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An inverse boundary value problem
for harmonic differential forms

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Abstract. We show that the full symbol of the Dirichlet to Neumann map of the k-form Laplace's equation on a Riemannian manifold (of dimension greater than 2) with boundary determines the full Taylor series, at the boundary, of the metric. This extends the result of Lee and Uhlmann for the case $k = 0$. The proof avoids the computation of the full symbol by using the calculus of pseudo-differential operators parametrized by a boundary normal coordinate and recursively calculating the principal symbol of the difference of boundary operators.

1. Introduction

While there many results on the uniqueness of recovery of the coefficients of an elliptic partial differential equation from boundary data in the case of a single partial differential equation, there are few results for systems of PDEs. One might expect that the complete boundary data for a system might be sufficient to recover multiple coefficients and yet the results to date have been in essentially scalar cases. Lee and Uhlmann showed that the full Taylor series, at the boundary, of a metric can be obtained from the total symbol of the Dirichlet to Neumann map of the scalar Laplace's equation. One might expect to obtain at least the same information from the Dirichlet to Neumann map associated with the Laplacian operator on k-forms. The 1-form, or vector Laplacian on 3-manifolds being the example with the most obvious applications. We will show here that the full symbol of the k-form Laplacian does indeed determine the Taylor series of the metric at the boundary, and hence under suitable assumptions an analytic metric can be recovered from this data. The method used, in common with [4], avoids the computation of the full symbol of the Dirichlet to Neumann map. Rather, by using the calculus of pseudo-differential operators parametrized by a boundary normal coordinate, we recursively calculate the principal symbol of the difference of boundary operators, checking that it vanishes to a suitable order. As well as being an natural extension of the scalar case and interesting in its own right, we hope that this paper will stimulate the use of the technique in other inverse boundary value problems for elliptic systems of equations of interest to applications.

The context for all will be a smooth compact orientable manifold with boundary M, equipped with a Riemannian metric g. We also assume dim $M = n > 2$. The metric tensor induces a volume form $\mu \in \Omega^n(M)$ and Hodge star isomorphism $*: \Omega^k(M) \to \Omega^{n-k}(M)$ is defined by the property

$$*\omega \wedge \omega = g(\omega, \omega)\mu, \tag{1.1}$$

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where the action of the metric is extended naturally to act on k-forms. We can consider the Hodge star on k-forms as a contraction of the tensor $g^{\sharp \otimes k} \otimes \mu$. Here $g^\sharp$ is the covariant metric tensor.

The total symbol of an operator P on functions on $\mathbb{R}^n$ is

$$p(x, \xi) = e^{-ix \cdot \xi} P(e^{ix \cdot \xi}).$$

A classical pseudo-differential operator of order m has a full symbol which is an asymptotic sum of terms $p_{m-j}(x, \xi)$ which are smooth in $\xi \neq 0$ and for $\lambda > 0$ are homogeneous of degree $m - j$

$$p_{m-j}(x, \lambda \xi) = \lambda^{m-j} p_{m-j}(x, \xi).$$

The principal symbol is p_m also denoted by $\sigma_m(P)$. The class of classical pseudo-differential operators is denoted by $\Psi\mathrm{DO}^m_{\mathrm{cl}}(\mathbb{R}^n)$. There are more general classes of pseudo-differential operators based on more general symbols, but we shall not need them here. These classes form a graded algebra under composition

$$\circ : \Psi\mathrm{DO}^m_{\mathrm{cl}} \times \Psi\mathrm{DO}^{m'}_{\mathrm{cl}} \to \Psi\mathrm{DO}^{m+m'}_{\mathrm{cl}}.$$

To obtain the principal symbol of the composite one just takes the product:

$$\sigma_{m+m'}(PQ) = \sigma_m(P)\sigma_{m'}(Q)$$

however the full symbol of the product is rather more complicated. Operators in $\Psi\mathrm{DO}^{-\infty}_{\mathrm{cl}} = \bigcap_{m \in \mathbb{R}} \Psi\mathrm{DO}^m_{\mathrm{cl}}$ are called smoothing operators. The full symbol of a pseudo-differential operator determines the operator modulo smoothing operators. For brief introduction to pseudo-differential operators we recommend the notes [3] and for more detail Shubin [10]. We note that the definition of pseudo-differential operators can be extended to smooth manifolds using coordinate charts. Here the principal symbol is invariantly defined as a function on the cotangent bundle while the total symbol depends on choice of coordinates.

Following [4] we will consider pseudo-differential operators on a smooth manifold Y depending smoothly on a parameter t. For our purposes we will have $Y = \partial M$ and t the normal distance from the boundary. We say that $P \in \Psi\mathrm{DO}^{m,r}(Y, \mathbb{R}^+)$ if it is a family of pseudo-differential operators of order m on Y, varying smoothly up to $t = 0$, and such that

$$P = \sum_{j=0}^{r} t^{r-j} P_j$$

with P_j a smooth family of operators on Y of order $m - j$. This definition extends naturally to operators on bundles, in our case the bundle of k-forms being the important example.

The symbol of $P \in \Psi\mathrm{DO}^{m,r}(Y, \mathbb{R}^+)$ is defined to be the vector

$$\left(\sigma_{m-j}(P_j) \right)_{j=0}^{r}$$

evaluated at $t = 0$. This is a vector of functions on the cotangent bundle of Y. For the case of an operator on a vector bundle, each of these functions is a field of enodmorphisms on the fibres of the bundle.

