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Dodson, CTJ

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A note on quantum chaology and gamma approximations to eigenvalue spacings for infinite random matrices

C.T.J. Dodson
School of Mathematics
University of Manchester
Manchester M13 9PL, UK

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Abstract

Quantum counterparts of certain simple classical systems can exhibit chaotic behaviour through the statistics of their energy levels. Gamma distributions do not precisely model the various analytic systems discussed here, but some features may be useful in studies of qualitative generic properties in applications to data from real systems which manifestly seem to exhibit behaviour reminiscent of near-random processes. We use known bounds on the distribution function for eigenvalue spacings for the Gaussian orthogonal ensemble (GOE) and show that gamma distributions, which have an important uniqueness property, can yield an approximation similarly good, except near the origin, to that of the widely used Wigner surmise. This has the advantage that then both the chaotic and non chaotic cases fit in the information geometric framework of the manifold of gamma distributions, which has been the subject of recent work on neighbourhoods of randomness for more general stochastic systems.

Keywords: Random matrices, quantum chaotic, eigenvalue spacing, statistics, gamma distribution, randomness, information geometry

1 Introduction

Berry introduced the term quantum chaology in his 1987 Bakerian Lecture [4] as the study of semi-classical but non-classical behaviour of systems whose classical motion exhibits chaos. He illustrated it with the statistics of energy levels, following his earlier work with Tabor [5] and related developments from the study of a range of systems. In the regular spectrum of a bound system with $n \geq 2$ degrees of freedom and n constants of motion, the energy levels are labelled by n quantum numbers, but the quantum numbers of nearby energy levels may be very different. In the case of an irregular spectrum, such as for an ergodic system where only energy is conserved, we cannot use quantum number labelling. This prompted the use of energy level spacing distributions to allow comparisons among different spectra [5]. It was known, eg from the work of Porter [13], that the spacings between energy levels of complex nuclei and atoms with n large are modelled by the spacings of eigenvalues of random matrices and that the Wigner distribution [18] gives a very good fit. It turns out that the spacing distributions for generic regular systems are negative exponential, that is random; but for irregular systems the distributions are skew and unimodal, at the scale of the mean spacing. Mehta [11] provides a detailed discussion of the numerical experiments and functional approximations to the energy level spacing statistics, Alt et al [1] compare eigenvalues from numerical analysis and from microwave resonator experiments, also eg. Bohigas et al [7] and Soshnikov [15] confirm certain universality properties.

Here we show that gamma distributions provide approximations comparable to the Wigner distribution at the scale of the mean; that may be useful because the gamma distribution has a well-understood and tractable information geometry [2, **?**] as well as the following important uniqueness property:

Theorem 1.1 (Hwang and Hu [9]) *For independent positive random variables with a common probability density function f, having independence of the sample mean and the sample coefficient of variation is equivalent to f being the gamma distribution.*

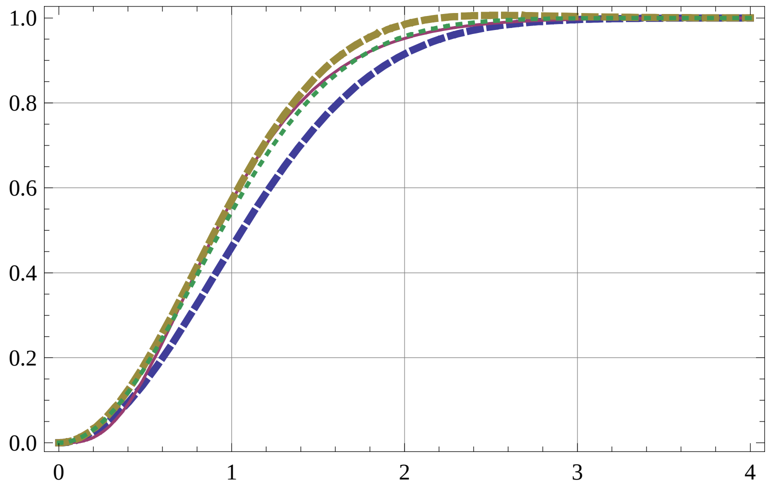

Figure 1: *The bounds on the normalized cumulative distribution function of eigenvalue spacings for the GOE of random matrices (2.3) (dashed), the Wigner surmise (2.1) (dotted) and the gamma distribution function (2.5) (solid).*

2 Eigenvalues of Random Matrices

The two classes of spectra are illustrated in two dimensions by bouncing geodesics in plane billiard tables: eg in the de-symmetrized 'stadium of Bunimovich' with ergodic chaotic behaviour and irregular spectrum on the one hand, and on the other hand in the symmetric annular region between concentric circles with non-chaotic behaviour, regular spectrum and random energy spacings [5, 7, 11, 4].

