again be finitely presented. We will use a fairly obvious notation, writing $[\overrightarrow{F}M]$ instead of $[F]$, with M as a dummy variable, in referring to such functors and their quotients.

• R/P^n: This point is isolated by the open set $[MP^n] \cap [\mathrm{ann}(P)/(MP^{n-1} \cap \mathrm{ann}(P))]$. We go through the details. The open set $[MP^n]$ contains exactly those indecomposable pure-injectives N satisfying $NP^n = 0$, namely $R/P, R/P^2, \dots, R/P^n$. The open set $[\mathrm{ann}(P)/(MP^{n-1} \cap \mathrm{ann}(P))]$ contains exactly those N with $\mathrm{ann}_N(P) \leq NP^{n-1}$ and, on consulting the list, one sees that this defines the set $\{R/P^n, R/P^{n+1}, \dots, R_{P^\infty}\}$. The intersection of these two open sets is exactly $\{R/P^n\}$, as claimed. Similar checks below are left to the reader.

• $\overline{R}_P$: First, there is a neighbourhood which excludes all points associated to the prime P apart from $\overline{R}_P$ itself, namely $[\mathrm{ann}(P)]$. Then, given finitely many non-zero primes $Q_1, \dots, Q_k$ different from P there is a neighbourhood of $\overline{R}_P$ which excludes all points associated to those primes, namely $\bigcap_{i=1}^k([M/MQ_i] \cap [\mathrm{ann}(Q_i)])$. We cannot exclude points associated to more than finitely many primes since, otherwise, looking at the Zariski-closed=Ziegler-open set complementary to such a basic neighbourhood, we could express a basic (so, 3.3, compact) Ziegler-open set as a union of infinitely many proper open subsets, one for each of the excluded primes. Therefore a basis consists of the sets given by 'finite localisation', i.e. removing all trace of finitely many other primes, then removing all other points associated to P.

If R has only finitely many primes then there is a minimal neighbourhood, $\{\overline{R}_P, Q\}$.

• R_{P^∞}: The comments for $\overline{R}_P$ apply here also (alternatively use duality, see after

3.8) and the sets $[M/MP] \cap \bigcap_{i=1}^{k}([M/MQ_i] \cap [\mathrm{ann}(Q_i)])$, where $Q_1, \ldots, Q_k$ are any non-zero primes of R different from P, form a basis of open neighbourhoods.

• Q: Again, 'finite localisation' allows us to remove all trace of any finitely many non-zero primes but, for the same reasons as before, no more. So, if R has only finitely many primes then Q is an open point.

Observe that Zar_R, provided R is not a division ring, is not compact: it is the union of the sets $[M/MP]$, $[\mathrm{ann}(P)]$ and the $[MP^n] \cap [\mathrm{ann}(P)/(MP^{n-1} \cap \mathrm{ann}(P))]$ for P prime and $n \geq 1$ and there is no finite subcover.

With this description to hand one may check that the following is true.

Proposition 5.3. *Let R be a PI Dedekind domain. The isolated points of Zar_R are precisely the points of finite length, except in the case where R has only finitely many primes, in which case the generic point Q also is isolated. Every point of Zar_R, apart from Q, is closed.*

Taking the simplest example, that of a local ring, say $k[X]_{(X)}$ or $k[[X]]$, one may compare Zar_R and $\mathrm{Spec}(R)$. The latter has two points: the maximal ideal, which is closed, together with the generic point, which is not. In Zar_R the maximal ideal is 'doubled' and the 'extra' points in Zar_R compared with $\mathrm{Spec}(R)$ - the finite length modules $k[X]/(X^n)$ - all are clopen.

As already remarked, since these rings are PI the Zariski spectrum in the usual sense is embedded *via* the indecomposable injective modules.

Denote by $\mathrm{Zar}_R^{\mathrm{f}}$ the open set of points of Zar_R of finite length. Then if R is a PI Dedekind domain with infinitely many primes $\mathrm{Zar}_R^{\mathrm{f}}$ is Zariski-dense in Zar_R. For, from the list above, every open neighbourhood of every infinite length point contains a point of finite length. In particular, $\mathrm{Zar}_R^{\mathrm{f}}$ is exactly the set of isolated points of Zar_R.

Denote by Zar_R^1 the set $\mathrm{Zar}_R \setminus \mathrm{Zar}_R^{\mathrm{f}}$ of points of infinite length and endow this with the topology inherited from Zar_R. Since $\mathrm{Zar}_R^{\mathrm{f}}$ coincides with the set of all isolated points, Zar_R^1 also equals the first Cantor-Bendixson derivative of Zar_R. From the description of open neighbourhoods we deduce that if R is a PI Dedekind domain, with division ring of quotients Q then the non-empty Zariski-open subsets of Zar_R^1 are exactly the cofinite sets which contain the generic point Q.

Now we turn to computing rings of definable scalars. The rings of definable scalars of individual points (see 4.8) are as follows.

• The ring of definable scalars of the module R/P^n (P a non-zero prime) is the ring R/P^n: for the module has finite endolength and hence (4.4) its ring of definable scalars coincides with its biendomorphism ring, which is R/P^n.

• If N is the P-adic or P-Prüfer module then the ring of definable scalars R_N is the localisation, $R_{(P)}$, of R at P. This follows by 4.8 for the Prüfer module and then directly, or using duality [20, 6.2] and 3.8, for the adic case.

• If N is the generic point, Q, then R_N is the ring Q, again by 4.4 since N has finite endolength and $\mathrm{Biend}(Q_R) = Q$ because $R \longrightarrow Q$ is an epimorphism of rings.

For any subset, $\mathcal{P}$, of the set, $\mathrm{maxspec}(R)$, of maximal ideals of R, let $U(\mathcal{P}) = \{N \in \mathrm{Zar}_R : P(N) \notin \mathcal{P}\}$. Denote by $R[\mathcal{P}^{-1}]$ the localisation of R at $\mathrm{maxspec}(R) \setminus \mathcal{P}$, that is, $R[S^{-1}]$ where $S = C(R) \setminus \{Q \in \mathrm{maxspec}(R) : Q \notin \mathcal{P}\}$ ($\setminus\{0\}$ if $\mathcal{P} = \mathrm{maxspec}(R)$). Since the canonical map $R \longrightarrow R[\mathcal{P}^{-1}]$ is a ring epimorphism, we have, see 4.5, that $U(\mathcal{P}) = \mathrm{Zg}_{R[S^{-1}]} \subseteq \mathrm{Zg}_R$ is a Ziegler-closed subset of Zg_R. In the case that $\mathcal{P}$ is a finite subset of $\mathrm{maxspec}(R)$ then $U(\mathcal{P})$ is Zariski-open, being, in the notation introduced earlier, $\bigcap_{P\in\mathcal{P}}([\mathrm{ann}(P)] \cap [M/MP])$.

