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The term “inverse problems” is itself a curious one as it does not have a tight mathematical definition. Nevertheless a very large class of what are generally agreed to be inverse problems involves going the *opposite way to nature*. We think of a “forward problem” as being a mathematical model for a problem solved naturally in the physical world. Given a body with a specified spatially varying conductivity one can apply a current density at the boundary and the resulting potential can be found by solving a Neumann problem for an elliptic partial differential equation. This problem is “solved” in the physical world and the voltage is well defined (up to a constant) and depends in a stable way on the conductivity and current. As our Neumann problem for an elliptic partial differential equation is a good model for the physical situation it shares these properties of existence, uniqueness and stability of solution. An example of an inverse problem would be to take measurements of the voltage at the boundary arising from the application of a number of patterns of current density on the boundary and attempt to deduce the conductivity in the interior. This problem often called Electrical Impedance Tomography (EIT: see Box) has application in medical diagnosis, industrial process monitoring and geophysical exploration.

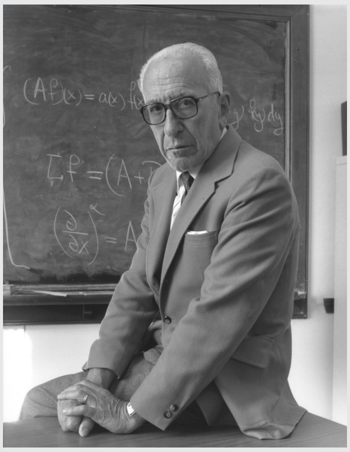
Another archetypal inverse problem concerns solving the heat equation backwards in time. There the temperature $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

on some domain Ω and we suppose it to be specified on the boundary of some domain, say $u = 0$ on $\partial\Omega$, and the temperature in the domain is known at some specif time T, $u(x,T) = f(x)$. One then seeks to deduce the temperature $u(x,0)$ in the domain at time zero. As time increases solutions of the heat equation become spatially smoother, and it is not surprising that solving backwards in time is unstable. The problem is closely related to de-blurring a blurred optical image, such as an out of focus photograph.

Box 1: Electrical Impedance Tomography

In electrical impedance tomography a domain Ω in two or three dimensional space has a (possibly complex) conductivity σ with $\mathrm{Re}(\sigma) > C > 0$. In an idealized case one can apply any current density j on the boundary $\partial\Omega$ and measure the potential $\phi|_{\partial\Omega}$ where
$\nabla \cdot \sigma \nabla \phi = 0$
with the boundary conditions $\sigma \partial\phi/\partial\mathbf{n} = j$ and $\int_{\partial\Omega} \phi = 0$. Here $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. The data can be considered as the Neumann-to-Dirichlet map $R_\sigma : j \mapsto \phi|_{\partial\Omega}$. The *Forward Problem* is then $\sigma \mapsto R_\sigma$, and the inverse problem to recover (information about) σ from (a sampled version of) of R_σ).



Alberto Calderón. 1920-1998

In mathematics it is often called 'the Calderón problem' after Alberto Calderón one of the major figures in analysis of PDEs in the 20th century. A conference paper in Brazil, for him a small side issue to his main work, prompted a large body of work on uniqueness of solution of this problem reviewed in [1] and still with interesting open problems. The importance of the mathematical theory underpinning practical applied problems cannot be underestimated. Also practical problems give rise to deep theoretical challenges and it is of note that Calderón, a key figure in mathematical analysis in the last century, originally trained as an engineer.

When talking to practically oriented engineers and experimental scientists they are inclined to think that questions of existence and uniqueness of solution, for example for boundary value problems for partial differential equations, are a somewhat esoteric pass-time of pure mathematicians. They know that physically there is a solution, and they can observe that it depends stably on parameters that they can vary. Of course there are some exceptions, such as chaotic dynamical systems, but they stand out as exceptions to a general rule. Hadamard declared[2] that a well posed problem should have a solution, that solution should be unique and solution should depend continuously on the parameters or coefficients. From the view point of a practical person considering a forward problem a failure of any of Hadamard's three conditions would mean that the model was incorrect. But the inverse problem is typically of human construction. We have chosen to try to find the conductivity from a knowledge of certain boundary data. The questions of existence of solution is then, given that our model is correct, is simply the same as asking if there is a conductivity consistent with our data. If there is not it is an indication of the inaccuracy of our data. The question of uniqueness of solution put another way is to ask if we have gathered sufficient data to be able to determine our unknowns unambiguously. Stability of solution is a serious issue, and many inverse problem http://badmanstropicalfish.com/abc/range6.html