Let u be a 1-form then the Bochner Laplacian (sometimes called the rough Laplacian) is the operator expressed in coordinates as $-\sum_{ij} g^{ij} u_{k;ij}$. The principal symbol in this case is gI where I is the identity on 1-forms. The formal adjoint with respect to a metric of the exterior derivative on k-forms is $\delta = (-1)^{(nk+n+1)} * \mathrm{d}*$. The Laplacian on differential forms (or Hodge Laplacian) is then $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$. The principal symbol of d is $\sigma_{\mathrm{d}}^1(\xi)\omega = \mathrm{i}\xi \wedge \omega$. Let $X \lrcorner \omega$ denote the contraction of the differential form ω with respect to the vector field X. We denote by $\xi^\sharp$ the vector field dual to the one form ξ. The principal symbol of δ is then $\sigma_\delta^1(\xi) = -\mathrm{i}\xi^\sharp \lrcorner$. We conclude using that contraction is an antiderivation on forms

$$\sigma_\Delta^2(\xi)\omega = \xi^\sharp \lrcorner (\xi \wedge \omega) + \xi \wedge (\xi^\sharp \lrcorner)\omega = g(\xi,\xi)\omega.$$

The connection between the Laplacian and the Bochner Laplacian, as well as an alternative way to calculate the principal symbol of the former, is given by the coordinate expression for the Laplacian

$$(\Delta u)_{i_1 \ldots i_k} = \sum_{ij} \Big(-g^{ij} u_{i_1 \ldots i_k;ij} + \sum_{\alpha=1}^{k} R_{i_\alpha}^{j} u_{i_1 \ldots i_{\alpha-1} j i_{\alpha+1} \ldots i_k}$$
$$+ \frac{1}{2} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} R^{ij}{}_{i_\beta i_\alpha} u_{i_1 \ldots i_{\alpha-1} j i_{\alpha+1} \ldots i_{\beta-1} i i_{\beta+1} \ldots i_k} \Big).$$

Note that for a flat metric the Laplacian and Bochner Laplacian coincide. A differential form u satisfying Laplace's equation $\Delta u = 0$ is called a harmonic form. On a compact manifold without boundary, this is equivalent to the condition that the form is a harmonic field, that is it is both exact, $\mathrm{d}u = 0$, and co-exact, $\delta u = 0$ as

$$\langle u, \Delta u \rangle = \|\mathrm{d}u\|^2 + \|\delta u\|^2 + \int_{\partial M} \delta u \wedge *u + \int_{\partial M} u \wedge *\mathrm{d}u. \tag{1.2}$$

However on manifolds with boundary there can be harmonic forms which are not harmonic fields.

Closely related systems of elliptic partial differential equations occur in electro-magnetics (the vector Helmholtz equation) and in linear elasticity.

In a linear elastic solid with metric tensor g and with no body forces, the displacement field u (a vector field) satisfies the equation $\mathrm{Div}(C L_u g) = 0$ where $u \mapsto L_u g$ is the Lie derivative of the metric which in components is $(L_u g)_{ij} = u_{i;j} + u_{j;i}$ (as usual a semi-colon indicates covariant differentiation with respect to following indices) and Div is its formal adjoint $a_{ij} \mapsto \sum_{jk} a_{ij;k} g^{jk}$. (All sums will be indicated explicitly.) The elastic tensor C is a field of automorphisms of the symmetric tensors on each fibre. The principal symbol of the elastic operator is C. For an isotropic solid $C = \lambda g \otimes g^\sharp + \mu I$ where I is the identity operator on symmetric tensor fields. The problem considered by [7] was the recovery of the Lamé parameters λ and μ for an isotropic solid. They also considered a related anisotropic problem for two-dimensional elastic media.

Nakamura and Uhlmann [8] derive a factorisation for the anisotropic linear elasticity operator in boundary normal coordinates (for the flat metric). This allows them to recover the full Taylor series of the 'surface impedance tensor', which is a function of C, but not the complete Taylor series of C. For a special case, transversely isotropic media, C can be recovered [9].

In electro-magnetic theory the electric field E and magnetic field H are naturally defined as 1-forms, as to take measurements of these fields one must integrate over curves. The resulting electric and magnetic fluxes, D and B are naturally two forms as one must integrate them over surfaces to make a

measurement. The material properties (for simplicity we consider a non-chiral, linear, insulating material) are the permittivity ε and permeability μ, these map one forms to two forms and are the Hodge star operators for an associated electric and magnetic Riemannian metric. Assuming all fields to be time harmonic with angular frequency ω and the electric charge density to be constant we have Maxwell's equations

$$dB = 0, \qquad dD = 0, \tag{1.3}$$

$$dE = -\mathrm{i}\omega\mu H, \qquad D = \varepsilon E, \tag{1.4}$$

$$dH = \mathrm{i}\omega\varepsilon E, \qquad B = \mu H. \tag{1.5}$$

In a physically artificial situation where $\mu = \varepsilon = *$ (obviously after units have been scaled) we notice that E and H salsify the vector Helmholtz equations $\Delta E = \omega^2 E$ and $\Delta H = \omega^2 H$.