We compute the presheaf of definable scalars. The following result is immediate from 4.10.

Corollary 5.4. *Let R be a PI Dedekind domain and let $\mathcal{P}$ be a finite subset of* $\mathrm{maxspec}(R)$. *Then the ring of definable scalars over the corresponding Zariski-open subset, $U(\mathcal{P})$, of* Zar_R *is the localisation, $R[\mathcal{P}^{-1}]$, of R. Furthermore, if $\mathcal{P} \subseteq \mathcal{P}'$, then the restriction map from $R_{U(\mathcal{P})}$ to $R_{U(\mathcal{P}')}$ is the natural embedding between these localisations of R.*

Corollary 5.4 gives all the information that we need to compute the sheafification, LDef_R, of the presheaf of definable scalars. We will describe, by way of example, the ring of definable scalars and the ring of sections, that is, the ring of *locally* definable scalars, over some basic open subsets of Zar_R.

• If $U = \{R/P_1^{n_1}, \dots, R/P_t^{n_t}\}$ where the primes $P_1, \dots, P_t$ are all distinct, then $\mathrm{LDef}_R(U) = R/P_1^{n_1} \times \cdots \times R/P_t^{n_t} = R/(P_1^{n_1} \dots P_t^{n_t})$.

• If $U = \{R/P^m, R/P^n\}$ with $n \geq m$ then $R_U = R/P^n$. For by 4.4, R_U is $\mathrm{Biend}(R/P^m \oplus R/P^n)$ and, noting that there is the endomorphism of this module projecting the second component on to the first, one easily computes that this is R/P^n. Since the set U has the discrete topology, $\mathrm{LDef}_R(U)$ is the direct product $R/P^m \times R/P^n$ and so we see that the presheaf of definable scalars is not a sheaf even on those open sets where it is defined.

• R_U for U an arbitrary finite set of finite length points is given by combining the above observations.

• Let $U = \mathrm{Zar}_R \setminus \{N_1, \dots, N_t\}$ where each N_i is an adic or Prüfer module. Then $R_U = R_V$ where V is the smallest set of the form $U(\mathcal{P})$ which contains U, that is, provided at least one of the P-adic, P-Prüfer is in U then P cannot be inverted over U. The presence in U of infinitely many finite length, hence isolated, points means that the ring of locally definable scalars, $\mathrm{LDef}_R(U)$, is rather large. It makes sense, therefore, to throw away these isolated points, see below.

• Consider the special (but illustrative) case, $R = k[X]$, $U = U(\langle X\rangle) \cup \{R/\langle X\rangle\}$. A module with support U which, being basic Zariski-open, also is

Ziegler-closed, is $k[X, X^{-1}] \oplus (k[X]/\langle X\rangle)$, so we have to compute the definable scalars on this module. By 4.3 this is the biendomorphism ring of a module of the form $M \oplus M'$ where M, respectively M', is a 'large enough' module in the definable subcategory generated by $k[X, X^{-1}]$, resp. by $k[X]/\langle X\rangle$. Since $\mathrm{Hom}(M, M') = 0 = \mathrm{Hom}(M', M)$ the endomorphism ring of this module is just the block-diagonal matrix ring $\mathrm{diag}(\mathrm{End}(M_R), \mathrm{End}(M'_R))$ and hence the biendomorphism ring is just the block-diagonal matrix ring $\mathrm{diag}(k[X, X^{-1}], k[X]/\langle X\rangle)$ - that is, the direct product of these rings.

Now we compute the sheaf of locally definable scalars restricted to the set, Zar^1_R, obtained by throwing away the finite length points. Since Zar^1_R is not Zariski-open, we need the following observations concerning restriction. Define LDef^1_R to be the inverse image sheaf of LDef_R under the inclusion of Zar^1_R in Zar_R: by definition (e.g. see [19, p. 65]) this is the sheaf associated to the presheaf which assigns to a relatively open subset $U \cap \mathrm{Zar}^1_R$ of Zar^1_R, where U is a Zariski-open subset of Zar_R, the direct limit of the rings $\mathrm{LDef}_R(V)$ as V ranges over Zariski-open subsets of Zar_R with $V \supseteq U \cap \mathrm{Zar}^1_R$. If $U \cap \mathrm{Zar}^1_R = \mathrm{Zar}^1_R \setminus \{N_1, \dots, N_t\}$ let V be the set of all points of Zar_R except those which belong to a prime P such that both the P-adic and P-Prüfer appear among $N_1, \dots, N_t$, that is, V is the smallest set of the form $U(\mathcal{P})$ which contains U. Then, by the computations above, this limit is already equal to R_V and hence this presheaf is already a sheaf. Thus we have the following description of LDef^1.

Proposition 5.5. *Let R be a PI Dedekind domain and let Zar^1_R be the set of infinite-length points, regarded as a subspace of Zar_R. Let LDef_R denote the sheaf of locally definable scalars over Zar_R. Then the inverse image sheaf, LDef^1_R, on Zar^1_R may be computed as follows. Given a Zariski-open subset U of Zar_R, let V be smallest set of the form $U(\mathcal{P})$ which contains U. Then $\mathrm{LDef}^1_R(U \cap \mathrm{Zar}^1_R) = \mathrm{LDef}_R(V) = R[\mathcal{P}^{-1}]$ and the restriction maps are those of LDef_R, that is, the canonical localisation maps.*

In particular, the ring of definable scalars, $R_{\mathrm{Zar}^1_R}$, of Zar^1_R is R itself.

The sheaf LDef^1_R is 'unseparated' in the sense that it contains points N,N' such that U is an open neighbourhood of N iff $(U \setminus \{N\}) \cup \{N'\}$ is an open neighbourhood of N': take the Prüfer and adic associated to any maximal prime. In order to recover the 'classical' situation we have to identify corresponding adic and Prüfer points. So let $\alpha : \mathrm{Zar}^1_R \longrightarrow \mathrm{Zar}^1_R$ be the map of 3.8 which interchanges the P-adic and P-Prüfer point for every P and which fixes the generic point.