It turns out that the mean spacing between eigenvalues of infinite symmetric real random matrices—the so called Gaussian Orthogonal Ensemble (GOE)—is bounded and therefore it is convenient to normalize the distribution to have unit mean. In fact, Wigner [16, 17, 18] had already surmised that the cumulative probability distribution function at the scale of the mean spacing should be of the form:

$$W(s) = 1 - e^{-\frac{\pi s^2}{4}} \tag{2.1}$$

which has unit mean and variance $\frac{4-\pi}{\pi} \approx 0.273$ with probability density function

$$w(s) = \frac{\pi}{2} s^2 \, e^{-\frac{\pi s^2}{4}}. \tag{2.2}$$

Remarkably, Wigner's surmise gave an extremely good fit with numerical simulations and with a variety of observed data from atomic and nuclear systems [18, 5, 4, 11].

From Mehta [11] p 171, we have bounds on the cumulative probability distribution function P for the spacings between eigenvalues of infinite symmetric real random matrices:

$$L(s) = 1 - e^{-\frac{1}{16}\pi^2 s^2} \leq \; P(s) \; \leq U(s) = 1 - e^{-\frac{1}{16}\pi^2 s^2}\left(1 - \frac{\pi^2 s^2}{48}\right). \tag{2.3}$$

Here the lower bound L has mean $\frac{2}{\sqrt{\pi}} \approx 1.13$ and variance $\frac{4(4-\pi)}{\pi^2} \approx 0.348$, and the upper bound U has mean $\frac{5}{3\sqrt{5}} \approx 0.940$ and variance $\frac{96-25\pi}{9\pi^2} \approx 0.197$.

The family of probability density functions for gamma distributions having unit mean is given by

$$f(s) = \kappa^{\kappa} \frac{s^{\kappa-1}}{\Gamma(\kappa)} e^{-s\kappa}, \quad \text{for } \kappa > 0 \tag{2.4}$$

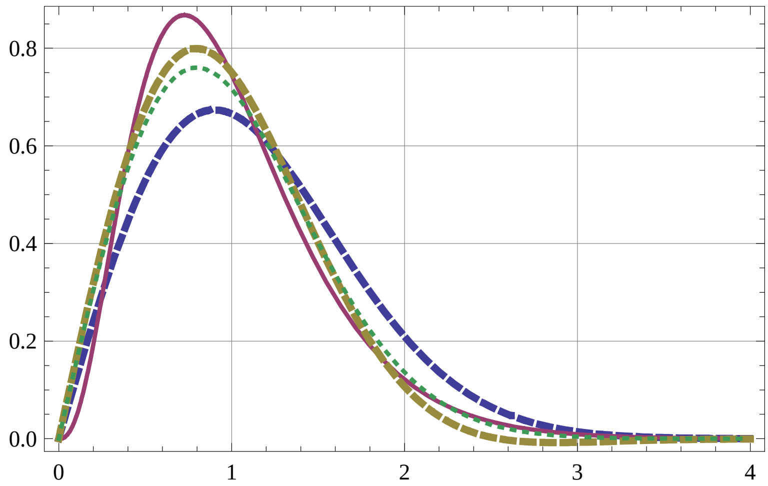

Figure 2: *Probabilty density function for gamma distribution with unit mean and same variance (2.6) (solid) as the Wigner surmised density (2.2) (dotted) and the probability densities for the bounds (2.8) (dashed) for the distribution of normalized spacings between eigenvalues for infinite symmetric real random matrices.*

and variance $\frac{1}{\kappa}$. The maximum entropy case has $\kappa = 1$ and corresponds to an underlying Poisson random event process; for $\kappa > 1$ the distributions are skew unimodular. The maximum likelihood gamma distribution, of unit mean and the same variance as Wigner's, has $\kappa = \frac{\pi}{4-\pi} \approx 3.660$ with cumulative probability distribution function given by

$$G(s) = 1 - \frac{\Gamma\left(\frac{\pi}{4-\pi}, \frac{\pi s}{4-\pi}\right)}{\Gamma\left(\frac{\pi}{4-\pi}\right)} \tag{2.5}$$

and probability density function

$$g(s) = \frac{e^{\frac{-\pi s}{4-\pi}} \left(\frac{\pi s}{4-\pi}\right)^{\frac{\pi}{4-\pi}}}{s\Gamma\left(\frac{\pi}{4-\pi}\right)}. \tag{2.6}$$