The main result we prove is an extension to k-forms for $k > 0$, of the result of Lee and Uhlmann [6] using a similar factorization in boundary normal coordinates. Initially, our notion of a Dirichlet-to-Neumann map is non-standard for $k \neq 0$. Employing the multi-index notation $I = (i_1, \ldots, i_k)$ we write a k-form as $u = \sum_I u_I \, dx_I$ where $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$. Following [6] we use a coordinate chart $(x_1, \ldots, x_n) = (x', x_n)$ where $x' = (x_1, \ldots, x_{n-1})$ is a chart on the boundary, and x_n is the distance to the boundary. We denote by ∂_n both a unit vector field normal to the boundary and its associated derivation on functions. We extend this to k forms by its actions on the components as functions $\partial_n u = \sum_I \partial_n u_I \, dx_I$. The operator $\Lambda_g : u|_{\partial M} \mapsto (\partial_n u)|_{\partial M}$, where $\Delta u = 0$, is linear while somewhat unnatural. We will discuss the relationship to more natural Neumann data for Laplace's equation in Section 2. Our main result is

Theorem 1.1. *Let M be a smooth compact orientable Riemannian manifold with boundary, with $\dim(M) = n > 2$ and metric g, and let k be an integer $0 \leqslant k \leqslant n$.*

(i) *The Dirichlet-to-Neumann map $\Lambda_g : u|_{\partial M} \mapsto (\partial_n u)|_{\partial M}$ for the k-form Laplace's equation $\Delta u = 0$ is a classical pseudo-differential operator of order one.*

(ii) *The Taylor series, at the boundary, of the metric in boundary normal coordinates is uniquely determined by the full symbol of Λ_g. For $0 < k < n$ only one diagonal component of the full symbol is needed corresponding to $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ but for $k = (n+1)/2$ the multi-index $I = (i_1, \ldots, i_k)$ must exclude n and for $k = (n-1)/2$, I must include n.*

Where our work differs from Lee and Uhlmann's is in its use of families of operators parameterized by the normal distance. The case $k = n$ is clearly equivalent to $k = 0$ so we need only consider the case $0 < k < n$. Lee and Uhlmann showed that the full Taylor series of a real analytic metric on a real analytic manifold, where the relative homotopy group of the boundary $\pi(M, \partial M)$ is trivial, determines the metric, provided the manifold is strongly convex or the metric can be extended analytically to a larger manifold without boundary. Recent work of Lassas and Uhlmann [5] removes these geometric and topological hypotheses showing that an analytic metric is determined throughout any connected analytic manifold by the Dirichlet-to-Neumann map. It remains to be seen if the same techniques can be applied to the general k-form case.

Only a small modification of the argument is needed to prove a version of Theorem 1.1 for the equivalent k-form Helmholtz problem at fixed frequency.

2. Boundary conditions

In a neighbourhood of the boundary where our boundary normal coordinates are defined, we can distinguish between *tangential* k-forms which have no $\mathrm{d}x_n$ in their coordinate expression, and *normal* forms which have a common factor $\mathrm{d}x_n$. Clearly the space of k-forms on this neighbourhood is the direct sum of the spaces of tangential and normal forms. The projection on to the tangential component is $\pi_{\mathrm{t}}(\omega) = \partial_n \lrcorner (\mathrm{d}x_n \wedge \omega)$ and on to the normal component $\pi_{\mathrm{n}}(\omega) = \mathrm{d}x_n \wedge (\partial_n \lrcorner \omega)$. From these expressions it is clear that $*\pi_{\mathrm{n}} = \pi_{\mathrm{t}}*$ and that the Hodge star of a tangential form is normal and *vice-versa*.

Let $i : \partial M \to M$ be the inclusion of the boundary. The tangential component of k-form $u \in \Omega^k(M)$ at the boundary is then the pull-pack to the boundary $i^*u \in \Omega^k(\partial M)$. The normal part of u at the boundary, can be determined uniquely from $i^* * u = *_\partial (\partial_n \lrcorner u)|_{\partial M} \in \Omega^{n-k}(\partial M)$ where $*_\partial$ is the induced Hodge-star on the boundary. Dirichlet data for harmonic k-forms consists of the both the tangential component and the normal component of the form at the boundary [1]. Note a possible source of confusion here. When considering forms on manifolds with a boundary i^*u is often thought of as the restriction of form to the boundary, and the pull back is often omitted, for example in Stokes' theorem. In Theorem 1.1 our Dirichlet data $u|_{\partial M}$ is the restriction to the boundary of a form, but in the sense of considering only base points on ∂M and no projection of the fibre on to the tangential component. This Dirichlet data together with the integrals of u on a basis of the relative homology group $H_k(M, \partial M)$ determines a unique solution to $\Delta u = 0$. For simplicity we will assume that the said integrals are specified to be zero. We note that for the case $k = 0$ one simply specifies the integral of u on each connected component of M with a non-empty boundary. The natural Neumann data [1] is the specification of $i^* * \mathrm{d}u$ and $i^* \delta u$. Note that as in the case $k = 0$ where Neumann data $i^* * \mathrm{d}u$ must have zero integral on the boundary, there are compatibility conditions for Neumann data [1]. A natural Dirichlet-to-Neumann mapping is therefore $\Pi_g : \Omega^k(\partial M) \times \Omega^{n-k}(\partial M) \to \Omega^{n-k-1}(\partial M) \times \Omega^{k-1}(\partial M)$ given by

$$\Pi_g(f_\tau, f_\nu) = (i^* * \mathrm{d}u, i^* \delta u),$$

where $\Delta u = 0$, $i^*u = f_\tau$, $i^* * u = f_\nu$. Here we use ν and τ as labels for the normal and tangential components in the sense of the domain and range of Π_g. We can now recast Theorem 1.1 in terms of this natural data.