Corollary 5.6. *The map $\alpha : \mathrm{Zar}^1_R \longrightarrow \mathrm{Zar}^1_R$ is a homeomorphism of order 2 and $\mathrm{LDef}^1_R \simeq \alpha^\star \mathrm{LDef}^1_R \simeq \alpha_\star \mathrm{LDef}^1_R$ where $\alpha^\star$, $\alpha_\star$ denote the inverse image and direct image sheaves respectively (see [19], [51]).*

Proof. From the description of the topology it is clear that α is a homeomorphism. For any basic Zariski-open set, U, of Zar^1_R we have, again by what has been said above, $R_U \simeq R_{\alpha U}$ and so the isomorphisms are direct from the definitions. $\square$

We can, therefore, form the quotient space Zar^1_R/α of α-orbits and the corresponding sheaf LDef^1_R/α over this space, thus obtaining a ringed space with centre isomorphic, *via* the identification of $\mathrm{Spec}(R)$ with $\mathrm{Spec}(C(R))$, to the structure sheaf over the commutative Dedekind domain $C(R)$.

5.2 The rep-Zariski spectrum of a PI hereditary order

A **PI hereditary order** is a hereditary ring which is an order in a simple artinian ring, equivalently a PI hereditary noetherian prime ring (see [29]). Here we note that the results of the previous section generalise to such rings. There is little to check since, by [36, Section 3], the description of the Ziegler spectrum, points and topology, and hence of the, dual, Zariski topology, is just as in the case of a PI Dedekind domain. In particular to every point, N, of Zar_R is associated a prime ideal, $P(N)$, of R. The only significant difference is that if the ring R is not a Dedekind prime ring then the map from $\mathrm{Spec}(R)$ to $\mathrm{Spec}(C(R))$ given by intersecting a prime ideal with the centre is not 1-1. So it is essential here to use $\mathrm{Spec}(R)$, rather than $\mathrm{Spec}(C(R))$, to parametrise the primes.

Example 5.7. Let R be the ring $\left(\begin{smallmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ - a non-maximal order in the simple artinian ring $A = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{smallmatrix}\right)$. For each non-zero prime $p \in \mathbb{Z}$, $p \neq 2$, we have the corresponding prime ideal $P_p = \left(\begin{smallmatrix} p\mathbb{Z} & 2p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} \end{smallmatrix}\right)$, and the corresponding p-adic and p-Prüfer modules, which may be regarded as $(\bar{\mathbb{Z}}_{(p)}, \bar{\mathbb{Z}}_{(p)})$ and $(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty})$ respectively, as well as the finite length indecomposable modules, R/P^n, associated to P.

Corresponding to the prime $p = 2$, there are two prime ideals of R, $P_1 = \left(\begin{smallmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right)$ and $P_2 = \left(\begin{smallmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ with corresponding simple modules $S_i = R/P_i$ and with $\mathrm{Ext}(S_i, S_{3-i}) \neq 0$ for $i = 1, 2$. The P_1-adic module has a unique infinite descending chain of submodules with simple composition factors S_1,S_2 alternating and starting with S_1. Dually the P_1-Prüfer module N is the injective module with socle S_1, $\mathrm{soc}(N/S_1) = S_2$, $(N/S_1)/\mathrm{soc}(N/S_1) \simeq N$. Similarly for P_2.

First we have to compute rings of definable scalars. These are obtained for primes P belonging to singleton cliques, in the sense of [7], by localising just as in the Dedekind prime case. For the other primes we use universal localisation, as in [7] (alternatively, as mentioned there, Goodearl's localisation from [16]), to obtain the corresponding Prüfer and adic modules. We give an example below for illustration. Beyond this, the description of the presheaf of definable scalars and the corresponding sheaf, both on Zar_R and on Zar^1_R, is as before, using the fact that if $R \longrightarrow S$ is an epimorphism of rings then, in addition to the induced homeomorphic embedding of Zar_S into Zar_R (4.5), there is induced an embedding of LDef_S, 'up to Morita equivalence' into LDef_R. This follows from the argument of [37, Section 8]. Recall that every universal localisation is an epimorphism of rings.

We continue Example 5.7, retaining the notation, and compute the various rings of definable scalars.

Corresponding to the prime P_p there is the ring of definable scalars $\left(\begin{smallmatrix} \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{smallmatrix}\right)$, which is a maximal order in A.

The rings of definable scalars corresponding to P_1 and P_2 may be computed using [16, first paragraphs of Section 2]. Explicitly, and adopting the notation of that paper, we remove the simple module S_1 by localising at $X_1 = \{P_2\} \cup \{P_p : p \neq 2\}$. Let $\mathcal{S}_1$ be the set of essential right ideals I of R such that none of R/P_2, R/P_p ($p \neq 2$) occurs as a composition factor of R/I. Note that $P_1 \in \mathcal{S}_1$. Let $R_{(1)}$ denote the localisation of R at the torsion theory which has $\mathcal{S}_1$ as dense set of right ideals. Then, [16], $R_{(1)} = \{a \in M_2(\mathbb{Q}) : aI \leq R \text{ for some } I \in \mathcal{S}_1\}$. One checks that $R_{(1)} = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$. Similarly, if $R_{(2)}$ denotes the ring obtained by localising away S_2 then one checks that $R_{(2)} = \left(\begin{smallmatrix} \mathbb{Z} & 2\mathbb{Z} \\ (1/2)\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$. Hence the ring of definable scalars at P_1 is $\left(\begin{smallmatrix} \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{smallmatrix}\right)$ and that at P_2 is $\left(\begin{smallmatrix} \mathbb{Z}_{(2)} & 2\mathbb{Z}_{(2)} \\ (1/2)\mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{smallmatrix}\right)$. Notice that, as rings, though not as R-algebras, these are isomorphic (by the map taking $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} a & 2b \\ c/2 & d \end{smallmatrix}\right)$).

One way of regarding this is that we have two epimorphisms from R to the maximal order $M_2(\mathbb{Z})$. The corresponding Ziegler-closed and, one may check, Zariski-open, sets cover Zar_R. So LDef_R is covered by two very much overlapping copies of '$M_2(\mathrm{LDef}_{\mathbb{Z}})$'.