In fact, κ is a geodesic coordinate in the Riemannian 2-manifold of gamma distributions with Fisher information metric [2]; arc length along this coordinate from $\kappa = a$ to $\kappa = b$ is given by

$$|\int_a^b \sqrt{\frac{d^2 \log(\Gamma(\kappa))}{d\kappa^2} - \frac{1}{\kappa}}\, d\kappa| \,. \tag{2.7}$$

Plotted in Figure 1 are the cumulative distributions for the bounds (2.3) (dashed), the maximum likelihood gamma distribution with unit mean (2.5) (solid), and the Wigner surmised distribution (2.1) (dotted). The gamma distribution and the Wigner surmise are very close together and within the bounds but they differ near the origin. It would be interesting to know if other irregular spectra exhibit energy spacings with different values of $\kappa > 1$. From a different standpoint, Berry and Robnik [6] gave a statistical model using a mixture of energy level spacing sequences from exponential and Wigner distributions.

The corresponding probability density functions in Figure 2 for the lower and upper bounds are the derivatives of the bounding cumulative distributions from (2.3), respectively,

$$l(s) = \frac{1}{2} e^{-\frac{\pi s^2}{4}} \pi s, \;\; u(s) = -\frac{1}{384} e^{-\frac{1}{16}\pi^2 s^2} \pi^2 s \left(\pi^2 s^2 - 64\right). \tag{2.8}$$

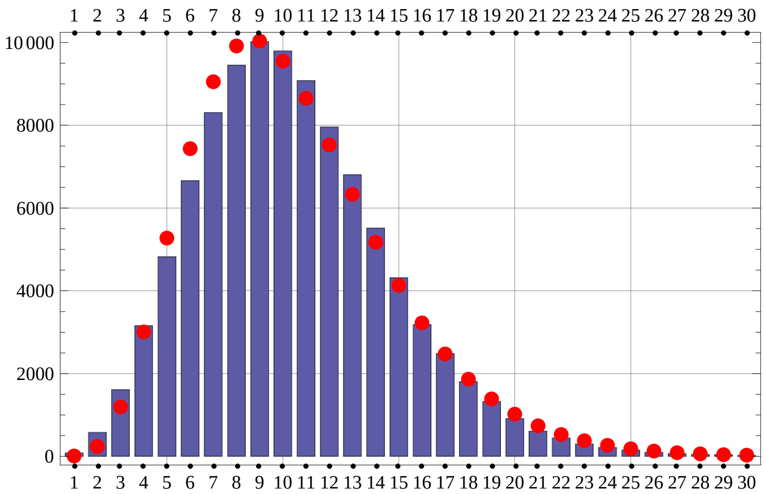

Figure 3: *Frequency histogram with unit mean for the spacings between the first 100,000 zeros of the Riemann zeta function from the tabulation of Odlyzko [12], and the corresponding points from the maximum likelihood gamma distribution which has* $\kappa = 5.427$*. The data has been collected into 30 bins each of width 0.1.*

3 Asymptotic Deviations

Berry has pointed out [3] that the gamma density (2.6) with $\kappa = \frac{\pi}{4-\pi}$ does not have the correct asymptotic linear behaviour at the origin for modelling the GOE case and that the behaviour near the origin is an important feature of the ensemble statistics of these matrices. Moreover, for the unitary ensemble (GUE) of complex hermitian matrices, near the origin, the behaviour is $\sim s^2$ and for the symplectic ensemble (GSM, representing half-integer spin particles with time-reversal symmetric interactions) it is $\sim s^4$.