Corollary 2.1. *Let M be a smooth compact orientable Riemannian manifold with boundary, with $\dim(M) = m > 2$ and metric g, and let k be an integer $0 \leqslant k \leqslant n$.*

  (i) *The natural Dirichlet-to-Neumann map Π_g for the k-form Laplace's equation defined above is a classical pseudo-differential operator of order 1.*
 (ii) *The full symbol of Π_g determines the Taylor series (at the boundary) of the metric in boundary normal coordinates. Furthermore for $k \notin \{(n-2)/2, (n-1)/2, n\}$ only the full symbol of the tangential part $\Pi_{g\,\tau\tau}$ is needed and for and $k \notin \{0, (n+1)/2, (n+2)/2\}$ only the full symbol of the normal part $\Pi_{g\,\nu\nu}$ is needed.*

The proof of Corollary 2.1 follows the proof of Theorem 1.1 in Section 3.

In the case of electromagnetics note that when formulated in differential forms Maxwell's equations (1.3), (1.4) are independent of the ambient Euclidean metric and thus invariantly defined. In the inverse boundary value problems for time harmonic Maxwell's equations one typically specifies the

boundary data invariantly as the tangential component of E and H. The isotropic case where both the electric and magnetic metrics are conformally flat, has been studied by [11,4].

By contrast in elasticity strain is a measure of the distortion of the Euclidean metric, and one seeks the elastic tensor with the ambient metric given. This problem is not diffeomorphism invariant.

3. Factorization and symbol calculation

We consider metrics to be equivalent if they are related by a diffeomorphism which fixes points on the boundary. Without loss of generality, therefore, we can assume that x_n is the boundary normal coordinate for both metrics. Later we will make a more specific choice for the coordinate chart on the boundary.

We use notation from [4], in particular $\Psi\mathrm{DO}^{m,l}$ denotes families of pseudo-differential operators, P_{x_n}, in x' such that the j term in the total symbol vanishes to order $l - j$ at $x_n = 0$, and $\mathrm{DO}^{m,r}$ is the class of such differential operators.

Let $\Delta' = \sum_{i,j=1}^{n-1} h^{ij} D_{x_i} D_{x_j}$. We have that

$$\Delta = \left(D_{x_n}^2 + \Delta'\right)I + E D_{x_n} + H(x, D_{x'}),$$

where H is a first order system in x' on the bundle of k-forms and E is a smooth endomorphism of the bundle of k-forms. We use the notation $|\xi| = \sqrt{g(\xi, \xi)}$ for a covector ξ.

Proposition 3.1. *There exists a $B(x, D_{x'}) \in \Psi\mathrm{DO}_{\mathrm{cl}}^1(\partial M; \mathbb{R}; \Omega^1(M))$ such that $\sigma_1(B) = |\xi'|_x I$ and*

$$\Delta = (D_{x_n} I + E + \mathrm{i}B)(D_{x_n} I - \mathrm{i}B),$$

modulo smoothing and B is unique modulo smoothing.

Proof. If we expand, we obtain

$$D_{x_n}^2 I + B^2 + E D_{x_n} - \mathrm{i}EB + \mathrm{i}[D_{x_n}, B].$$

Taking the principal symbol of B as specified we obtain an error, in $\Psi\mathrm{DO}_{\mathrm{cl}}^1$. Now suppose we have chosen B_j such that the error, F_j, is in $\Psi\mathrm{DO}_{\mathrm{cl}}^{1-j}$. Let $B_{j+1} = B_j + C$ with C in $\Psi\mathrm{DO}_{\mathrm{cl}}^{-j}$. Upon expanding we then obtain an extra term $CB_j + B_j C + E_j$ with E_j of order $-j$. Taking $\sigma_{-j}(C) = -\frac{1}{2}|\xi'|_x^{-1}\sigma_{1-j}(F_j)$. We have achieved an error one order better. Inducting and summing, we achieve an error in $\Psi\mathrm{DO}_{\mathrm{cl}}^{-\infty}$.

As the choice at each stage was forced, B is unique. $\square$

The importance of this factorization is that $B(0, D_{x'})$ is equal modulo smoothing terms to Λ_g. We will summarise the argument which is identical to that given by [6] for the 0-form case. Given a harmonic k-form u, let $v = (D_{x_n} I - \mathrm{i}B)u$ so that $(D_{x_n} I + E + \mathrm{i}B)v = 0$. These are both generalised heat equations, the second with 'time' reversed. As both are smoothing we see that $\partial_n u = Bu + \text{smooth terms}$ and hence $\Lambda_g = B \bmod \Psi\mathrm{DO}_{\mathrm{cl}}^{-\infty}$. This proves Theorem 1.1 part (i) and part (ii) will follow if we can show that two metrics g_1, g_2 with identical full symbols of B at the boundary must agree to infinite order at the boundary. Rather than calculating the full symbol of B, we use the calculus of pseudo-differential operators parameterised by the normal distance. The advantage is that we need only calculate principal symbols.

We want to compare the Laplacians associated to two different metrics assumed equal up to order l in the normal coordinate. Of course it is immediate from the principal symbol of B that the metrics agree on the boundary so we can take $g_1 - g_2 = x_n^l k$ for some $l > 0$.

First we compare the Hodge star operators. By definition $\omega \wedge *\omega = g(\omega, \omega)\mu$ and we see that

$$*_1 = *_2 + x_n^l \alpha,$$

where α is a smooth homomorphism from Ω^k to Ω^{n-k}.

Lemma 3.1. *If Δ_j is the Laplacian on k-forms associated with g_j then*

$$\Delta_2 - \Delta_1 = x_n^{l-1} F D_{x_n} + A,$$

where F is a smooth endomorphism and $A \in \mathrm{DO}^{2,l}$.

Proof. The Laplacian is defined by $*\mathrm{d} * \mathrm{d}$ and $\mathrm{d} * \mathrm{d}*$ where d is independent of the metric and $*_2 = *_1 + x_n^l \alpha$.