Proposition 5.8. *Let R be a hereditary order. Then the presheaf of definable scalars over Zar^1_R is already a sheaf.*

Proof. Take an open cover $\{U_i\}_i$ of Zar^1_R, say $U_i = \mathrm{Zar}^1_R \setminus Y_i$, where Y_i is any finite subset of Zar^1_R which does not contain the generic Q, and let elements $s_i \in R_i = R_{U_i}$ be such that on $U_i \cap U_j = \mathrm{Zar}^1_R \setminus \{Y_i \cup Y_j\}$ we have $s_i = s_j = s$, say. Here we identify all the rings R_i with subrings of the full, simple artinian, quotient ring of R, so this equality makes sense. We have $s \in R_i \cap R_j$ and this equals $R_{U_i \cup U_j}$ since a prime P satisfies $P.R_i \cap R_j = R_i \cap R_j$ iff both the P-Prüfer and P-adic modules lie in both Y_i and Y_j, which is so iff $P.R_{U_i\cup U_j} = R_{U_i \cup U_j}$. So now taking any finite subcover, say $U_1, \dots, U_n$, we deduce that $s_1 = \cdots = s_n = s \in R = R_{\mathrm{Zar}^1_R}$. Thus s is a global section which restricts on each U_i to s_i and is already in the presheaf. The argument for an arbitrary basic open set is similar, indeed reduces to this case by localisation. □

Proposition 5.9. *Suppose that R is a hereditary order. Let $\mathrm{Spec}R$ denote the space of prime ideals of R with the Zariski topology and let $\pi : \mathrm{Zar}^1_R \longrightarrow \mathrm{Spec}R$ be the map which sends $N \in \mathrm{Zar}^1_R$ to $P(N)$, the prime ideal of R associated with N. Then the direct image, $\pi_\star \mathrm{LDef}^1_R$, of LDef^1_R is a sheaf on $\mathrm{Spec}R$ which sends* *an open set* $U = \mathrm{Spec}R \setminus Y$*, where* Y *is a finite subset of* $\mathrm{maxspec}R$*, to the localisation, in the sense of [16], of* R *at* U *and* $\pi_\star \mathrm{LDef}^1_R$ *may be identified with* LDef^1_R/α *where* α *is the homeomorphism interchanging corresponding Prüfer and adic points.*

Proof. The direct image of a sheaf is always a sheaf (e.g. [51]) and the description of the sheaf $\pi_\star \mathrm{LDef}R^1$ and its identification with $\mathrm{LDef}R^1/\alpha$ follows from the previous discussion. □

The underlying space, $\mathrm{Spec}R$, of this sheaf may also be identified with the space $\mathrm{Spec}_s R$ (based on the set of simple modules) from [7], *via* $S \mapsto \mathrm{ass}E(S)$. The sheaf $\mathrm{Spec}_s R$ in [7] may, therefore be identified with the centre of LDef^1_R/α.

5.3 The rep-Zariski spectrum of a tame hereditary artin algebra

If R is an artin algebra then every indecomposable module of finite length is a point of pinj_R. Therefore, one should not expect to be able to give a complete description of Zar_R for algebras which are of wild representation type. Nevertheless, one may aim to describe parts of this space and, over some tame rings, one may hope to give a complete description of the topological space and of the associated (pre-)sheaf of rings.

Throughout this section let R be a tame hereditary artin algebra which is not of finite representation type. (If R is of finite representation type then the topology is discrete and rings of definable scalars are, 4.4, just biendomorphism rings.) Also, without loss of generality, assume that R is **connected** (not a proper direct product of rings).

We recall some facts about the points of Zar_R and then we recall the description of the topology from [36], [47] (refer to these for more detail). For terms not defined here one may consult, say, [46].

Proposition 5.10. *[38, 2.21] Let* N *be an indecomposable finitely generated module over the artin algebra* R*. Then* N *is both open and closed in* Zar_R*.*

Proof. It is known [33, 13.1], and follows from the existence of almost split sequences over artin algebras, that N is Ziegler-open, hence Zariski-closed.

It is also the case that, over any ring, every point of finite endolength is Ziegler-closed (essentially this goes back to [11, Theorem 13]) so $\{N\}^c$ is both Ziegler-open and Ziegler-closed, hence compact, hence, by the description of the Ziegler topology, of the form (F) for some F. Therefore $\{N\} = [F]$, as required. □

Indeed, the open points are exactly those of finite length. These fall into three disjoint sets: the set, $\mathbf{P}$, of preprojective points, the set, $\mathbf{R}$, of regular points, and the set, $\mathbf{I}$, of preinjective points.

The set $\mathbf{R}$, regarded as part of the Auslander-Reiten quiver of R, is a disjoint union of 'tubes', each containing a finite number of quasisimple modules at its base. All but finitely many tubes are homogeneous, that is, contain just one quasisimple. (In Example 5.7 the prime $p = 2$ gives, in an analogous context, a non-homogeneous tube, with two quasisimples, S_1 and S_2.) We denote the tube (regarded as a set of modules) to which the quasisimple module S belongs by $\mathbf{T}(S)$. Denote the coray of epimorphisms in $\mathbf{T}(S)$ ending at S by $\mathbf{E}(S)$ and the ray of monomorphisms in $\mathbf{T}(S)$ beginning with S by $\mathbf{M}(S)$. The terms 'ray' and 'coray' here refer to the structure of the Auslander-Reiten quiver. To each quasisimple S is associated the S-adic module $P(S)$ which is the inverse limit of $\mathbf{E}(S)$ and the S-Prüfer module $E(S)$ which is the direct limit of $\mathbf{M}(S)$.

The modules $P(S)$ and $E(S)$, for S a quasisimple regular module, are points of the spectrum and the only other infinite-dimensional point is the generic module Q. We denote by Zar_R^1 the set or space of infinite-dimensional (=non-isolated) points. Thus, to each point N of $\mathbf{R} \cup \mathrm{Zar}_R^1$, apart from the generic point, we have an associated quasisimple module which we denote $S(N)$. In this context the quasi-simples play the role that primes did in the previous examples (this is more than an analogy, see [7]). As in those cases, the process of 'finite localisation', i.e. removal of all trace of finitely many primes/quasisimples lies behind the description of neighbourhood bases. Given a set, $\mathcal{S}$, of quasisimple modules, let $U(\mathcal{S})$ denote the set consisting of the generic point and all points of $\mathbf{R} \cup \mathrm{Zar}_R^1$ which are associated to some quasisimple *not* in $\mathcal{S}$.

The papers [36], [47] describe the Ziegler, rather than the rep-Zariski, topology but, since the latter may be defined in terms of the former (3.5), one easily deduces the following.

Theorem 5.11. *[39] Let R be a tame hereditary artin algebra. A basis of open sets for Zar_R is as follows.*
As for every artin algebra, the finite-dimensional points are open.
If N is S-adic or S-Prüfer then the sets of the form $\{N\} \cup U(\mathcal{S})$ where $\mathcal{S}$ is a finite set of quasisimples, form a basis of open neighbourhoods for N.
The sets of the form $U(\mathcal{S})$ where $\mathcal{S}$ is a finite set of quasisimples, form a basis of open neighbourhoods for the generic point G.