From (2.6) we see that at unit mean the gamma density behaves like $s^{\kappa-1}$ near the origin, so linear behaviour would require $\kappa = 2$ which gives a variance of $\frac{1}{\kappa} = \frac{1}{2}$. This may be compared with variances for the lower bound l, $\frac{4(4-\pi)}{\pi^2} \approx 0.348$, the upper bound u, $\frac{96-25\pi}{9\pi^2} \approx 0.197$, and the Wigner distribution w, $\frac{4-\pi}{\pi} \approx 0.273$. We note that the maximum likelihood gamma distributions fitted to the lower and upper bounding distributions have, respectively, $\kappa_L = \frac{\pi}{4-\pi} \approx 3.660$ and $\kappa_U = \frac{25\pi}{96-25\pi} \approx 4.498$.

The author is indebted to Rudnick [14] for pointing out that the GUE eigenvalue spacing distribution is rather closely followed by the distribution of zeros for the Riemann zeta function. The spacings between the first 100,000 zeros provided by Odlyzko [12], normalized here to unit mean, and a maximum likelihood gamma distribution are shown in Figure 3; the actual mean spacing was ≈ 0.749 and the variance was ≈ 0.429 so the gamma distribution has $\kappa = 5.427$.

Remark It is clear that gamma distributions do not precisely model the analytic systems discussed here, but some features may be useful in studies of qualitative generic properties in applications to data from real systems. The gamma distribution provides a similarly good model to the Wigner distribution, except near the origin, for the spacings between GOE eigenvalues of infinite symmetric real random matrices. It would be interesting to investigate whether data from real atomic and nuclear systems has generally the property that the sample coefficient of variation is independent of the mean. That by Theorem 1.1 is an information-theoretic distinguishing property of the gamma distribution.

References

[1] H. Alt, C. Dembrowski, H.D. Graf, R. Hofferbert, H. Rehfield, A. Richter and C. Schmit. Experimental versus numerical eigenvalues of a Bunimovich stadium billiard: A comparison. *Phys. Rev. E* 60, 3 (1999) 2851-2857.

[2] Khadiga Arwini and C.T.J. Dodson. Information geometric neighbourhoods of randomness and geometry of the McKay bivariate gamma 3-manifold. *Sankhya: Indian Journal of Statistics* 66, 2 (2004) 211-231.

[3] M.V. Berry. Private communication. 2008.

[4] M.V. Berry. Quantum Chaology. *Proc. Roy. Soc. London A* 413, (1987) 183-198.

[5] M.V. Berry and M. Tabor. Level clustering in the regular spectrum. *Proc. Roy. Soc. London A* 356, (1977) 373-394.

[6] M.V. Berry and M. Robnik. Semiclassical level spacings when regular and chaotic orbits coexist. *J. Phys. A Math. General* 17, (1984) 2413-2421.

[7] O. Bohigas, M.J. Giannoni and C. Schmit. Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws. *Phys. Rev. Lett.* 52, 1 (1984) 1-4.

[8] C.T.J. Dodson. Quantifying galactic clustering and departures from randomness of the inter-galactic void probablity function using information geometry. http://arxiv.org/abs/astro-ph/0608511 (2006).

[9] T-Y. Hwang and C-Y. Hu. On a characterization of the gamma distribution: The independence of the sample mean and the sample coefficient of variation. *Annals Inst. Statist. Math.* 51, 4 (1999) 749-753.

[10] S. Kullback. **Information and Statistics**, J. Wiley, New York, 1959.

[11] Madan Lal Mehta. **Random Matrices** 3^{rd} Edition, Academic Press, London 2004.

[12] A. Odlyzko. Tables of zeros of the Riemann zeta function.
`http://www.dtc.umn.edu:80/~odlyzko/zeta_tables/index.html` .

[13] C.F. Porter. **Statistical Theory of Spectra: Fluctuations** Edition, Academic Press, London 1965.

[14] Z. Rudnick. Private communication. 2008. Cf. also Z. Rudnick. What is Quantum Chaos? *Notices A.M.S.* 55, 1 (2008) 33-35.

[15] A. Soshnikov. Universality at the edge of the spectrum in Wigner random matrices. *Commun. Math. Phys.* 207 (1999) 697-733.

[16] E.P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Annals of Mathematics* 62, 3 (1955) 548-564.

[17] E.P. Wigner. On the distribution of the roots of certain symmetric matrices. *Annals of Mathematics* 67, 2 (1958) 325-327.

[18] E.P. Wigner. Random matrices in physics. *SIAM Review* 9, 1 (1967) 1-23.