The result follows simply by observing that in $\mathrm{d} *_2 \mathrm{d} *_2$ and $*_2 \mathrm{d} *_2 \mathrm{d}$ terms not in Δ_1 will vanish to order l at $x_n = 0$ unless d is applied to the x_n^l term. If d is applied once to such a term we obtain a first order differential operator vanishing to order $l-1$ and if twice a zeroth order operator vanishing to order $l-2$. This is the result desired – we know there are no second order terms in D_{x_n} from our expression for the principal symbol. □

Lemma 3.2. *Let Δ_j be factored as in Proposition 3.1 with E_j, B_j the corresponding terms. We then have that,*

$$B_2 - B_1 \in \Psi\mathrm{DO}^{1,l}.$$

Proof. Let $C = B_2 - B_1$. As the principal symbols of B_1, B_2 agree at $x_n = 0$ so we have that C is in $\Psi\mathrm{DO}^{1,1}$.

Note that $E_2 = E_1 + x_n^{l-1} F$. Expanding the factorizations for Δ_2, Δ_1 and subtracting we have that,

$$\Delta_2 - \Delta_1 = \mathrm{i}[C, D_{x_n}] + B_1 C + C B_1 + x_n^{l-1} F D_{x_n} - x_n^{l-1}\mathrm{i}F(B + C) + C^2.$$

After cancelling the $x_n^{l-1} F D_{x_n}$, we have that

$$\mathrm{i}[C, D_{x_n}] + B_1 C + C B_1 - \mathrm{i}x_n^{l-1} F(B + C) + C^2 \in \Psi\mathrm{DO}^{2,l}.$$

If $C \in \Psi\mathrm{DO}^{1,r}$ with $1 \leqslant r < l$ then we have that $[C, D_{x_n}] \in \Psi\mathrm{DO}^{1,r-1}$, $B_1 C + C B_1 \in \Psi\mathrm{DO}^{2,r}$, $C^2 \in \Psi\mathrm{DO}^{2,2r}$, $x_n^{l-1}\mathrm{i}FB \in \Psi\mathrm{DO}^{1,l-1}$, and $x_n^{l-1}\mathrm{i}FC \in \Psi\mathrm{DO}^{1,r+l-1}$. Taking the residue modulo $\Psi\mathrm{DO}^{2,r}$ we have that $\mathrm{i}[C, D_{x_n}] + B_1 C + C B_1$ is congruent to zero modulo $\Psi\mathrm{DO}^{2,r}$. Recall that

$$C = \sum_{j \leqslant r} x_n^{r-j} C_j$$

with $C_j \in \Psi\mathrm{DO}_{\mathrm{cl}}^{1-j}$. The only term of second order is $C_1 B + B C_1$ so we deduce that the principal symbol of C_1 vanishes at $x_n = 0$. Let c_j denote the principal symbol of C_j at $x_n = 0$. We then have that

$$(r - j)c_j + 2|\xi'|_x c_{j-1} = 0$$

for each j. Iterating we conclude that $c_j = 0$ for each j which proves that $C \in \Psi\mathrm{DO}^{1,r+1}$. Repeating, the result follows. $\square$

It follows from this lemma that $B_2 - B_1$ restricted to $x_n = 0$ is C_l a pseudo-differential operator of order $1 - l$. Our main result will follow if we can compute the principal symbol of this operator and show that it determines k, the lead term of $g_1 - g_2$.

Now

$$\Delta_2 - \Delta_1 = x_n^l P_2 + x_n^{l-1} P_1 + x_n^{l-2} P_0 + x_n^{l-1} F D_{x_n}$$

with P_j a differential operator in x' of order j and we know from our principal symbol computation that P_2 is equal to $\sum \tilde{k}_{ij} D_{x_i} D_{x_j}$ where $\tilde{k} = -h^{-1}kh^{-1}$ where $g = \mathrm{d}x^2 + h(x', \mathrm{d}x') + \mathcal{O}(x_n)$.

Arguing as above with $C = B_2 - B_1$ we have

$$\mathrm{i}[C, D_{x_n}] + B_1 C + C B_1 - \mathrm{i}x_n^{l-1} F(B + C) + C^2 = x_n^l P_2 + x_n^{l-1} P_1 + x_n^{l-2} P_0. \tag{3.1}$$

So modulo $\Psi\mathrm{DO}^{2,l}$ we have,

$$\mathrm{i}[C, D_{x_n}] + B_1 C + C B_1 - \mathrm{i}x_n^{l-1} F B_1 = x_n^l P_2 + x_n^{l-1} P_1 + x_n^{l-2} P_0. \tag{3.2}$$

Let $C = \sum_{j=0}^{l} x_n^{l-j} C_j$ with $C_j \in \Psi\mathrm{DO}_{\mathrm{cl}}^{1-j}$, let c_j be the principal symbol of C_j. We have,

$$2|\xi'|_x c_0 = \sum_{i,j<n} \tilde{k}_{ij}\xi_i\xi_j,$$

$$2|\xi'|_x c_1 + l c_0 = \sigma_1(P_1)(\xi') + \mathrm{i}|\xi'|_x F,$$

$$2|\xi'|_x c_2 + (l-1)c_1 = \sigma_0(P_0),$$

$$2|\xi'|_x c_{2+j} + (l-j-1)c_{1+j} = 0, \quad \text{for } 1 < j \leqslant l-2.$$

We therefore deduce that

$$c_l = K_l \big(|\xi'|_x\big)^{-l-1} \sum_{i,j<n} \tilde{k}_{ij}\xi_i\xi_j I + L_l|\xi'|^{-l}\sigma_1(P_1)(\xi') + \mathrm{i}L_l|\xi'|_x^{1-l}F + M_l\sigma_0(P_0)|\xi'|_x^{1-l}, \tag{3.3}$$

where K_l, L_l, M_l are computable non-zero constants. (Of course, M_0, $M_1 = 0$.)