In particular, it follows that the sets $\mathbf{P}$ and $\mathbf{I}$ are Zariski-closed so do not figure in the description of neighbourhood bases of the infinite-dimensional points.

The approach of [39] is based on the following result.

Theorem 5.12. *([49], [7]) Let R be a tame hereditary artin algebra.*
(a) Let $\mathcal{S}$ be any set of quasisimple modules. Then the universal localisation, $R_{\mathcal{S}}$, of R at $\mathcal{S}$ is a hereditary PI order which is a subring of the simple artinian ring A obtained by taking $\mathcal{S}$ to be the set of all quasisimple modules.
(b) The localisation $R \longrightarrow R_{\mathcal{S}}$ is an epimorphism of rings, and the image of the inclusion functor $\text{Mod-}R_{\mathcal{S}} \longrightarrow \text{Mod-}R$ is the full subcategory of all modules M

which are orthogonal to $\mathcal{S}$ in the sense that $\mathrm{Ext}^1(S, M) = 0 = \mathrm{Hom}(S, M)$ *for all* $S \in \mathcal{S}$.

We have from 4.5 that the intersection of Mod-$R_{\mathcal{S}}$ with pinj_R is the Ziegler-closed subset $U(\mathcal{S})$ which, if $\mathcal{S}$ is finite, is basic Zariski-open, being defined by the conditions $\mathrm{Ext}^1(S, -) = 0 = \mathrm{Hom}(S, -)$ for $S \in \mathcal{S}$, and the ring of definable scalars for this Zariski-open set is just $R_{\mathcal{S}}$. Using this, we may deduce the next result.

Corollary 5.13. *Let R be a tame hereditary artin algebra. A basis for the Zariski topology on $\mathbf{R} \cup \mathrm{Zar}^1_R$ is the collection of sets of the form $\{N\}$ with $N \in \mathbf{R}$ together with the sets $U(\mathcal{S})$ where $\mathcal{S}$ ranges over those finite sets of quasisimple modules which, without loss of generality, contain all quasisimples from at least one tube.*

The restriction $\mathrm{Def}_R \restriction (\mathbf{R} \cup \mathrm{Zar}^1_R)$ of the presheaf of definable scalars is given on this basis by sending $U(\mathcal{S})$ to the localisation $R \longrightarrow R_{\mathcal{S}}$ and sending any open subset of $U(\mathcal{S})$ to the ring of definable scalars of the corresponding subset, regarded as an open subset the rep-Zariski spectrum of the hereditary PI order $R_{\mathcal{S}}$ (see Section 5.2).

In particular, the ring of definable scalars of any member N of $\mathbf{R} \cup \mathrm{Zar}^1_R$ may be computed by choosing a set $\mathcal{S}$ of quasisimples which contains all quasisimples from at least one tube and does not contain the associated quasisimple module $S(N)$, localising R at $\mathcal{S}$ to obtain the hereditary order $R_{\mathcal{S}}$ which, by choosing $\mathcal{U}$ to contain all but at most one quasisimple from each inhomogeneous tube, may be assumed to be a Dedekind prime ring and then computing the ring of definable scalars of N, regarded as an $R_{\mathcal{S}}$-module.

Now let R be the Kronecker algebra $\tilde{A}_1(k)$ - the path algebra over a field k of the quiver $1 \rightrightarrows 2$. We give a more explicit description of the sheaf LDef^1_R as a sheaf of hereditary (in fact, maximal) orders.

By 5.12, the full "quotient ring" of $\tilde{A}_1(k)$ is the ring, $M_2(k(X))$, of 2×2 matrices over the function field $k(X)$. In order to maintain the symmetry between the arrows α and β of $\tilde{A}_1(k)$ we represent $k(X)$ in the form $k(X_0, X_1)_0$ where the subscript denotes the 0-grade part of the quotient field of the graded ring $k[X_0, X_1]$. Then there is a natural embedding of $\tilde{A}_1(k)$ into $M_2(k(X_0, X_1)_0)$ which takes e_1 to $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, e_2 to $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, α to $\left(\begin{smallmatrix}0&X_0\\0&0\end{smallmatrix}\right)$ and β to $\left(\begin{smallmatrix}0&X_1\\0&0\end{smallmatrix}\right)$ and, under this embedding, the ring A of 5.12 may be identified with $M_2(k(X_0, X_1)_0)$.

Let S_0 (respectively S_1) be the quasisimple module which satisfies $S_0X_1 = 0$ (resp. $S_1X_0 = 0$). Let R_i denote the localisation of R at S_i ($i = 0, 1$). Let D_i be the Zariski-open subset of Zar^1_R, $D = \mathrm{Zar}^1_R \cap [M/MX_{1-i}] \cap [\mathrm{ann}(X_{1-i})]$ (notation as in Section 5.1). Then the localisation map $R \longrightarrow R_i$ identifies D_i with $\mathrm{Zar}^1_{R_i}$ and $\mathrm{LDef}^1_R \restriction D_i$ with $\mathrm{LDef}^1_{R_i}$. Each of R_0,R_1 is isomorphic as a ring to the polynomial ring over k in one indeterminate and the universal localisation $R_{0,1}$ of R at $\{S_0, S_1\}$ (which corresponds to the intersection $D_0 \cap D_1$) is a ring isomorphic to $k[T, T^{-1}]$. It is straightforward to compute these as subalgebras of $M_2(k(X_0, X_1)_0)$ and one obtains:

$$R_0 = \begin{pmatrix} k[X_0X_1^{-1}] & kX_1 \oplus X_0k[X_1^{-1}X_0] \\ X_1^{-1}k[X_0X_1^{-1}] & k[X_1^{-1}X_0] \end{pmatrix};$$

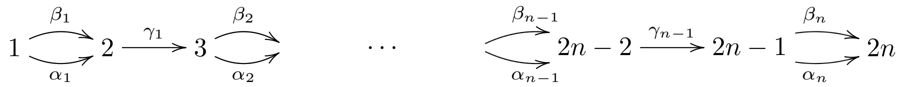
$$R_1 = \begin{pmatrix} k[X_1X_0^{-1}] & kX_0 \oplus X_1k[X_0^{-1}X_1] \\ X_0^{-1}k[X_1X_0^{-1}] & k[X_0^{-1}X_1] \end{pmatrix};$$

$$R_{0,1} = \begin{pmatrix} k[X_0X_1^{-1}, X_1X_0^{-1}] & X_1k[X_0^{-1}X_1] \oplus X_0k[X_1^{-1}X_0] \\ X_1^{-1}k[X_0X_1^{-1}] \oplus X_0^{-1}k[X_1X_0^{-1}] & k[X_1^{-1}X_0, X_0^{-1}X_1] \end{pmatrix} = \begin{pmatrix} k(X_0,X_1)_0 & k(X_0,X_1)_1 \\ k(X_0,X_1)_{-1} & k(X_0,X_1)_0 \end{pmatrix}$$

(where subscript denotes degree in the graded ring $k(X_0, X_1)$).