Figure 1: Peter's Elephantnose fish (*Gnathonemus petersii*) a weakly electric fish that locates its prey using its own natural EIT system. (Copyrighted image not in e-print version see for example link above)

arising in applications are unstable. Often the underlying cause is that the forward problem is governed by a compact operator (also called completely continuous) between function spaces. The inverse of such an operator, if it exists, is unbounded. A consequence of this fact of functional analysis is that the art of solving inverse problems in industrial or other practical situations often involves a close collaboration between that mathematician and the people who presented the problem. One has to initiate a dialogue to establish what is already known that can safely be included as an *a priori* assumption, details of the accuracy of the measurements made, and importantly what is actually required (as this is often less than they initially ask for). On this last point an example in practice might be that a physician asks for a detailed picture of the internal organs of the thorax, not possible from electrical measurements from a few electrodes, but actually they might be happy with a few useful parameters indicating the volume of air in a small number of regions of the lung, which is much more feasible. For details of medical EIT see [3]. Typical industrial applications of EIT and its close relatives include monitoring flow in pipes, mixing of fluids in process vessels and precipitate in large filter. Again in these cases what is required is less than an accurate image, but might be an identification of the flow regimes, the uniformity of the mixture or the distribution and depth of the precipitate. For up-to-date details of process tomography see the proceedings [4].

I have given the impression that EIT is a human invention but as an aside I would like to point out that something very similar is done by weakly electric fish in the murkey waters of the Amazon and Zambezi rivers. They use a sophisticated electrical sensing system to locate their prey and to avoid predators. One widely studied example is the Peter's Elephantnose fish (*Gnathonemus petersii* see figure), which is sufficiently common an aquarium fish that you might easily be able to visit one in captivity. The fish has an a dipole current source in its body and hundreds of voltage sensors on its surface. The prey has a conductivity different to the water and the fish essentially does EIT, except that instead of multiple current sources it swims around using its single dipole source[5]. It would be interesting to understand how their brain processes the electrical data to solve the inverse problem, and indeed what assumptions about their surroundings are implicitly used.

Although many inverse problems arising in practical situations are nonlinear, often a linear approximation is at least a good start and in many cases adequate. Frequently the linear problem is equivalent to the solution of a first kind integral equation of the form

$$g(x) = \int_a^b k(x,y)f(y)\,dy$$

where some discrete sampled version of g is measured and we are to deduce information about f. For a continuous kernel function k the integral operator will be a compact operator on $L^2(a,b)$ and so if it has an inverse that inverse will be unbounded. This is one of the common ways in which an inverse problem violates Hadamard's third condition. After discretization, for example using a quadrature rule, one obtains linear system

$$\mathbf{g} = \mathbf{K}\mathbf{f}$$

where now $\mathbf{g}, \mathbf{f}$ are vectors and $\mathbf{K}$ a matrix. The illposed nature of the continuum version of the inverse problem reveals itself in its discrete counterpart in the illconditioning of the matrix $\mathbf{K}$. It is interesting that we generally teach our engineering and science undergraduates to invert square matrices when in so many practical situations the need arises to deduce some parameters of a system from some measurements that are linear combinations of those parameters. All experimentalists are well aware that it would not be a good idea to attempt to deduce one parameter from only one measurement so almost instinctively would take several measurements. When faced with a system of equations the common case then is that it is overdetermined as more measurements have been made than there are parameters to be identified. The over determined, and hence in the presence of errors in the measurement, inconsistent case is therefore very commonly encountered so there is a very good case for educating engineers and scientists in the treatment of such problems. The simplest approach of course is to find the least squares solution, that is the $\mathbf{f}$ that minimizes

$$||\mathbf{K}\mathbf{f} - \mathbf{g}||^2$$

which is in the case of $\mathbf{K}$ an $m \times n$ matrix of rank n, $m \geq n$, is given by