We want to show that $c_l = 0$ implies that $\tilde{k}_{ij} = 0$ for all i, j.

If the principal symbol $c_l = 0$, we have taking any component rr of the symbol that

$$K_l \sum_{ij} \sum \tilde{k}_{ij}(x)\xi_i\xi_j + L_l|\xi'|\sigma_1(P_1)_{rr}(x, \xi') + |\xi'|^2\big(\mathrm{i}L_l F_{rr} + C_l\sigma_0(P_0)_{rr}\big) = 0.$$

As P_1 is a differential operator $\sigma_1(P_1)_{rr}(x, \xi')$ is linear in ξ' and the final two terms are independent of ξ'. Since the middle term is not smooth, unless zero, as $\xi' \to 0$ and the other terms are smooth, we deduce that $\sigma_1(P_1) = 0$ and so we have,

$$K_l \sum_{i,j<n} \sum \tilde{k}_{ij}(x)\xi_i\xi_j + |\xi'|^2\big(iL_lF_{rr} + C_l\sigma_0(P_0)_{rr}\big) = 0.$$

This shows that k_{ij} must be a scalar multiple of the identity matrix. Or more invariantly that $k(x)$ is a scalar multiple of $h(x)$. To see that k is actually zero, we need to compute more precisely.

For convenience we now reduce to the case of a Euclidean background metric. We first prove

Lemma 3.3. *Let*

$$g_1 = h + x_n m,$$
$$g_2 = h + x_n m + x_n^l r,$$

and let $*_j$ be the Hodge $*$ operator associated to g_j. We then have that $*_1 - *_2$ modulo $\mathcal{O}(x_n^{l+1})$ is independent of m.

Proof. Let μ be a volume form of h and μ_j a volume form for g_j. We have by definition $h(\mu, \mu) = 1$ and $g_j(\mu_j, \mu_j) = 1$. We then have that $\mu_2 = (1 + x_n m(\mu, \mu) + x_n^l r(\mu, \mu))^{-1/2}\mu$ and similarly for g_1. It is now clear that

$$\mu_2 = \frac{1}{\sqrt{1 + x_n m(\mu, \mu) + x_l^l r(\mu, \mu)}}\mu$$

and

$$\mu_1 = \frac{1}{\sqrt{1 + x_n m(\mu, \mu)}}\mu.$$

For any $\nu, \omega \in \Omega^k(M)$

$$\nu \wedge (*_2 - *_1)\omega = \frac{(h + x_n m + x_n^l r)(\nu, \omega)}{\sqrt{1 + x_n m(\mu, \mu) + x_n^l r(\mu, \mu)}} - \frac{(h + x_n m)(\nu, \omega)}{\sqrt{1 + x_n m(\mu, \mu)}}$$

$$= \left(x_n^l\left(r(\nu, \omega) - \frac{1}{2}r(\mu, \mu)h\right) + \mathcal{O}(x_n^{l+1})\right)\mu$$

which does not involve m. $\square$

We also have,

Lemma 3.4. *Let $g_2 = dx^2 + x_n^l r + m$, and $g_1 = dx^2 + m$, where m vanishes to second order at the origin. Let $*_j$ be the Hodge $*$ operator of g_j. We then have that $*_2 - *_1$ is independent of m modulo terms of the form $x_n^l t + x_n^{l+1}w$ where t vanishes to second order at the origin and w is smooth.*

Proof. Let μ be the volume form for dx^2. Arguing as above we have that,

$$\nu \wedge (*_2 - *_1)\omega = \frac{(dx^2 + m + x_n^l r)(\nu, \omega)}{\sqrt{1 + m(\mu, \mu) + x_n^l r(\mu, \mu)}} - \frac{(dx^2 + m)(\nu, \omega)}{\sqrt{1 + m(\mu, \mu)}}, \tag{3.4}$$

which modulo terms vanishing appropriately at $x = 0$ equals the bilinear form

$$(dx^2 + m + x_n^l r)\left(1 - \frac{1}{2}(m(\mu, \mu) + x_n^l r(\mu, \mu))\right) - (dx^2 + m)\left(1 - \frac{1}{2}m(\mu, \mu)\right), \tag{3.5}$$

which upon expanding modulo appropriately vanishing terms equals,

$$x_n^l r - \frac{1}{2}x_n^l r(\mu, \mu) \, dx^2, \tag{3.6}$$

which does not involve m. $\quad\square$

Now fix a point p where we will calculate the principal symbol of the difference of the Dirichlet to Neumann maps and show that it being zero implies that the next term of the difference of the metrics also vanishes there. We take geodesic normal coordinates about p in the boundary and then extend normally with respect to g_1. Now we fix a point on the boundary p where we will calculate the principal symbol of the difference of the Dirichlet to Neumann maps and show that it being zero implies that the next term of the difference of the metrics also vanishes there. We choose x' to be a Riemann normal coordinate system on the boundary and then extend normally with respect to g_1. In particular, we have

$$g = dx_n^2 + h(x, dx'),$$

and on the boundary $h((x', 0), dx') = dx'^2 + \mathcal{O}(x'^2)$.