5.4 Other examples

Let R be the k-path algebra of one of the quivers Λ_n ($n \geq 2$) shown:



with relations $\beta_i\gamma_i = 0 = \gamma_i\alpha_{i+1}$

The Ziegler and rep-Zariski spectra of these algebras were described in [6], with Λ_2 being treated in full detail, the others more briefly. To give these details would take some technical setting up so we refer the reader to that paper and make only a few remarks.

Corresponding to the n subquivers isomorphic to the Kronecker quiver $\tilde{A}_1$ there are n rep-Zariski-open subsets each homeomorphic to $\mathrm{Zar}_{\tilde{A}_1}$. In particular there are the n corresponding generic points (and no other generic points). There are some other, discretely parametrised, infinite-dimensional points which 'link' these n open subsets (in the same way that they are linked in the quiver). After the (open and closed) finite-dimensional points are removed, all other points are generic, 'linking', adic or Prüfer. All these infinite-dimensional points, except the generics, are closed. Roughly, after removing the finite-dimensional points, we have n double-except-for-generics copies of the projective line over k with some $\mathbb{N}$-parametrised families of points linking them into a chain.

In particular, for each of these algebras the space, Zar^1, of infinite-dimensional points is one-dimensional in an algebraic-geometric sense. It is conjectured in that paper that for a finite-dimensional algebra of infinite representation type the 'algebraic-geometric' dimension of the space Zar^1 will be either 1 or ∞; the latter in the sense that it embeds algebraic varieties of arbitrarily high dimension.

It is shown in [37, Section 9] that wild algebras do have algebraic-geometric dimension ∞ in this sense.

We also mention the first Weyl algebra, $A_1(k)$ where k is a field of characteristic zero: this is a wild algebra [1] and the comment just above applies. One may, however, look at parts of the spectrum, as is done, among other things, in [43], where the relative topologies on inj_R, on the torsionfree indecomposable pure-injective modules, and on closures of some tubes are described for a class of algebras ('generalised Weyl algebras') which includes $A_1(k)$.

6 The spectrum of a commutative coherent ring

Throughout this section R will be a commutative ring.

In the re-interpretation in Section 4.4 of $\mathrm{Spec}(R)$ as a topology on the set, inj_R, of indecomposable injective R-modules when R is a commutative noetherian ring we used the noetherian hypothesis in the pairing up of indecomposable injectives with prime ideals.

In the general commutative case the association $P \mapsto E(R/P)$ gives only an injection of $\mathrm{Spec}(R)$ into inj_R.

The first example shows that $\mathrm{Spec}(R) \longrightarrow \mathrm{inj}_R$ need not be surjective.

Example 6.1. Let $R = k[X_n(n \in \omega)]$ be a polynomial ring, over a field k, in infinitely many commuting indeterminates. It is easily checked that R is coherent. Let $I = \langle X_n^{n+1} : n \in \omega \rangle$. Clearly I is not prime but $E = E(R/I)$ is an indecomposable injective. To see this, it is enough to show that R/I is uniform and this may be shown as follows. First note that a polynomial $\sum a_\nu X^\nu$ (each multi-index ν occurring at most once) is in I iff each of its monomial factors is in I. Let x_i denote the image in R/I of X_i. Let $p \in R \backslash I$. A short inductive (on the number of monomials) argument shows that there is a multiple of p whose image in R/I has the form $x_1 x_2^2 \dots x_n^n \neq 0$ for some n. Hence any two non-zero elements of R/I have a common multiple of this form so R/I is uniform and $E(R/I)$ is indecomposable.

On the other hand, E does not have the form $E(R/P)$ for any prime P. This follows from 6.2 since it is easy to see that $P(E)$, as defined below, is the maximal ideal $\langle X_n : n \in \omega \rangle$ and that $E(R/\langle X_n : n \in \omega \rangle)$ has non-zero socle whereas $E(R/I)$ has zero socle. For if $p \in R \backslash I$, say $p \in k[X_0, \dots, X_n]$, then $(p+I)X_{n+1}$ generates a non-zero proper submodule of the submodule generated by $p+I$. Hence E is not isomorphic to $E(R/P(E))$ so, by 6.2, E does not have the form E_P for any prime ideal P.

Let E be any indecomposable injective R-module. Set $P = P(E)$ to be the sum of annihilator ideals of non-zero elements of E. Since E is uniform these ideals are closed under finite sum so the only issue is whether the sum, $P(E)$, of them all is itself an annihilator ideal.

As before we use the notation E_P to denote $E(R/P)$.

Lemma 6.2. *If $E \in \mathrm{inj}_R$ then $P(E)$ is a prime ideal. The module E has the form E_P for some prime ideal P iff the set of annihilator ideals of non-zero elements of E has a maximal member, namely $P(E)$, in which case $E = E_{P(E)}$.*

Proof. Suppose that $rs \in P(E)$. Then, by definition of $P(E)$ there is $a \in E$, $a \neq 0$ such that $ars = 0$. Then either $ar = 0$, in which case $r \in P(E)$, or

$ar \neq 0$ and hence $s \in P(E)$. This shows that $P(E)$ is prime. So, if $P(E)$ is an annihilator ideal then $E = E_{P(E)}$.

If $E = E(R/P)$ then $P \leq P(E)$, by definition of the latter. Suppose there were $r \in P(E) \setminus P$. Let $b \in E$ be non-zero with $br = 0$ and let $a \in E$ be such that $\mathrm{ann}_R(a) = P$. By uniformity of E there is a non-zero element $c \in aR \cap bR$, say $c = at$ with $t \in R$. Since $cr = 0$ we have $atr = 0$, hence $tr \in P$ and hence $t \in P$ (impossible since $c = at \neq 0$) or $r \in P$ - contradiction. So $P = P(E)$. □

Before examining the relation between $E \in \mathrm{inj}_R$ and $E_{P(E)} \in \mathrm{inj}_R$ we address the issue of which topology we should be using on inj_R.