$$\left(\mathbf{K}^T\mathbf{K}\right)^{-1}\mathbf{K}^T\mathbf{g}.$$

Ideally the problem of estimating $\mathbf{f}$ from noisy $\mathbf{g}$ should be treated statistically, but it is easy to justify the least squares solution as a maximum likelihood estimate assuming identically distributed, zero mean, Gaussian noise in the components of the data vector g_i. While this technique is not generally taught to engineers, and I argue that it should be, along with some elements of the statistics of dealing with multivariate data. Fortunately many engineers are familiar with programs such as MATLAB with a built in least squares solver that is easy to use. So now the case where $\mathbf{K}$ is illconditioned the simplest thing to is to replace the least squares minimization by the minimization of the function

$$||\mathbf{K}\mathbf{f} - \mathbf{g}||^2 + \alpha^2||\mathbf{L}\mathbf{f}||^2$$

where the matrix $\mathbf{L}$ controls the smoothness of the solution (for example a first order difference operator) and the *regularization parameter* α controls the trade-off between fitting the data and having a wildly oscillating or unbounded solution. This idea is sufficiently simple that it often reinvented, but in the mathematical literature it is known as Tikhonov-Phillips regularization, and in the statistical literature as Ridge Regression. Many practical inverse problems that arise in industry can be satisfactorily solved by this simple method, which has an explicit solution given by

$$\left(\mathbf{K}^T\mathbf{K} + \alpha^2\mathbf{L}^T\mathbf{L}\right)^{-1}\mathbf{K}^T\mathbf{g}.$$

Regularization methods such as this appear rather ad-hoc. A more rigorous approach is probabilistic. The error in the data is treated as a random variable, and the *a priori* knowledge about the unknown vector is also represented by a probability distribution (the prior distribution). Bayes' theorem is then used to calculate the *posterior distribution*, the probability distribution of the solution given the prior and the data. In a simple case in which the prior is multivariate Gaussian with a covariance matrix proportional to $(L^T L)^{-1}$ and the errors are Gaussian with covariance a scalar multiple of the identity matrix, the Tikhonov regularized solution is then the maximum of the posterior distribution. The probabilistic approach gives much more as one can assign probabilities to sets of solutions, and can then assign probabilities to answers to important questions such as "does this person have breast cancer" or "is there a foreign body in this food". The book of Tarantola[6] gives an excellent introduction to regularization and the probabilistic approach to inverse problems.

Box 2: A.N. Tikhonov


A.N. Tikhonov, 1906-1993

While the famous Russian mathematician A.N. Tikhonov is perhaps best known for his contribution to functional analysis and in inverse problems for his eponymous regularization method it is much less well known that also worked directly on industrial problems, including electrical prospecting a precursor of geophysical EIT. He contributed to the location of a large copper ore deposit in the Soviet Union.

Other examples of industrial inverse problems that I have been involved with include the determination of the depth dependence of the director field in a liquid crystal cell from polarized light measurements [7], capacitance tomography for monitoring flow in pipes [9], and a problem in which the ocean current is estimated from the position of a cable towed behind a geophysical survey ship [8]. In all three cases Tikhonov-Phillips regularization played an important part in their solution.

For a mathematician working with industry often a little help, such as teaching engineers to do basic regularization methods, goes a long way. Indeed it is often the case that it helps early on in ones relationship with an industrial partner to give them something that results in a noticeable improvement in what they aim to do. This builds trust and mutual respect often leading to a collaboration that reveals some more difficult, and to the mathematician more interesting, problems. Unfortunately many people from outside mathematics, be they from industry, academic science or medicine rather expect that if they go to a mathematician with their problem they (1) will be humiliated for not knowing the problem is trivial (2) will explain the problem to the mathematician who will then go on to solve another more "interesting" problem or (3) will explain the problem and the mathematician will solve it, but the solution will be unintelligible. One suspects that these fears are not entirely the result of prejudice. Successful partnerships with industry and indeed with other application communities are founded on a dialogue between the mathematician and the owner of the problem in which the mathematical formulation of the problem is refined and the solution communicated in an accessible form. This takes time and effort and is not what all mathematicians want to do. But the rewards in the end include the satisfaction of solving a problem that makes a practical difference.

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