We then have, using Lemma 3.4 that,

$$g_1 = dx^2 + m,$$

and

$$g_2 = dx^2 + m + x_n^l r,$$

where $m = x_n^l t$ with t vanishing to second order at $x' = 0$. Let $g_1' = dx^2$, and $g_2' = dx^2 + x_n^l r$. We then have that $*_j - *_j' = x_n \alpha_j + \beta_j$ for $j = 1, 2$ with β_j vanishing at $x' = 0$ to second order. We also have from our lemmas that

$$(*_1 - *_2) - (*_1' - *_2') = x_n^l \gamma$$

with γ vanishing to second order at $x' = 0$.

It is now clear that when computing the lead term of $d *_2 d(*_2 - *_1)$ and its appropriate permutations at the point p that we can replace $*_j$ by $*_j'$, without changing the value. So to finish our theorem we take $g_1 = dx^2$ and $g_2 = dx_n^2 + (1 + x_n^l \lambda(x')) dx'^2$, with λ a smooth function.

Let us consider the action of $*_2$ on normal and tangential forms. For a multi-index $I = (i_1, i_2, \ldots, i_k)$ we use the convention $u_I \, \mathrm{d}x_I = u_{i_1 i_2 \ldots i_k} \, \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_k}$. We will denote by I' the complimentary multi-index with $I' = (j_1, \ldots, j_{n-k})$ where $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is an even permutation of $1, \ldots, n$.

For the metric g_1, $(\partial_1, \ldots, \partial_n)$ is an oriented orthonormal frame and $(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$ an oriented orthonormal co-frame. We get an orthonormal frame for g_2 by dividing each ∂_i by $(1 + \lambda(x')x_n^l)^{1/2}$ except for ∂_n. Similarly $\eta_i = (1 + \lambda(x')x_n^l)^{1/2} \, \mathrm{d}x_i$ for $i < n$ and $\eta_n = \mathrm{d}x_n$ is an orthonormal coframe for g_2. Applying $*_2$ to a typical normal basis k-form $\mathrm{d}x_I$ (with $n \in I$), we have

$$*_2(\mathrm{d}x_I) = \left(1 + \lambda(x')x_n^l\right)^{-(k-1)/2} *_2(\eta_I) = \left(1 + \lambda(x')x_n^l\right)^{-(k-1)/2}(\eta_{I'})$$

$$= \left(1 + \lambda(x')x_n^l\right)^{(n+1)/2-k}(\mathrm{d}x_{I'}).$$

For a typical basis element of $\Omega_t^k(\partial M)$, we have $n \notin I$ and

$$*_2(\mathrm{d}x_I) = \left(1 + \lambda(x')x_n^l\right)^{-k/2} *_2(\eta_I) = \left(1 + \lambda(x')x_n^l\right)^{-k/2}(\eta_{I'}) = \left(1 + \lambda(x')x_n^l\right)^{(n-1)/2-k}(\mathrm{d}x_{I'}).$$

Modulo x_n^{l+1}, we have that on normal k-forms $(*_2 - *_1) = x_n^l((n+1)/2 - k)\lambda *_1$ and on tangential k-forms $(*_2 - *_1) = x_n^l((n-1)/2 - k)\lambda *_1$.

To prove the theorem we only have to consider the action of the difference of the operators on a particular k-form $u_I \, \mathrm{d}x_I$.

We first establish that there is no contribution from

$$*_2\mathrm{d} *_2 \mathrm{d} - *_1\mathrm{d} *_1 \mathrm{d} = (*_2 - *_1)\,\mathrm{d} *_2 \mathrm{d} + *_1\mathrm{d}(*_2 - *_1)\,\mathrm{d}.$$

Any first order term in the first term of the RHS will be in $\mathrm{DO}^{1,l}$ and therefore not contribute to the principal symbol. For the second term in the RHS, we have,

$$*_1\mathrm{d}(*_2 - *_1)\,\mathrm{d}(u_I \, \mathrm{d}x_I) = *_1\mathrm{d}\left(\left(\frac{n \mp 1}{2} - k\right)x_n^l\lambda(x') *_1 \sum_{j \notin I} \frac{\partial u_I}{\partial x_j} \, \mathrm{d}x_j \wedge \mathrm{d}x_I\right)$$

plus terms involving x_n^{l+1} which will not contribute. Now consider the second d, if it applies to u_I we get a second order term which we already understand; if it applies to x_n we get a term involving $\mathrm{d}x_n$ which will then have no $\mathrm{d}x_n$ component on applying $*_1$ again. Thus we get no sufficiently low order contribution to either the zeroth order term or to the coefficient of D_{x_n}.

Moving on to

$$\mathrm{d} *_2 \mathrm{d} *_2 - \mathrm{d} *_1 \mathrm{d}*_1 = \mathrm{d}(*_2 - *_1)\,\mathrm{d} *_2 + \mathrm{d} *_1 \mathrm{d}(*_2 - *_1),$$

looking at the first term on the RHS we can equally compute with $\mathrm{d}(*_2 - *_1)\mathrm{d}*_1$ as the difference will be in a non-contributory residue class.