For I an ideal of R set $D(I) = \{P \in \mathrm{Spec}(R) : I \nleq P\} = \bigcup_{r \in I} D(r)$ - a typical Zariski-open subset of $\mathrm{Spec}(R)$. Also set $D^{\mathrm{m}}(I) = \{E \in \mathrm{inj}_R : (R/I, E) = 0\}$ ("m" for "morphism"). Since $D^{\mathrm{m}}(I) \cap D^{\mathrm{m}}(J) = D^{\mathrm{m}}(I \cap J)$ (for the non-immediate direction, note that any morphism from $R/(I \cap J)$ to E extends, by injectivity of E, to one from $R/I \oplus R/J$) these form a basis for a topology on inj_R. If R is coherent and we use only finitely generated ideals then we obtain exactly the Gabriel-Zariski topology on inj_R.

The argument near the beginning of Section 4.4 shows that this topology on inj_R, when restricted to $\mathrm{Spec}(R)$ regarded as embedded in inj_R, coincides with the Zariski topology.

Corollary 6.3. *For any ideal I we have $D^{\mathrm{m}}(I) \cap \mathrm{Spec}(R) = D(I)$.*

Recall that for I any right ideal of a ring R and $r \in R$ there is an isomorphism $R/(I : r) \simeq (rR + I)/I$, where $(I : r) = \{s \in R : rs \in I\}$, induced by sending $1 + (I : r)$ to $r + I$.

Theorem 6.4. *Let R be commutative coherent, let E be an indecomposable injective module and let $P(E)$ be the prime ideal defined before. Then E and $E_{P(E)}$ are topologically indistinguishable in Zg_R and hence also in Zar_R.*

Proof. Let I be such that $E = E(R/I)$. For each $r \in R \setminus I$, by the remark just above, the annihilator of $rR + I \in E$ is $(I : r)$ and so, by definition of $P(E)$, $(I : r) \leq P(E)$. The natural projection $(rR + I)/I \simeq R/(I : r) \longrightarrow R/P(E)$ extends to a morphism from E to $E_{P(E)}$ which is non-zero on $r + I$. Forming the product of these morphisms as r varies over $R \setminus I$, we obtain a morphism from E to a product of copies of $E_{P(E)}$ which is monic on R/I, hence is monic. Therefore E is a direct summand of a product of copies of $E_{P(E)}$ and so is in the definable subcategory generated by $E_{P(E)}$. Therefore $E \in \text{Zg-cl}(E_{P(E)})$ (this conclusion required no assumption on R beyond commutativity).

For the converse, take a basic Ziegler-open neighbourhood of $E_{P(E)}$: by (the proof of) 3.12 this has the form (J/I) for a pair, $I < J$ of finitely generated ideals of R. Now, $E_{P(E)} \in (J/I)$ means that there is a non-zero morphism $f : J/I \longrightarrow E_{P(E)}$. Since $R/P(E)$ is essential in $E_{P(E)}$ the image of f has non-zero intersection with $R/P(E)$ so there is an ideal J', without loss of generality finitely generated, with $I < J' \leq J$ and such that the restriction, f', of f to

J'/I is non-zero (and contained in $R/P(E)$). Since $R/P(E) = \varinjlim R/I_\lambda$, where I_λ ranges over the annihilators of non-zero elements of E, and since J'/I is finitely presented, f' factorises through one of the maps $R/I_\lambda \longrightarrow R/P(E)$. In particular, there is a non-zero morphism $J'/I \longrightarrow E$ and hence, by injectivity of E, an extension to a morphism $J/I \longrightarrow E$, showing that $E \in (J/I)$. Therefore $E_{P(E)} \in \text{Zg-cl}(E)$, as required. □

Corollary 6.5. *Suppose that R is commutative coherent. Then the embedding $\text{Spec}(R) \to \text{inj}_R$ given by $P \mapsto E_P$ induces a homeomorphism at the level of topology (i.e. between the lattices of open sets) between the Zariski topology on $\text{Spec}(R)$ and the Gabriel-Zariski topology on inj_R.*

Corollary 6.6. *Let R be a commutative coherent ring and let $P \in \text{Spec}(R)$. Then the closure of E_P in the Gabriel-Zariski topology on inj_R is $\{E \in \text{inj}_R : P(E) \geq P\}$.*

Recall that if E is an injective module then we denote by $\text{cog}(E)$ the hereditary torsionfree class cogenerated by E, that is, all those modules which embed in a power of E. If E' is an indecomposable injective in $\text{cog}(E)$ then, since it is a direct summand of a direct product of copies of E, it is in the definable subcategory generated by E and hence is a member of $\text{supp}(E) \subseteq \text{Zg}_R$. In particular, if E is indecomposable then $E' \in \text{Zg-cl}(E)$ and hence $E \in \text{Zar-cl}(E')$. The first half of the proof of 6.4 shows that $E \in \text{Zg-cl}(E_{P(E)})$ whether or not R is coherent. It also shows the following.

Lemma 6.7. *If $I \leq J$ are (right) ideals of an arbitrary ring R then $E(R/I) \in \text{cog}(E(R/J))$.*

Corollary 6.8. *If R is commutative and $P, Q \in \text{Spec}(R)$ then $E_P \in \text{cog}(E_Q)$ iff $P \leq Q$.*

For the direction "⇒" note that if $f : R/P \to E_Q^\kappa$ is an embedding into a power of E_Q then, projecting $f(1+P)$ to some component where it is non-zero and recalling that Q is the maximal annihilator of non-zero elements of E_Q, we deduce $P \leq Q$.

Proposition 6.9. *Let E be an indecomposable injective module over the commutative coherent ring R. Then the torsion theory cogenerated by E is of finite type iff $E = E_P$ for some prime P.*

Proof. (⇐) By 6.3, for I any ideal of R we have $E_P \in D^{\text{m}}(I)$ iff $E_P \in D(I)$, that is, iff $(R/I, E_P) = 0$ (i.e. R/I is E_P-torsion) iff $I \not\leq P$. This last is so iff some finitely generated ideal $I' \leq I$ satisfies $I' \not\leq P$. So each E_P-dense ideal contains a finitely generated E_P-dense ideal, as required.