First considering the normal case $n \in I$ we compute

$$\mathrm{d} *_1 u = \sum_{j \notin I'} \partial_j u_I \, \mathrm{d}x_j \wedge \mathrm{d}x_{I'}.$$

Now consider

$$d(*_2 - *_1)(\partial_j u_I \, dx_j \wedge dx_{I'}) = \left(\frac{n \pm 1}{2} - k\right) d(x_n^l \lambda \partial_j u_I *_1 (dx_j \wedge dx_{I'}))$$

$$= \left(\frac{n \pm 1}{2} - k\right)(l x_n^{l-1} \lambda \partial_j u_I \, dx_n + x_n^l \, d(\lambda \partial_j u_I)) \wedge *_1(dx_j \wedge dx_{I'}).$$

Where the $+$ holds for $j = n$ and the $-$ otherwise, but for this to have a dx_I component we must have $j = n$. The second term in the final bracket has a coefficient x_n^l so can only contribute to the second order part which we already understand. We do have a contribution to F from

$$\left(\frac{n+1}{2} - k\right) l x_n^{l-1} \lambda \partial_n u_I \, dx_I. \tag{3.7}$$

We are left with the contribution of $d *_1 d(*_2 - *_1)$. If we apply d to $(*_2 - *_1)(u_I \, dx_I)$, we can drop the terms where d falls on a λ as these are in $\Psi DO^{1,l}$. So on applying $d(*_2 - *_1)$ to $u_I \, dx_I$ we are left with,

$$\left(\frac{n+1}{2} - k\right)\left(x_n^l \lambda \sum_{j \in I} \partial_j u_I \, dx_j \wedge dx_{I'} + (l-1) x_n^{l-1} \lambda u_I \, dx_n \wedge dx_{I'}\right).$$

On applying $*_1$, we get

$$\left(\frac{n+1}{2} - k\right)\left(x_n^l \lambda \sum_{j \in I} \partial_j u_I \, dx_{I_j} + (l-1) x_n^{l-1} \lambda u_I \, dx_{I_n}\right),$$

where I_j is simply I with j deleted.

We now apply d again to get the final contributions. We get another contribution to F identical to Eq. (3.7) so that

$$F = 2il\left(\frac{n+1}{2} - k\right)\lambda$$

and an $x_n^{l-2} u_I$ which gives

$$\sigma_0(P_0) = l(l-1)\left(\frac{n+1}{2} - k\right)\lambda.$$

Substituting back in to Eq. (3.3) we see that if $c_l = 0$ then $\tilde{k}_{ij} = 0$ and Theorem 1.1 is proved for the case for $k \neq (n+1)/2$. Now we consider the case of tangential data that is $u_I \, dx_I$ with $n \notin I$.

A similar argument applies to a tangential form, with $k \neq (n-1)/2$ and Theorem 1.1 is proved.

Proof of Corollary 2.1. First observe that given the induced metric on the boundary the data $(i^*u, i^* *u)$ determines $u|_{\partial M}$. In a neighbourhood's of the boundary a k-form u can be expressed as

$$u = \sum_{|I|=k, \, n \notin I} u_I \, dx_I + \sum_{|J|=k-1, \, n \notin J} u_{(n,J)} \, dx_n \wedge dx_J$$

so that

$$\partial_n \lrcorner \, \mathrm{d}u = \sum_{|I|=k, \, n \notin I} \partial_n u_I \, \mathrm{d}x_I$$

and

$$i^* * \mathrm{d}u = *_\partial (\partial_n \lrcorner \, \mathrm{d}u)|_{\partial M} = *_\partial \pi_\mathrm{t} \Lambda_g(u|_{\partial M}).$$

This shows that $\pi_\tau \Pi_g$ is a pseudo-differential operator of order 1. Notice now that for a harmonic k-form u, $v = *u$ is a harmonic $n - k$ form for which both the tangential and normal parts of Dirichlet and Neumann data are exchanged $i^* v = i^* * u$, $i^* * v = \pm i^* u$, $i^* * \mathrm{d}v = \pm i^* \delta u$ and $i^* \delta v = \pm i^* \, \mathrm{d}u$. It follows that $\pi_\nu \Pi_g$ is also a pseudo-differential operator of order 1. Thus we have proved part (i) of Corollary 2.1. For part (ii) notice that for $k = 0$ the result is proved by [6], and here $\delta u = 0$ identically so the normal part of the Neumann data gives us no information. Similarly for $k = n$ the tangential part of the Neumann data vanishes. For the case $0 < k < n$ and $k \neq (n - 1)/2$ where any tangential-tangential diagonal component of Λ_g determines the Taylor series we require, it is clear that the Taylor series is determined as long as we have $*_\partial$. We have the principal symbol $\sigma_1(\Pi_{g\tau\tau})(\xi) = *_\partial |\xi|_g$ and to finish the proof for this case we show that this determines $*_\partial$ at each point on the boundary.

Fix a point on ∂M and choose any multi-indices I_0 and J_0 such that $g_0(\xi, \xi) := (\sigma_1(\Pi_{g\tau\tau})(\xi)_{I_0 J_0})^2$ is a non-zero quadratic function of ξ. Now $g(\xi, \xi) = \alpha g_0(\xi, \xi)$ where $\alpha = 1/(*_{I_0 J_0})^2$ is to be determined. Let $*_{0\partial}$ be the Hodge star on k-forms on the boundary determined by g_0 then $*_\partial = \alpha^{k-(n-1)/2} *_{0\partial}$ and $\sigma_1(\Pi_{g\tau\tau})(\xi)_{I_0 J_0} = \alpha^{k-(n-2)/2} *_{0\partial} |\xi|_{g_0}$. As g_0 is known α is determined provided $k \neq (n - 2)/2$, hence we have g at the boundary and $*_\partial$.

For the case $k \notin \{0, (n + 1)/2, (n + 2)/2\}$ we simply apply the above argument to boundary data for $*u$. As one of the conditions on k must hold it is certainly true that the full symbol for the complete Π_g determines the Taylor series, at the boundary, of g.

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