(⇒) If E cogenerates a torsion theory of finite type then, by the proof of the second half of 6.4, we have $E_{P(E)} \in \text{cog}(E)$. For there, taking $J = R$, it is shown that if I is a finitely generated ideal with $\text{Hom}(R/I, E) = 0$, i.e. with

R/I E-torsion, then $\mathrm{Hom}(R/I, E_{P(E)}) = 0$ hence $E_{P(E)} \in \mathrm{cog}(E)$. Therefore there is an embedding $R/P(E) \longrightarrow E^\kappa$ for some index set κ. It follows that $\mathrm{ann}_E P(E) \neq 0$ and hence $E \simeq E_{P(E)}$, as required. □

Thus $\mathrm{Spec}(R)$ may be identified within inj_R as those modules which cogenerate torsion theories of finite type.

Theorem 6.10. *If R is any right coherent ring then a subset of* inj_R *is Ziegler-closed iff it has the form $\mathcal{F} \cap \mathrm{inj}_R$ where $\mathcal{F}$ is the torsionfree class for some torsion theory of finite type.*

Proof. A torsionfree class $\mathcal{F}$ is a definable subcategory of Mod-R iff the corresponding torsion theory is of finite type ([32], [24]) and so, by 4.2, any subset of the form given will be Ziegler-closed.

For the converse if $X \subseteq \mathrm{Zg}_R$ is closed then, by 4.2 it has the form $\mathcal{X} \cap \mathrm{inj}_R$ for some definable subcategory, $\mathcal{X}$, of Mod-R. We may replace $\mathcal{X}$ by its closure, $\mathcal{X}'$, under arbitrary submodules which is, note, again a definable subcategory and $\mathcal{X}' \cap \mathrm{inj}_R = X$. Note that $\mathcal{X}' = \mathrm{cog}(X)$ and so, as a definable torsionfree class, it is of finite type by [32], [24]. □

Theorem 6.11. *Let R be commutative coherent and let $X \subseteq \mathrm{inj}_R$ be Ziegler-closed. Then X is irreducible in the Ziegler topology iff $X = \mathrm{cog}(E_P) \cap \mathrm{inj}_R$ for some prime ideal, P, of R.*

Proof. By 6.10 there is a torsionfree class, $\mathcal{F}$, of finite type with $\mathcal{F} \cap \mathrm{inj}_R = X$. Let $\mathcal{I}$ be the set of annihilators of non-zero element of members of $\mathcal{F}$. If $\{I_\lambda\}_\lambda$ is a chain of members of $\mathcal{I}$ with their union=sum equal to I, say, then, since $\mathcal{F}$, being of finite type, is closed under direct limits, there is $M \in \mathcal{F}$ and $a \in M$ with $a \neq 0$ and $aI = 0$. So by Zorn's Lemma every $I \in \mathcal{I}$ is contained in a maximal member of $\mathcal{I}$. Denote the set of these maximal members by $\mathcal{P}$. The argument used in 6.2 shows that all ideals in $\mathcal{P}$ are prime.

Choose $P_0 \in \mathcal{P}$ and set $E_0 = E_{P_0}$ and $E' = \bigoplus\{E_P : P \in \mathcal{P}, P \neq P_0\}$. By 6.7 (and comments before that) $\mathrm{supp}(E_0) \cup \mathrm{supp}(E') = X$ (because R is coherent the support of any injective module will consist of injective modules, see [44, 4.4]). So, by irreducibility of X, either $X = \mathrm{supp}(E_0)$, which equals $\mathrm{cog}(E_0) \cap \mathrm{inj}_R$ by 6.9 and 6.10, as required, or $X = \mathrm{supp}(E')$. But, in the latter case we would have $E_0 \in \mathrm{cog}E'$ and hence there would be an embedding of the form $f : R/P_0 \longrightarrow \prod\{E_P^{\kappa(P)} : P \in \mathcal{P}, P \neq P_0\}$ with, say $(1 + P_0) \longrightarrow (e_\lambda)_\lambda$. Some e_λ would be non-zero and so P_0 would be (properly!) contained in $\mathrm{ann}_R(e_\lambda)$ - contradicting $P_0 \in \mathcal{P}$. □

Proposition 6.12. *Let R be commutative coherent. A subset V of* inj_R *is rep-Zariski-closed and irreducible iff there is a prime ideal Q of R such that $V = \{E : P(E) \geq Q\}$.*

Proof. This is just the usual description of irreducible closed subsets of $\mathrm{Spec}(R)$ combined with 6.4. □

Corollary 6.13. *Let R be commutative coherent. Then there are natural bijections between the following:*
(i) the set of irreducible Ziegler-closed subsets of inj_R*;*
(ii) the set of irreducible Zariski-closed subsets of $\mathrm{Spec}(R)$*;*
(ii) the points of $\mathrm{Spec}(R)$*;*
(iii) the set of irreducible rep-Zariski-closed subsets of inj_R*;*
given by $\{E : P(E) \leq Q\} \sim \{P : P \geq Q\} \sim Q \sim \{E : P(E) \geq Q\}$.

7 Appendix: pp conditions

Many of the references use terminology derived from model theory: here we explain, briefly, the main item of terminology, namely "pp formula". In the context of this paper, it is most convenient to think of this as meaning simply a finitely generatead subfunctor of the forgetful functor or of one of its finite powers.

Every finitely generated subfunctor of the n-th power, $(R^n, -)$, of the forgetful functor from mod-R to **Ab** has the following form. Fix a homogeneous R-linear system of equations with $m \geq n$ indeterminates: to every module M we may associate the solution set in M of this system - this will be a subgroup of M^m; consider the image of this solution set under the projection of M^m onto the first n coordinates - this image is a subgroup of M^n. That's how the functor works on objects and the action on morphisms is the obvious one. That's all. A pp formula is basically such a system of equations, together with the specification of projecting on to (say) the first n coordinates.

Every finitely presented functor in (mod-R, **Ab**) is a quotient, F/F', of two such subfunctors of some power of the forgetful functor: the model-theoretic terminology corresponding to such a quotient is "pp-pair".

The functorial terminology is better in some regards: a formula is really a presentation rather than the functor being presented and for theorems (as opposed to calculations) one does not usually need to refer to presentations.

For more explanation, or for other terms, see the introductions to various of the references or, e.g. the expository paper [40]. A book, [41], on all this, and more, is in preparation but, for a fast introduction (as opposed to a comprehensive treatment), the existing literature is better.

I finish by mentioning some relevant papers in connection with derived and triangulated categories, namely [3] where the rep-Zariski spectrum appears in connection with the spectrum of the cohomology ring of the group algebra of a finite group, [26] where the Ziegler spectrum (and hence, implicitly the rep-Zariski spectrum) for compactly generated triangulated categories is defined and [13] and its successors [14], [15], [12], where the relation between the rep-Zariski and Ziegler spectra is exploited (in particular 6.4 is used).

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